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Some generalizations of the class of analytic functions with respect to k-symmetric points

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Abstract

In this article, we introduce a new class $M_s^k[\alpha,\beta,\lambda]$ which generalizes the various classes of analytic functions with respect to k-symmetric points. Naturally, the class $M_s^k[\alpha,\beta,\lambda]$ combines the classes $S_s^k(\alpha,\beta)$ and $C_s^k(\alpha,\beta)$. We also study the coefficient estimates and obtain some inequalities related to the coefficients of functions in these classes. We develop the integral representation, inclusions and convolution conditions for the functions in the class $M_s^k[\alpha,\beta,\lambda]$.

Keywords: Convex functions, Starlike functions, Symmetric points, subordination.

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1. Introduction and preliminaries

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. A function f is said to be subordinate to a function g written as $f \prec g$, if there exists a schawarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$.

The classes of starlike and convex univalent functions are defined as

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$$S^* = \left\{ f : f \in A \; and \; Re \, \frac{zf'(z)}{f(z)} > 0, \; z \in E \right\},$$

$$C = \left\{ f : f \in A \, and \; Re \frac{(zf'(z))'}{f'(z)} > 0, \quad z \in E \right\}.$$

A function which is analytic in the open unit disc E is said to be starlike with respect to the symmetric point if it satisfies

$$\operatorname{Re} \frac{(zf'(z))}{f(z) - f(-z)} > 0, \; z \in E.$$

The class S_s^* of starlike functions with respect to symmetric points was introduced and studied by Sakaguchi, see [5]. The class $S_s^k[\alpha, \beta]$ consists of functions $f \in S$ satisfying the inequality

$$\left| \frac{2zf'(z)}{f_k(z)} - 1 \right| < \beta \left| \frac{2\alpha zf'(z)}{f_k(z)} + 1 \right| \; for \; 0 \leqslant \alpha \leqslant 1, \; 0 \leqslant \beta \leqslant 1, \; z \in E,$$

where f_k is defined as

$$f_k(z) = \frac{1}{k} \sum_{m=0}^{k-1} \varkappa^{-m} f(\varkappa^m z) \tag{1.2}$$

with $\varkappa^k = 1$ and $k \geqslant 1$ a fixed positive integer. Similarly the class$\}C_s^k[\alpha, \beta]$ is defined by:

$$C_s^k[\alpha, \beta] = \left\{ f \in S \text{ and } \left| \frac{(\mathrm{zf'(z)})'}{\mathrm{f_k'(z)}} - 1 \right| < \beta \left| \frac{\alpha \, (\mathrm{zf'(z)})'}{\mathrm{f_k'(z)}} + 1 \right| \right\},$$

where $0 \leqslant \alpha \leqslant 1, \; 0 < \beta \leqslant 1$ and f_k is given in (1.2) . The classes $S_s^k[\alpha, \beta]$ and $C_s^k[\alpha, \beta]$ were defined by Wang [8] and Gao and Zhou [2] respectively. These classes are further studied in [7, 9]. Keeping in view the above mentioned classes, we define the following subclass of analytic function with respect to k-symmetric point.

1.1. Definition. A function $f \in S$ is in the class $M_s^k[\alpha, \beta, \lambda]$ if it satisfies the following condition:

$$\left| (1-\lambda) \frac{zf'(z)}{f_k(z)} + \lambda \frac{(zf'(z))'}{f_k'(z)} - 1 \right| < \beta \left| \alpha \left[(1-\lambda) \frac{zf'(z)}{f_k(z)} + \lambda \frac{(zf'(z))'}{f_k'(z)} \right] + 1 \right|, \tag{1.3}$$

where f_k is defined by (1.2), $0 \leqslant \alpha \leqslant 1, \; 0 < \beta \leqslant 1, \; k \geqslant 1$ a fixed positive integer, $\lambda \in \mathbb{R}$ and $z \in E$.

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Special Cases

(i). For $k = 2$, the class $M_s^k[\alpha, \beta, \lambda]$ reduces to the class $M[\alpha, \beta, \lambda]$ consisting of univalent functions which satisfy the condition:

$$\left|(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\frac{(zf'(z))'}{f'(z)}-1\right|<\beta\left|\alpha\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\frac{(zf'(z))'}{f'(z)}\right]+1\right|,$$

where $0\leqslant\alpha\leqslant 1,\ 0<\beta\leqslant 1,\ k\geqslant 1,\ \lambda\in\mathbb{R}$ and $z\in E$.
(ii). For $\lambda=1$, the class $M_s^k[\alpha,\beta,\lambda]$ yields the class $C_s^k(\alpha,\beta)$ introduced and studied by Wang [8].
(iii). When $\lambda=0$, the class $M_s^k[\alpha,\beta,\lambda]$ produces the class $S_s^k(\alpha,\beta)$ studied by Gao and Zhou [2].
(iv). For $k=2,\ \lambda=1$, we obtain the class $C_s(\alpha,\beta)$.
(v). Taking $k=2,\ \lambda=0,\ M_s^k[\alpha,\beta,\lambda]$ reduces to the class $S_s^*(\alpha,\beta)$, see [6].
(vi). For $k=2,\ \lambda=0,\ \alpha=\beta=1,\ M_s^k[\alpha,\beta,\lambda]$ reduces to the class S_s^* [5].

In the following, we have some useful lemmas.

1.2. Lemma. [1] *Suppose that the function φ is convex univalent in E with $\varphi(0)=1$ and*

$$\operatorname{Re}(\beta\varphi(z)+\gamma)>0 \text{ for } \beta,\gamma\in\mathbb{C},\ \mathrm{z}\in\mathrm{E}.$$

If p is analytic in E with $p(0)=1$, then

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\varphi(z)\ \textit{implies}\ p(z)\prec\varphi(z),\ z\in E.$$

1.3. Lemma. [4] *Let $\beta,\gamma\in\mathbb{C}$ and φ be a convex, univalent function with*

$$\operatorname{Re}(\beta\varphi(z)+\gamma)>0,\ z\in E.$$

Also let $h\in A: h(z)\prec\varphi(z)$. If $p\in P$ and

$$p(z)+\frac{zp'(z)}{\beta h(z)+\gamma}\prec\varphi,\ \textit{then}\ p(z)\prec\varphi(z).$$

1.4. Lemma. [6] *Let G be analytic in E and let*

$$\left|\frac{1-G(z)}{1+\eta G(z)}\right|<\delta \qquad (1.4)$$

$z\in E,\ 0\leqslant\eta\leqslant 1,\ 0<\delta\leqslant 1$ with $G(0)=1$. Then

$$G(z)=\frac{1-z\varphi(z)}{1+\eta z\varphi(z)}, \qquad (1.5)$$

where φ is analytic in E and $|\varphi(z)|\leqslant\delta$ for $z\in E$. Conversely any function G given by (1.5) is analytic in E and satisfies (1.4).

1.5. Lemma. [3] *Let F be analytic and convex in E. If $f,\ g\in A$ and $f,\ g\prec F$. Then*

$$\sigma f+(1-\sigma)g\prec F,\quad 0\leqslant\sigma\leqslant 1.$$

2. Main Results

2.1. Theorem. *Let* $F_k(z) = (1-\lambda)\frac{zf'(z)}{f_k(z)} + \lambda\frac{(zf'(z))'}{f_k'(z)}$, $\lambda \in \mathbb{R}$, $k \geqslant 1$, f *and* f_k *defined by (1.1) and 1.2 respectively. Then* $F_k(z) = 1 + \sum_{j=2}^{\infty} c_j z^{j-1} + \ldots \in P$ *for* $c_j = \left[(1-\lambda(1-j))j + ((1-\lambda)j + \lambda - (1+j))\, d_j\right] a_j$, *where* $c_j \leqslant 2$.

Proof. Here we let

$$F_k(z) = (1-\lambda)\frac{zf'(z)}{f_k(z)} + \lambda\frac{(zf'(z))'}{f_k'(z)}. \tag{2.1}$$

It can easily follows from (1.2) that

$$\begin{aligned} f_k(z) &= \frac{1}{k}\sum_{m=0}^{k-1} \varkappa^{-m} f(\varkappa^m z) = \frac{1}{k}\sum_{m=0}^{k-1} \varkappa^{-m}\left[\varkappa^m z + \sum_{j=2}^{\infty} a_j \left(\varkappa^m z\right)^j\right] \\ &= z + \sum_{l=2}^{\infty} a_{(l-1)k+1} z^{(l-1)k+1} = z + \sum_{j=2}^{\infty} a_j d_j z^j, \end{aligned} \tag{2.2}$$

where $d_j = \frac{1}{k}\sum\limits_{m=0}^{k-1} \varkappa^{(j-1)m} = \left\{\begin{array}{ll} 1, & j = (l-1)k+1 \\ 0, & j \neq (l-1)k+1 \end{array}\right\}$, $\varkappa^k = 1$. On combining (2.1) and (2.2) , we have

$$\begin{aligned} F_k(z) &= \frac{(1-\lambda)zf'(z)f_k'(z) + \lambda(f'(z) + zf''(z)))f_k(z)}{f_k(z)f_k'(z)} \\ &= \frac{(1-\lambda)(z + \sum\limits_{j=2}^{\infty} ja_j z^j)(1 + \sum\limits_{j=2}^{\infty} ja_j d_j z^{j-1}) + \lambda(1 + \sum\limits_{j=2}^{\infty} j^2 a_j z^{j-1})(z + \sum\limits_{j=2}^{\infty} a_j d_j z^j)}{(z + \sum\limits_{j=2}^{\infty} a_j d_j z^j)(1 + \sum\limits_{j=2}^{\infty} ja_j d_j z^{j-1})} \\ &= \frac{1 + \sum\limits_{j=2}^{\infty}\left[\left((1-\lambda+j\lambda)\, ja_j + (1-\lambda)ja_jd_j + \lambda a_j d_j\right) z^{j-1} + j^2 a_j^2 d_j z^{2j-2}\right]}{1 + \sum\limits_{j=2}^{\infty}\left[(1+j)\, a_j d_j z^{j-1} + ja_j^2 d_j z^{2j-2}\right]} \\ &= \left[1 + \sum_{j=2}^{\infty}\left[\left((1-\lambda+j\lambda)\, ja_j + (1-\lambda)ja_jd_j + \lambda a_j d_j\right) z^{j-1} + j^2 a_j^2 d_j z^{2j-2}\right]\right] \times \\ &\quad \left[1 - \sum_{j=2}^{\infty}\left[(1+j)\, a_j d_j z^{j-1} + ja_j^2 d_j z^{2j-2}\right] + \ldots\right] \\ &= 1 + \sum_{j=2}^{\infty}\left[(1-\lambda+j\lambda)\, j + ((1-\lambda)j + \lambda - (1+j))\, d_j\right] a_j z^{j-1} + \ldots \\ &= 1 + \sum_{j=2}^{\infty} c_j z^{j-1} + \ldots. \end{aligned}$$

Thus $F_k(z) = 1 + \sum\limits_{j=2}^{\infty} c_j z^{j-1} + ... \in P$,

where $c_j = \left[(1-\lambda+j\lambda)\,j + ((1-\lambda)j + \lambda - (1+j))\,d_j\right] a_j$ such that $|c_j| \leqslant 2$. ∎

2.2. Theorem. *Let $0 \leqslant \alpha \leqslant 1,\ 0 < \beta \leqslant 1,\ k \geqslant 1,\ \lambda \in \mathbb{R}$, and $z \in E$. Then the function $f \in M_s^k[\alpha, \beta, \lambda]$ if and only if*

$$(2.3) \quad (1-\lambda)\frac{zf'(z)}{f_k(z)} + \lambda\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

where f_k is given in (1.2).

Proof. Let $f \in M_s^k[\alpha, \beta, \lambda]$. Then from (1.3), we have

$$\left|(1-\lambda)\frac{zf'(z)}{f_k(z)} + \lambda\frac{(zf'(z))'}{f_k'(z)} - 1\right| < \beta\left|\alpha\left[(1-\lambda)\frac{zf'(z)}{f_k(z)} + \lambda\frac{(zf'(z))'}{f_k'(z)}\right] + 1\right|.$$

Taking F_k as defined in (2.1) we write

$$|F_k(z) - 1|^2 < \beta^2\,|\alpha F_k(z) + 1|^2$$

or

$$\left(1-\alpha^2\beta^2\right)|F_k(z)|^2 - 2(1+\alpha\beta^2)\operatorname{Re} F_k(z) < \beta^2 - 1.$$

If $\alpha \neq 1$ or $\beta \neq 1$, then we have

$$|F_k(z)|^2 - 2\left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)\operatorname{Re} F_k(z) + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2 < \frac{\beta^2-1}{1-\alpha^2\beta^2} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2$$

or

$$\left|F_k(z) - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|^2 < \frac{\beta^2(1+\alpha)^2}{\left(1-\alpha^2\beta^2\right)^2}.$$

This represents the disk with center at $\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}$ and radius $\frac{\beta(1+\alpha)}{\left(1-\alpha^2\beta^2\right)}$. Also the function $\omega(z) \prec \varphi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ maps the unit disk onto the disk

$$\left|\omega - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{\left(1-\alpha^2\beta^2\right)}.$$

From here we notice that, $F_k(E) \subset \varphi(E),\ F_k(0) = \varphi(0)$ and φ is univalent in E. Therefore, we write

$$F_k(z) \prec \varphi(z) = \frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, let $F_k(z) \prec \frac{1+\beta z}{1-\alpha\beta z}$. Then using subordination, we have

$$(2.4) \quad F_k(z) = \frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)},$$

where $\omega \in \Omega$. From (2.4), we write

$$|F_k(z)-1| = \left|\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}-1\right| = \left|\frac{1+\beta\omega(z)-1+\alpha\beta\omega(z)}{1-\alpha\beta\omega(z)}\right|$$

$$|F_k(z)-1| = \beta\left|\frac{(1+\alpha)\,\omega(z)}{1-\alpha\beta\omega(z)}\right|. \tag{2.5}$$

Also

$$|\alpha F_k(z)+1| = \left|\frac{\alpha+\alpha\beta\omega(z)}{1-\alpha\beta\omega(z)}+1\right| < \beta\left|\frac{(1+\alpha)}{1-\alpha\beta\omega(z)}\right|. \tag{2.6}$$

Using (2.5) in (2.6) , we have

$$|F_k(z)-1| < \beta\,|\alpha F_k(z)+1|\,,\ where\ |\omega(z)|<1\ for\ all\ z\in E.$$

This implies that

$$\left|(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}-1\right| < \beta\left|\alpha\left[(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}\right]+1\right|.$$

Hence, $f \in M_s^k[\alpha,\beta,\lambda]$. ∎

2.3. Theorem. *Let $f \in M_s^k[\alpha,\beta,\lambda]$. Then $f_k \in M_s[\alpha,\beta,\lambda]$ and also $f_k \in S_s^k(\alpha,\beta)$.*

Proof. Let $f \in M_s^k[\alpha,\beta,\lambda]$. Then by Theorem 2.2, we have

$$(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

where $0\leqslant\alpha\leqslant1,\ 0<\beta\leqslant1,\ k\geqslant1$ (is fixed positive integer), $\lambda\in\mathbb{R}$ and f_k is defined by (1.2). Now using subordination, we write

$$(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)} = \frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)},\ \omega(0)=0 \text{ and } |\omega(\mathrm{z})|<1. \tag{2.7}$$

On replacing z by $\varkappa^m z$, where $m = 0,1,2,...,k-1$ and $\varkappa^k = 1$ in (2.7) , we have

$$(1-\lambda)\frac{\varkappa^m zf'(\varkappa^m z)}{f_k(\varkappa^m z)}+\lambda\frac{f'(\varkappa^m z)+\varkappa^m zf''(\varkappa^m z)}{f_k'(\varkappa^m z)} = \frac{1+\beta\omega(\varkappa^m z)}{1-\alpha\beta\omega(\varkappa^m z)}. \tag{2.8}$$

From (1.2), we write $f_k(\varkappa^m z) = \varkappa^m f_k(z)$ and $f_k'(\varkappa^m z) = f_k'(z)$. These results along with (2.8) yield

$$(1-\lambda)\frac{\varkappa^m zf'(\varkappa^m z)}{\varkappa^m f_k(z)}+\lambda\frac{f'(\varkappa^m z)+\varkappa^m zf''(\varkappa^m z)}{f_k'(z)} = \frac{1+\beta\omega(\varkappa^m z)}{1-\alpha\beta\omega(\varkappa^m z)}.$$

Taking summation from $m=0$ to $k-1$, we write

$$(1-\lambda)\frac{1}{k}\sum_{m=0}^{k-1}\frac{\varkappa^m zf'(\varkappa^m z)}{\varkappa^m f_k(z)}+\frac{\lambda}{f_k'(z)}(\frac{1}{k}\sum_{m=0}^{k-1}f'(\varkappa^m z)+\frac{1}{k}\sum_{m=0}^{k-1}\varkappa^m zf''(\varkappa^m z))$$

$$=\quad \frac{1}{k}\sum_{m=0}^{k-1}\left(\frac{1+\beta\omega\left(\varkappa^m z\right)}{1-\alpha\beta\omega\left(\varkappa^m z\right)}\right)$$

or

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}=\frac{1}{k}\sum_{m=0}^{k-1}\left(\frac{1+\beta\omega\left(\varkappa^m z\right)}{1-\alpha\beta\omega\left(\varkappa^m z\right)}\right)$$

or

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}=\frac{1}{k}\sum_{m=0}^{k-1}\left(\frac{1+\beta\omega\left(\varkappa^m z\right)}{1-\alpha\beta\omega\left(\varkappa^m z\right)}\right)\in P[\alpha,\beta],$$

where $P[\alpha,\beta]$ is a convex set and containing function $p(z)\prec\frac{1+\beta z}{1-\alpha\beta z}$. This implies that

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

which implies that $f_k\in M_s[\alpha,\beta,\lambda]$. Now, let $h(z)=\frac{zf_k'(z)}{f_k(z)}$. After some manipulation, we have

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}=h\left(z\right)+\lambda\frac{zh'\left(z\right)}{h\left(z\right)}. \tag{2.9}$$

This implies that

$$h\left(z\right)+\lambda\frac{zh'\left(z\right)}{h\left(z\right)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Using Lemma 1.2, we obtain

$$h\left(z\right)\prec\frac{1+\beta z}{1-\alpha\beta z}. \tag{2.10}$$

Hence, $f_k\in S_s^k\left(\alpha,\beta\right)$. ∎

2.4. Theorem. *Let* $0\leqslant\alpha\leqslant1$, $0<\beta\leqslant1$, $k\geqslant2$ *(fixed positive integer),* $\lambda>0$. *Then* $M_s^k\left(\alpha,\beta,\lambda\right)\subset S_s^k\left(\alpha,\beta\right)\subset S$.

Proof. For $f\in M_s^k\left[\alpha,\beta,\lambda\right]$, we have

$$(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Now, we let

$$p\left(z\right)=\frac{zf'(z)}{f_k(z)}\ \ and\, h(z)=\frac{zf_k'(z)}{f_k(z)},$$

where, h and p satisfy the conditions described in Lemma 1.3. Therefore

$$(2.11)\quad (1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}=p(z)+\lambda\frac{zp'(z)}{h(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

By using (2.10) ,(2.11) and Lemma 1.3, we obtain

$$p(z)=\frac{zf'(z)}{f_k(z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

which implies that , $f\in S_s^{(k)}(\alpha,\beta)\subset S$ or $M_s^k(\alpha,\beta,\lambda)\subset S_s^{(k)}(\alpha,\beta)\subset S.$ ∎

2.5. Theorem. *Let* $0\leqslant\alpha\leqslant 1$ *and* $0<\beta\leqslant 1,\ 0\leqslant\lambda_1<\lambda_2$. *Then* $M_s^k[\alpha,\beta,\lambda_2]\subset M_s^k[\alpha,\beta,\lambda_1]$.

Proof. Suppose that $f\in M_s^k[\alpha,\beta,\lambda_2]$. Then by Theorem 2.2, we have

$$h_1(z)=(1-\lambda_2)\frac{zf'(z)}{f_k(z)}+\lambda_2\frac{(zf'(z))'}{f_k'(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Also from Theorem 2.4, we write

$$h_2(z)=\frac{zf'(z)}{f_k(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Now

$$\begin{aligned}(1-\lambda_1)\frac{zf'(z)}{f_k(z)}+\lambda_1\frac{(zf'(z))'}{f_k'(z)} &= \left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{zf'(z)}{f_k(z)}+\frac{\lambda_1}{\lambda_2}\left\{(1-\lambda_2)\frac{zf'(z)}{f_k(z)}+\lambda_2\frac{(zf'(z))'}{f_k'(z)}\right\}\\ &= \left(1-\frac{\lambda_1}{\lambda_2}\right)h_2(z)+\frac{\lambda_1}{\lambda_2}h_1(z)\end{aligned}$$

Since $\frac{1+\beta z}{1-\alpha\beta z}$ is convex set, therefore by using Lemma 1.5 we get the required result. ∎

2.6. Theorem. *Let* $f\in M_s^k[\alpha,\beta,\lambda]$. *Then*

$$f_k(z)=\left[\frac{1}{\lambda}\int_0^z\frac{1}{u}\left\{u.\exp\sum_{m=0}^{k-1}\int_0^u\frac{(1+\alpha)\beta\omega(t)}{t(k-\alpha\beta\omega(t))}dt\right\}^{\frac{1}{\lambda}}du\right]^{\lambda},$$

where $\omega\in\Omega$. *For* $\lambda=0$,

$$f(z)=\int_0^z\frac{f_k(\lambda)}{\lambda}\frac{1+\beta\omega(\lambda)}{1-\alpha\beta\omega(\lambda)}d\lambda.$$

Proof. Let $f\in M_s^k[\alpha,\beta,\lambda]$. Then from Theorem 2.2, we have

$$(1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Using subordination, we obtain

$$(2.12)\quad (1-\lambda)\frac{zf'(z)}{f_k(z)}+\lambda\frac{(zf'(z))'}{f_k'(z)}=\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)},$$

where ω is analytic in E with $\omega(0)=0$ and $|\omega(z)|<1$. Now replacing z by $\varkappa^m z$, where $m=0,1,2...,k-1$, $\varkappa^k=1$, using (1.2) with $f_k(\varkappa^m z)=\varkappa^m f_k(z)$ and $f_k'(\varkappa^m z)=f_k'(z)$ and then taking summation for $m=0,1,2...,k-1$ in (2.12), we obtain

$$\frac{1-\lambda}{k}\left\{\sum_{m=0}^{k-1}\frac{\varkappa^m zf'(\varkappa^m z)}{\varkappa^m f_k(z)}\right\}+\frac{\lambda}{k}\frac{1}{f_k'(z)}\left\{\sum_{m=0}^{k-1}f'(\varkappa^m z)+\sum_{m=0}^{k-1}\varkappa^m zf''(\varkappa^m z)\right\}$$

$$=\frac{1+\frac{\beta}{k}\sum\limits_{m=0}^{k-1}\omega(\varkappa^m z)}{1-\frac{\alpha\beta}{k}\sum\limits_{m=0}^{k-1}\omega(\varkappa^m z)}.$$

This implies that

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}=\left(k+\beta\sum_{m=0}^{k-1}\omega(\varkappa^m z)\right)\Big/\left(k-\alpha\beta\sum_{m=0}^{k-1}\omega(\varkappa^m z)\right)$$

or

$$(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}-\frac{1}{z}=\left((1+\alpha)\beta\sum_{m=0}^{k-1}\omega(\varkappa^m z)\right)\Big/z\left(k-\alpha\beta\sum_{m=0}^{k-1}\omega(\varkappa^m z)\right).$$

Integrating from 0 to z, we write

$$\log\left\{\frac{(f_k(z))^{(1-\lambda)}(zf_k'(z))^{\lambda}}{z}\right\}=\int_0^z\frac{(1+\alpha)\beta\sum\limits_{m=0}^{k-1}\omega(\varkappa^m\zeta)}{\zeta(k-\alpha\beta\sum\limits_{m=0}^{k-1}\omega(\varkappa^m\zeta))}d\zeta$$

or

$$(2.13)\quad \left[\frac{zf_k'(z)}{f_k(z)}\right]^{\lambda}f_k(z)=z\exp\sum_{m=0}^{k-1}\int_0^z\frac{(1+\alpha)\beta\omega(\varkappa^m\zeta)}{\zeta(k-\alpha\beta\omega(\varkappa^m\zeta))}d\zeta.$$

Let

$$(2.14)\quad F_k(z)=\left[\frac{zf_k'(z)}{f_k(z)}\right]^{\lambda}f_k(z),\quad F_k(0)=0,\ F_k'(0)=1.$$

Since f_k is λ-convex and if λ is not an integer, then we can select a suitable branch, so that F_k is analytic in E. Logarithmic differentiation of (2.14) yields

$$\frac{zF_k'(z)}{F_k(z)}=(1-\lambda)\frac{zf_k'(z)}{f_k(z)}+\lambda\frac{(zf_k'(z))'}{f_k'(z)}.$$

Hence, F_k is starlike in E. Now we solve (2.14) for f_k by assuming that $\lambda \neq 0$. (The case when $\lambda = 0$ gives $F_k(z) = f_k(z)$). Formal manipulations leads to the solution

$$(2.15)\quad f_k(z) = \left[\frac{1}{\lambda} \int_0^z \frac{[F_k(\zeta)]^{\frac{1}{\lambda}}}{\zeta} d\zeta \right]^{\lambda}.$$

By using (2.13) and (2.15), we have

$$\left[\frac{zf_k'(z)}{f_k(z)} \right]^{\lambda} f_k(z) = z.\exp \sum_{m=0}^{k-1} \int_0^z \frac{(1+\alpha)\,\beta\omega\,(\varkappa^m \zeta)}{\zeta\,(k - \alpha\beta\omega\,(\varkappa^m \zeta))} d\zeta,$$

which implies that

$$f_k(z) = \left[\frac{1}{\lambda} \int_0^z \frac{1}{u} \left[u.\exp \left\{ \sum_{m=0}^{k-1} \int_0^u \frac{(1+\alpha)\,\beta\omega\,(t)}{t\,(k - \alpha\beta\omega\,(t))} dt \right\} \right]^{\frac{1}{\lambda}} du \right]^{\lambda}.$$

This is the required integral representation for f_k when $f \in M_s^k[\alpha, \beta, \lambda]$. It can be easily verified that for $\lambda = 0$,

$$f(z) = \int_0^z \frac{f_k(\lambda)}{\lambda} \frac{1 + \beta\omega\,(\lambda)}{1 - \alpha\beta\omega\,(\lambda)} d\lambda.$$

∎

2.7. Theorem. *Let $f \in M_s^k[\alpha, \beta, \lambda]$. Then we have*

$$f(z) = \int_0^z \frac{1+c}{\lambda\,[f_k(\lambda)]^c} \int_0^{\varkappa^m \lambda} [f_k(t)]^c\, f_k'(t) \frac{1 + \beta\omega\,(t)}{1 - \alpha\beta\omega\,(t)} dt d\lambda,$$

where f_k is given in (1.2), *$c = \frac{1}{\lambda} - 1 : \lambda \neq 0$ and ω is analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$. If $\lambda = 0$, then we have*

$$f(z) = \int_0^z \frac{f_k(\lambda)}{\lambda} \frac{1 + \beta\omega\,(\lambda)}{1 - \alpha\beta\omega\,(\lambda)} d\lambda.$$

Proof. Let $f \in M_s^k[\alpha, \beta, \lambda]$. Then

$$(2.16)\quad (1-\lambda) \frac{zf'(z)}{f_k(z)} + \lambda \frac{(zf'(z))'}{f_k'(z)} = \frac{1 + \beta\omega\,(z)}{1 - \alpha\beta\omega\,(z)},$$

where $\omega \in \Omega$. Multiplying both sides of (2.16) by $\lambda^{-1}\,[f_k(z)]^c\, f_k'(z)$, where $c = \frac{1}{\lambda} - 1 : \lambda \neq 0$, we get

(2.17)

$$czf'(z)\,[f_k(z)]^{c-1}\, f_k'(z) + [f_k(z)]^c\,(zf'(z))' = (1+c)\,[f_k(z)]^c\, f_k'(z) \frac{1 + \beta\omega\,(z)}{1 - \alpha\beta\omega\,(z)}.$$

The left hand side of (2.17) is the exact differential of $zf'(z)\left[f_k(z)\right]^c$. On integrating both sides of (2.17), we obtain

$$f'(z)=\frac{1+c}{z\left[f_k(z)\right]^c}\int_0^z\left[f_k(\zeta)\right]^c f_k'(\zeta)\frac{1+\beta\omega\left(\zeta\right)}{1-\alpha\beta\omega\left(\zeta\right)}d\zeta$$

or

$$f'(z)=\frac{1+c}{z\left[f_k(z)\right]^c}\int_0^{\varkappa^m z}\left[f_k(t)\right]^c f_k'(t)\frac{1+\beta\omega\left(t\right)}{1-\alpha\beta\omega\left(t\right)}dt.$$

This implies that

$$f(z)=\int_0^z\frac{1+c}{\lambda\left[f_k(\lambda)\right]^c}\int\limits_0^{\varkappa^m\lambda}\left[f_k(t)\right]^c f_k'(t)\frac{1+\beta\omega\left(t\right)}{1-\alpha\beta\omega\left(t\right)}dtd\lambda.$$

If $\lambda=0$, then we have

$$f'(z)=\frac{f_k(z)}{z}\frac{1+\beta\omega\left(z\right)}{1-\alpha\beta\omega\left(z\right)},$$

which implies that

$$f(z)=\int_0^z\frac{f_k(\lambda)}{\lambda}\frac{1+\beta\omega\left(\lambda\right)}{1-\alpha\beta\omega\left(\lambda\right)}d\lambda.$$

∎

2.8. Theorem. *Let* $f\in M_s^k[\alpha,\beta,\lambda]$. *Then*

$$\frac{1}{z}\left\{f\left(z\right)*\left(\frac{z}{\left(1-z\right)^2}-\frac{1+\beta e^{j\theta}}{\left(1-\alpha\beta e^{j\theta}\right)}h\left(z\right)\right)\right\}\neq 0,$$

where $0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\alpha\leqslant 1,\ 0<\beta\leqslant 1,\ \lambda>0$ *and* $z\in E$.

Proof. Let $f\in M_s^k\left(\alpha,\beta,\lambda\right)$. Then by using Theorem 2.3, we have

$$f\in S_s^{(k)}(\alpha,\beta),$$

which implies that for $0\leqslant\theta\leqslant 2\pi$, we write

$$\frac{zf'(z)}{f_k(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}$$

or

$$\frac{zf'(z)}{f_k(z)}\neq\frac{1+\beta e^{j\theta}}{1-\alpha\beta e^{j\theta}}.$$

Therefore

$$\frac{1}{z}\left\{zf'(z)-\left(\frac{1+\beta e^{j\theta}}{1-\alpha\beta e^{j\theta}}\right)f_k(z)\right\}\neq 0. \tag{2.18}$$

For $zf'(z) = f(z) * \frac{z}{(1-z)^2}$ and $f_k(z) = z + \sum_{j=0}^{\infty} a_j c_j z^j = (f*h)(z)$, the inequality (2.18) yields

$$\frac{1}{z}\left\{f(z) * \frac{z}{(1-z)^2} - \left(\frac{1+\beta e^{j\theta}}{1-\alpha\beta e^{j\theta}}\right)(f*h)(z)\right\} \neq 0$$

or

$$\frac{1}{z}\left\{f(z) * \frac{z}{(1-z)^2} - \left(\frac{1+\beta e^{j\theta}}{1-\alpha\beta e^{j\theta}}\right)(f*h)(z)\right\} \neq 0$$

or

$$\frac{1}{z}\left\{f(z) * \left(\frac{z}{(1-z)^2} - \frac{1+\beta e^{j\theta}}{(1-\alpha\beta e^{j\theta})} h(z)\right)\right\} \neq 0.$$

∎

2.9. Theorem. *Let* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *If for* $0 \leqslant \alpha \leqslant 1,\ 0 < \beta \leqslant 1,\ k \geqslant 1,$ *and* $\lambda \geqslant 0$, *we have*

(2.19)
$$\sum_{\substack{j=2 \\ j\neq lk+1}}^{\infty} \chi(\alpha,\beta,\lambda,j)\, j\, |a_j| + \sum_{j=1}^{\infty} \chi_1(\alpha,\beta,\lambda,j,k)\, |a_{jk+1}| + \sum_{j=1}^{\infty} \chi_2(\alpha,\beta,\lambda,j,k)\, |a_{jk+1}|^2 < \beta(1+\alpha) - 2$$

where $\chi_1(\alpha,\beta,\lambda,j,k = \{(1-\lambda)(jk+1)+\lambda\}(1-\alpha\beta) + (1-\beta)(2+jk)$, $\chi(\alpha,\beta,\lambda,j) = ((1-\lambda)+\lambda \mathrm{j})(1-\alpha\beta)\, m$ *and*
$\chi_2(\alpha,\beta,\lambda,j,k) = \{(1-\alpha\beta)(jk+1) + (1-\beta)\}(\mathrm{jk}+1)$. *Then* $f \in M_s^k[\alpha,\beta,\lambda]$.

Proof. Let $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and f_k be given in (1.2) for $z \in E$. Then, we have

$$\begin{aligned} (2.20)\ M &= \left|(1-\lambda)zf'(z)f_k'(z) + \lambda(zf'(z))'f_k(z) - f_k(z)f_k'(z)\right| - \\ &\quad \beta\left|\alpha((1-\lambda)zf'(z)f_k'(z) + \lambda(zf'(z))'f_k(z)) + f_k(z)f_k'(z)\right|. \end{aligned}$$

Also for $\varkappa^k = 1$ and $k > 1$,

(2.21)
$$f_k(z) = \frac{1}{k}\sum_{m=0}^{k-1} \varkappa^{-m} f(\varkappa^m z) = z + \sum_{j=2}^{\infty} a_j d_j z^j, \text{ where } d_j = \left\{\begin{array}{ll} 1, & j = (l-1)k+1 \\ 0, & j \neq (l-1)k+1 \end{array}\right\}$$

From (2.20) , (2.21) and for $|z| = r < 1$, we have

$$
\begin{aligned}
M &\leqslant (1-\lambda)(1-\alpha\beta)\left|zf'(z)f_k'(z)\right| + \lambda(1-\alpha\beta)\left|(zf'(z))'f_k(z)\right| + (1-\beta)\left|f_k(z)f_k'(z)\right| \\
&< (1-\lambda)(1-\alpha\beta) + \lambda(1-\alpha\beta) + (1-\beta) \\
&\quad +(1-\lambda)(1-\alpha\beta)\sum_{j=2}^{\infty} j\,|a_j| + \lambda(1-\alpha\beta)\sum_{j=2}^{\infty} j^2\,|a_j| \\
&\quad +(1-\lambda)(1-\alpha\beta)\sum_{j=2}^{\infty} j\,|a_j d_j| + \lambda(1-\alpha\beta)\sum_{j=2}^{\infty} |a_j d_j| + (1-\beta)\sum_{j=2}^{\infty}(1+j)\,|a_j d_j| \\
&\quad +(1-\lambda)(1-\alpha\beta)\sum_{j=2}^{\infty} j^2\,|a_j|\,|a_j d_j| + \lambda(1-\alpha\beta)\sum_{j=2}^{\infty} j^2\,|a_j|\,|a_j d_j| + (1-\beta)\sum_{j=2}^{\infty} j\,|a_j d_j|^2
\end{aligned}
$$

or

$$
\begin{aligned}
M &< 2-\beta(\alpha+1) + \sum_{\substack{j=2 \\ j\neq lk+1}}^{\infty} \left((1-\lambda)(1-\alpha\beta)\,j + \lambda(1-\alpha\beta)\,j^2\right)|a_j| + \\
&\quad \sum_{j=1}^{\infty} \left(\{(1-\lambda)(jk+1)+\lambda\}(1-\alpha\beta) + (1-\beta)(2+jk)\right)|a_{jk+1}| + \\
&\quad \sum_{j=1}^{\infty} \{[(1-\alpha\beta)(jk+1) + (1-\beta)](jk+1)\}\,|a_{jk+1}|^2 .
\end{aligned}
$$

For $M < 0$, we obtain (2.19) . Hence, $f \in M_s^k[\alpha,\beta,\lambda]$. ∎

2.10. Theorem. *Let f and f_k be defined by* (1.1) *and* (1.2)*. Suppose that $f \in M_s^k[\alpha,\beta,\lambda]$ and F_k is given by (2.1). Then*

$$
\begin{aligned}
&\left|(1+(j-1)\lambda)\,a_j d_j - (\lambda j + (1-\lambda))\,j a_j\right|^2 \\
<\;& \beta\,|\alpha+1|^2 + \beta\sum_{k=2}^{j-1}\Big[\Big(\alpha^2(1-\lambda)^2\,|1+d_k|^2 + \alpha^2\lambda^2\,|k|^2 + 2\alpha(1-\lambda)\,|1+|d_k||\,\alpha\lambda\,|k|\Big)\,|k a_k|\Big] \\
&+2\beta\sum_{k=2}^{j-1}\left(\alpha(1-\lambda)\,|1+d_k| + \alpha\lambda\,|k|\right)|\alpha\lambda+1+k|\,|d_k|\,|a_k| \\
&+\sum_{k=2}^{j-1}\Big[\beta\left(|\alpha k + d_k|^2\,|k|^2 + |\alpha\lambda+1+k|^2\right) - \sum_{k=2}^{j-1}|1+(k-1)\lambda|^2\Big]\,|a_k|^2\,|d_k|^2 \\
&+\sum_{k=2}^{j-1}|\lambda k + (1-\lambda)|^2\,|k|^2\,|a_k|^2 - 2\sum_{k=2}^{j-1}|1+(k-1)\lambda|\,|\lambda k+(1-\lambda)|\,|k|\,|a_k|^2\,|d_k|\Big].
\end{aligned}
$$

For the proof of this theorem, we use Lemma 1.4, (1.2) *and follow the same lines as in Theorem 2.9.*

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