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On commutativity of prime gamma rings with derivation

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Abstract

Let M be a weak Nobusawa Γ-ring and γ be a nonzero element of Γ. In this paper, we find a relation between Γ-rings and rings, and give some commutativity conditions on Γ-rings by using this relation. Also, we prove that any Γ-ring M in the sense of Nobusawa with a nonzero element γ in the center of M-ring Γ is γ-prime if and only if M is Γ-prime. As a consequence, we show that the semiprimeness (semisimpleness) of the ring $(M, +, \cdot_\gamma)$ for any $\gamma \in \Gamma$ implies the Γ-semiprimeness(Γ-semisimpleness) of the Γ-ring M.

Keywords: gamma ring, prime Γ-ring, k-derivation, commutativity, γ-radical.

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1. Introduction

Let M and Γ be additive Abelian groups. M is said to be a Γ-ring in the sense of Barnes [3] if there exists a mapping $M \times \Gamma \times M \to M$ satisfying these two conditions for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(1) $(a + b)\,\alpha c = a\alpha c + b\alpha c$
$a(\alpha + \beta)c = a\alpha c + a\beta c$
$a\alpha\,(b + c) = a\alpha b + a\alpha c$

(2) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)$

In addition, if there exists a mapping $\Gamma \times M \times \Gamma \to \Gamma$ such that the following axioms hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(3) $(a\alpha b)\,\beta c = a\,(\alpha b\beta)\,c$

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(4) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$, where $\alpha \in \Gamma$

then M is called a Γ-ring in the sense of Nobusawa [17]. If a Γ-ring M in the sense of Barnes satisfies only the condition (3), then it is called weak Nobusawa Γ-ring [11].

We assume that all gamma rings in this paper are weak Nobusawa gamma ring unless otherwise specified.

Let M be a Γ-ring. M is said to be a Γ-prime gamma ring if $a\Gamma M\Gamma b = 0$ with $a, b \in M$ implies either $a = 0$ or $b = 0$ [15]. M is Γ-simple if $M\Gamma M \neq 0$ and M has no ideals (0) and M itself [15].

$C_M = \{\alpha \in \Gamma \mid \alpha m\beta = \beta m\alpha, \forall m \in M, \beta \in \Gamma\}$ is called the center of M-ring Γ and $C_\gamma = \{c \in M \mid c\gamma m = m\gamma c, \forall m \in M\}$ with $\gamma \in \Gamma$ is called the γ-center of Γ-ring M.

Recall that from [9], an additive mapping $d : M \to M$ is called a derivation on M if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$. Note that $d = 0$ when d is defined on a prime weak Nobusawa Γ-ring M. So, in this paper we consider k-derivations that has been defined by Kandamar [10] on any gamma ring M.

In this work, we first obtain some commutativity conditions on the γ-prime Γ-ring M with k-derivations and prove that M is γ-prime if and only if M is Γ-prime where γ is a nonzero element in the center of M-ring Γ in the sense of Nobusawa. Then, we also show that if there exists a nonzero element γ in C_M in a Nobusawa Γ-ring M, then (0) is Γ-prime ideal if and only if (0) is γ-prime ideal. Finally, we study the relation between semiprimeness (semisimpleness) of the ring $(M, +, \cdot_\gamma)$ and Γ-semiprimeness (Γ-semisimpleness) of the Γ-ring M where $\gamma \in \Gamma$.

2. Relation between Γ-rings and rings up to γ

We now give some definitions that have been firstly defined by Arslan and Kandamar in [1].

2.1. Definition. Let M be a Γ-ring, γ be a nonzero element of Γ and I be an additive subgroup of M.

(i) M is said to be γ-commutative if $x\gamma y = y\gamma x$ for all $x, y \in M$.

(ii) I is said to be a γ-subring of M if $x\gamma y \in I$ for all $x, y \in I$.

(iii) I is said to be a γ-left ideal(resp. γ-right ideal) of M if $m\gamma a \in I$(resp. $a\gamma m \in I$) for all $m \in M$, $a \in I$. If I is both γ-left and γ-right ideal then I is called a γ-ideal of M.

(iv) I is said to be a γ-prime ideal if $A\gamma B$ implies $A \subseteq I$ or $B \subseteq I$ for any γ-ideals A and B of M.

(v) I is said to be a γ-Lie ideal of M if $[x, m]_\gamma = x\gamma m - m\gamma x \in I$ for all $x \in I$ and $m \in M$.

2.2. Definition. A Γ-ring M is called a γ-prime gamma ring if there exists a nonzero element γ in Γ such that $a\gamma M\gamma b = 0$ with $a, b \in M$ implies either $a = 0$ or $b = 0$.

2.3. Definition. A Γ-ring M is called a γ-simple if $M\gamma M \neq 0$ and M has no γ-ideal besides the (0) and itself.

2.4. Lemma. *Let M be a Γ-ring. Then the following holds:*

(i) *If M is a γ-prime gamma ring, then M is Γ-prime.*

(ii) *If M is a γ-simple gamma ring, then M is Γ-simple.*

Proof. (i) Let M be a γ-prime gamma ring and $a\Gamma M\Gamma b = 0$ for any $a, b \in M$. Therefore, we have $a\gamma M\gamma b = 0$. Since M is a γ-prime gamma ring, we get $a = 0$ or $b = 0$. Hence, the γ-primeness of M implies the Γ-primeness of M.

(ii) It is clear from the definitions of γ-simple and Γ-simple gamma rings.

□

2.5. Proposition. *Let M be a Γ-ring and γ be a nonzero element of Γ. Then the Abelian group M with a binary operation $\cdot_\gamma$ defined by $a \cdot_\gamma b = a\gamma b$ for all $a, b \in M$ is a ring.*

Proof. It is clear from the definition of the gamma ring. □

According to the Proposition 2.5, the Abelian group M can be made into a ring by defining binary operations for all $\gamma \in \Gamma$. We denote this ring by $(M, +, \cdot_\gamma)$.

It is obvious that a γ-ideal of a Γ-ring M is an ideal of the ring $(M, +, \cdot_\gamma)$. Conversely, every ideal of the ring $(M, +, \cdot_\gamma)$ is a γ-ideal of the Γ-ring M. Similarly γ-Lie ideals of the Γ-ring M and Lie ideals of the ring $(M, +, \cdot_\gamma)$ is same. Also, if d is a k-derivation of the Γ-ring M and $k(\gamma) = 0$, then d is a derivation of the ring $(M, +, \cdot_\gamma)$. Thus, we can adapt all of the known results for the ring $(M, +, \cdot_\gamma)$ to the Γ-ring M. For instance, the commutativity of the ring $(M, +, \cdot_\gamma)$ is equal to the γ-commutativity of the Γ-ring M. Similarly one can say the primeness (semiprimeness) of the ring $(M, +, \cdot_\gamma)$ is the same as the γ-primeness (γ-semiprimeness) of the Γ-ring M. We give some results below.

2.6. Theorem. *Let M be a γ-prime gamma ring and d_1, d_2 be nonzero k_1, k_2-derivations of M such that $k_1(\gamma) = k_2(\gamma) = 0$ respectively. If char$M \neq 2$ and $d_1 d_2$ is $k_1 k_2$-derivation of M, then $d_1 = 0$ or $d_2 = 0$.*

Proof. By the hypothesis $d_1 \neq 0$, $d_2 \neq 0$ and $d_1 d_2$ are derivations of the prime ring $(M, +, \cdot_\gamma)$. Also the characteristic of the ring $(M, +, \cdot_\gamma)$ is different from 2. Therefore by [18, Theorem 1] one of the derivations d_1 and d_2 is zero in the ring $(M, +, \cdot_\gamma)$. □

2.7. Corollary. *Let M be a γ-prime gamma ring of characteristic not 2 and d be a 0-derivation of M such that $d^2 = 0$. Then $d = 0$.*

Proof. Let M is a γ-prime gamma ring. Then M is a Γ-prime gamma ring by Lemma 2.4. Since $d^2 = 0$ is a derivation on M, we get $d = 0$ by Theorem 2.6. □

2.8. Theorem. *Let M be a gamma ring and d be a k-derivation of M such that $k(\gamma) = 0$ and $d^3 \neq 0$. Then the γ-subring generated by $d(m)$ for all m in M contains a nonzero γ-ideal of M.*

Proof. Since d is a derivation of the ring $(M, +, \cdot_\gamma)$ and $d^3 \neq 0$, the subring generated by $d(m)$ for all m in M contains a nonzero ideal of $(M, +, \cdot_\gamma)$ by [6, Theorem 1]. Therefore the γ-subring generated by $d(m)$ for all m in M contains a nonzero γ-ideal of M. □

2.9. Corollary. *Let M be a Γ-ring, d be a nonzero 0-derivation on M such that $d^3 \neq 0$. Then, the subring A of M generated by all $d(a\alpha b)$, with $\alpha \in \Gamma$ and $a, b \in M$, contains a nonzero ideal of M.*

Another proof of Corollary 2.9 can be found in [19].

2.10. Theorem. *Let M be a γ-prime gamma ring and d be a nonzero k-derivation of M such that $k(\gamma) = 0$. Then M is γ-commutative if one of the following conditions holds:*

(i) $[a, d(a)]_\gamma \in C_\gamma$ *for all* $a \in M$.
(ii) *char$M \neq 2$ and* $[d(M), d(M)]_\gamma \subset C_\gamma$.
(iii) *char$M \neq 2$ and* $d^2(M) \subset C_\gamma$.
(iv) *d_1, d_2 are nonzero k_1, k_2-derivations of M such that $k_1(\gamma) = k_2(\gamma) = 0$ respectively, char$M \neq 2$ and $d_1 d_2(M) \subset C_\gamma$.*

Proof. (i) By the hypothesis d is a nonzero derivation of the prime ring $(M,+,\cdot_\gamma)$. Since $[a,d(a)]$ is in the center of the ring $(M,+,\cdot_\gamma)$ for all $a\in M$, the ring $(M,+,\cdot_\gamma)$ is commutative by [18, Theorem 2]. Therefore the gamma ring M is γ-commutative since commutativity of $(M,+,\cdot_\gamma)$ requires γ-commutativity of Γ-ring M.

(ii) By the hypothesis d is a nonzero derivation of the prime ring $(M,+,\cdot_\gamma)$, the characteristic of the ring M is different from 2 and $[d(M),d(M)]_\gamma$ is contained in the center of the ring M. Hence M is commutative as a ring by [13, Theorem 2]. Therefore M is γ-commutative.

(iii) By the hypothesis d is a nonzero derivation of the prime ring $(M,+,\cdot_\gamma)$, the characteristic of the ring M is different from 2 and $d^2(M)$ is contained in the center of the ring M. Hence M is commutative as a ring by [13, Theorem 3]. Therefore M is γ-commutative.

(iv) By the hypothesis d_1 and d_2 are nonzero derivations of the prime ring $(M,+,\cdot_\gamma)$. Also the characteristic of the ring $(M,+,\cdot_\gamma)$ is different from 2 and $d_1d_2(M)$ is contained in the center of the ring M. Hence M is commutative as a ring by [13, Theorem 4]. Therefore M is γ-commutative.

□

2.11. Corollary. *Let M be a γ-prime gamma ring for all nonzero elements γ in Γ and d be a nonzero 0-derivation on M. Then M is Γ-commutative if one of the following conditions holds:*

(i) $[a,d(a)]_\gamma\in C_\gamma$ *for all* $a\in M$ *and* $\gamma\in\Gamma$.
(ii) *char*$M\neq 2$ *and* $[d(M),d(M)]_\gamma\subset C_\gamma$ *for all* $\gamma\in\Gamma$.
(iii) *char*$M\neq 2$ *and* $d^2(M)\subset C_\gamma$ *for all* $\gamma\in\Gamma$.
(iv) d_1, d_2 *are nonzero 0-derivations of* M, *char*$M\neq 2$ *and* $d_1d_2(M)\subset C_\gamma$ *for all* $\gamma\in\Gamma$.

2.12. Theorem. *Let M be a γ-prime gamma ring of characteristic not 2 and U be a γ-Lie ideal of M. If $U\nsubseteq C_\gamma$, then there exists a γ-ideal K of M such that $[K,M]_\gamma\subseteq U$ but $[K,M]_\gamma\nsubseteq C_\gamma$.*

Proof. U is a Lie ideal of the prime ring $(M,+,\cdot_\gamma)$ that is not contained in the center of the ring M and the characteristic of the ring M is different from 2 by hypothesis. Hence, there exists an ideal K of $(M,+,\cdot_\gamma)$ such that $[K,M]\subseteq U$ and $[K,M]$ is not contained in the center of the $(M,+,\cdot_\gamma)$ by [4, Lemma 1]. Therefore, there exists an ideal K of Γ-ring M such that $[K,M]_\gamma\subseteq U$ but $[K,M]_\gamma\nsubseteq C_\gamma$. □

2.13. Theorem. *Let M be a γ-prime gamma ring of characteristic not 2 and U be a γ-Lie ideal of M such that $U\nsubseteq C_\gamma$. If d_1, d_2 are nonzero k_1, k_2-derivations of M such that $k_1(\gamma)=k_2(\gamma)=0$ respectively and $d_1d_2(U)=0$, then $d_1=0$ or $d_2=0$.*

Proof. By the hypothesis, d_1 and d_2 are nonzero derivations of the prime ring $(M,+,\cdot_\gamma)$ and U is a Lie ideal of M that is not contained in the center of the ring M. Also the characteristic of the ring $(M,+,\cdot_\gamma)$ is different from 2 and $d_1d_2(U)=0$. Hence $d_1=0$ or $d_2=0$ by [4, Theorem 4]. □

2.14. Theorem. *Let M be a γ-prime gamma ring of characteristic not 2, U be a γ-Lie ideal of M and d be a k-derivation of M such that $k(\gamma)=0$. Then U is contained in the γ-center of M if one of the following conditions holds:*

(i) $d^2(U)=0$.
(ii) $d\neq 0$ *and* $d^2(U)\subset C_\gamma$.

(iii) d_1, d_2 *are nonzero* k_1, k_2*-derivations of* M *such that* $k_1(\gamma) = k_2(\gamma) = 0$ *respectively and* $d_1 d_2(U) \subset C_\gamma$.

Proof. It is similar to the proof of Theorem 2.10. □

2.15. Corollary. *Let* M *be a* γ*-prime gamma ring of characteristic not 2 for all nonzero elements* γ *in* Γ, U *be a* γ*-Lie ideal of* M *and* d *be a* 0*-derivation of* M. *Then* U *is contained in the center of* M *if one of the following conditions holds:*

(i) $d^2(U) = 0$.
(ii) $d \neq 0$ *and* $d^2(U) \subset C_\gamma$ *for all* $\gamma \in \Gamma$.
(iii) d_1, d_2 *are nonzero* 0*-derivations of* M *and* $d_1 d_2(U) \subset C_\gamma$ *for all* $\gamma \in \Gamma$.

2.16. Theorem. *Let* M *be a* γ*-prime gamma ring of characteristic not 2 and* U *be a* γ*-Lie ideal of* M *such that* $U \nsubseteq C_\gamma$. *If* d_1 *and* d_2 *are nonzero* k_1 *and* k_2*-derivations of* M *such that* $k_1(\gamma) = k_2(\gamma) = 0$ *respectively and* $d_1 d_2(U) \subset C_\gamma$, *then* $d_1 = 0$ *or* $d_2 = 0$.

Proof. By the hypothesis d_1 and d_2 are nonzero derivations of the prime ring $(M, +, \cdot_\gamma)$ and U is a Lie ideal of M that is not contained in the center of M. Also the characteristic of the ring $(M, +, \cdot_\gamma)$ is different from 2 and $d_1 d_2(U)$ is contained in the center of M. Hence $d_1 = 0$ or $d_2 = 0$ by [2, Theorem 6]. □

3. γ-Radicals of Gamma Rings

Radicals of Γ-rings has been investigated by a number of authors. Barnes [3] defined prime radicals and proved some properties for gamma rings by methods similar to those of McCoy[16]. Coppage and Luh [5] introduced the notions of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ-rings and Barnes' prime radical was studied further. Kyuno [12] also studied prime radicals of gamma rings and showed relations between radicals of gamma rings and radicals of its operator rings.

We define γ-prime radical, strongly γ-nilpotent radical, γ-Levitzki nil radical and γ-Jacobson radical for Γ-rings and show their relations with the radicals of Γ-rings in the literature.

Let M be a gamma ring and $S \subseteq M$. S is said to be a γ-m-system if $S = \emptyset$ or $(a)_\gamma \gamma (b)_\gamma \cap S \neq \emptyset$ for any $a, b \in M$. Here, $(a)_\gamma$ is the set of all elements of the form $ka + m\gamma a + a\gamma x + \sum_{i=1}^{n} u_i \gamma a \gamma v_i$ for $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $m, x, u_i, v_i \in M$.

Proofs of the below results are obvious from the relation given in Section 2. So we omit their proofs.

3.1. Proposition. *Let* M *be a gamma ring and* P *be a* γ*-ideal of* M. *Then* P *is a* γ*-prime ideal if and only if the complement of* P *is a* γ*-*m*-system.*

Let A be a γ-ideal of a Γ-ring M. Then the set of all elements m in M such that every γ-m-system in M which contains m meets A is called γ-prime radical of the γ-ideal A and is denoted by $\mathfrak{B}_\gamma(A)$. γ-prime radical of zero γ-ideal is called γ-prime radical of the Γ-ring M and is denoted by $\mathfrak{B}_\gamma(M)$. In fact, the prime radical of the ring $(M, +, \cdot_\gamma)$ is equal to $\mathfrak{B}_\gamma(M)$.

3.2. Theorem. *If* A *is a* γ*-ideal in the* Γ*-ring* M, *then* $\mathfrak{B}_\gamma(A)$ *coincides with the intersection of all the* γ*-prime ideals in* M *which contain* A.

3.3. Corollary. γ*-prime radical of a* Γ*-ring* M *is the intersection of all the* γ*-prime ideals in* M.

An element a in M is called strongly γ-nilpotent if there exists a positive integer n such that $(a\gamma)^n a = 0$. A subset L of M is called strongly γ-nil if all of the elements in L are strongly γ-nilpotent. A subset S of M is called strongly γ-nilpotent if there exists a positive integer m such that $(S\gamma)^m S = 0$.

The strongly γ-nilpotent radical of M is the sum of all strongly γ-nilpotent ideals of M and is denoted by $\mathfrak{S}_\gamma(M)$.

3.4. Proposition. *If A and B are any strongly γ-nilpotent ideals in a Γ-ring M, then $A+B$ is also a strongly γ-nilpotent ideal in M.*

3.5. Corollary. *The strongly γ-nilpotent radical of a Γ-ring M is a strongly γ-nil ideal in M.*

A subset S of M is called γ-locally nilpotent if for any finite subset F of S there exists a positive integer n such that $(F\gamma)^n F = 0$.

The γ-Levitzki nil radical of M is the sum of all γ-locally nilpotent ideals of M and is denoted by $\mathfrak{L}_\gamma(M)$.

An element a in M is called γ-right quasi regular if there exist $b \in M$ such that $a + b + a\gamma b = 0$. A subset S of M is called γ-right quasi regular if all of the elements in S are γ-right quasi regular.

The γ-Jacobson radical of M is the set of all $a \in M$ such that the principal γ-ideal generated by a is γ-right quasi regular and is denoted by $\mathfrak{J}_\gamma(M)$. In fact, the Jacobson radical of the ring $(M, +, \cdot_\gamma)$ is equal to $\mathfrak{J}_\gamma(M)$.

4. Main Results

Not all of the properties of a ring holds for a gamma ring. For example, let d be a k-derivation of γ-prime gamma ring M of characteristic not 2. If $k(\gamma) \neq 0$, then the hypothesis $d^2 = 0$ does not imply $d = 0$.

4.1. Example. Let $M = \left\{ \begin{pmatrix} a & b & a \\ c & r & c \end{pmatrix} \mid a, b, c, r \in \mathbb{Z} \right\}$, Γ be the set of all 3×2 matrices over $\mathbb{Z}$ and $\gamma = \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$. Then, M is a γ-prime Γ-ring of characteristic not 2. Define $d : M \to M$, $d\begin{pmatrix} a & b & a \\ c & r & c \end{pmatrix} = \begin{pmatrix} -b & 0 & -b \\ -r & 0 & -r \end{pmatrix}$ and $k : \Gamma \to \Gamma$, $k\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u_{11} + u_{31} & u_{12} + u_{32} \\ 0 & 0 \end{pmatrix}$.

It can be shown that d is a k-derivation and $k(\gamma) \neq 0$. Moreover, it is easy to see that $d \neq 0$ but $d^2 = 0$.

This example also shows that if d is a k-derivation on the Γ-prime gamma ring of characteristic not 2 such that $d^2 = 0$, then d may not be the zero derivation. In such a case, k^2 must be equal to zero as proved in the next theorem.

4.2. Theorem. *Let M be a γ-prime gamma ring in the sense of Nobusawa of characteristic not 2 and d be a k-derivation. If $d^2 = 0$, then either $d = 0$ or $k^2 = 0$.*

Proof. Let $k(\gamma) = 0$. Then, the k-derivation d on M is also a derivation for the ring $(M, +, \cdot_\gamma)$. Therefore, $d = 0$ by [18, Theorem 1]. Now, let $k(\gamma) \neq 0$. By hypothesis we have $d^2(d(x)\beta d(y)) = 0$ for all $x, y \in M$ and $\beta \in \Gamma$. Expanding this we get $d(x)k^2(\beta)d(y) = 0$. Replacing β by $\beta d(z)\alpha$ we have $d(x)k(\beta)d(z)k(\alpha)d(y) = 0$ since char$M \neq 2$. Replacing β by $\beta d(m)\delta$ we get $d(x)k(\beta)d(m) = 0$ since M is Γ-prime

Nobusawa Γ-ring by Lemma 2.4. If we replace x by $d(x)\alpha y$ in the last equation we have $d(x)k(\alpha)y = 0$ or $zk(\beta)d(m) = 0$. If $d(x)k(\alpha)y = 0$, then replacing α by $\alpha mk(\delta)$ we get $d(x)\alpha mk^2(\delta)y = 0$ for all $x, m, y \in M$ and $\alpha, \delta \in \Gamma$. Then, $d = 0$ or $k^2 = 0$ since M is a prime Nobusawa Γ-ring. If we consider the case $zk(\beta)d(m) = 0$, same result can be obtained similarly.

□

4.3. Theorem. *Let M be a Γ-ring in the sense of Nobusawa and γ be a nonzero element of Γ. If $\gamma \in C_M$, then M is γ-prime gamma ring if and only if M is Γ-prime.*

Proof. If M is γ-prime gamma ring then M is Γ-prime by Lemma 2.4. Let M is a Γ-prime gamma ring, $a\gamma M\gamma b = 0$ for any $a, b \in M$ and $a \neq 0$. Then we have $a\Gamma M\gamma M\gamma b = 0$. Since M is a Γ-prime $M\gamma M\gamma b = 0$. Thus $M\gamma M\Gamma b\gamma M = 0$. Hence we get $b = 0$ since M is a Γ-prime gamma ring and $\gamma \in C_M$. Therefore, M is γ-prime. □

4.4. Theorem. *The prime radical of a Γ-ring M is contained in γ-prime radical of M.*

Proof. Let x be an element of $\mathfrak{B}(M)$, the prime radical of M. Suppose that $x \notin \mathfrak{B}_\gamma(M)$. Then, there is a γ-m-system S which contains x such that $0 \notin S$. Therefore, there is an m-system in M which contains x but not contains 0 since S is also an m-system. This contradicts with $x \in \mathfrak{B}(M)$. Hence, if x is an element of $\mathfrak{B}(M)$, then x must be in $\mathfrak{B}_\gamma(M)$. □

4.5. Theorem. *The strongly nilpotent radical of a Γ-ring M is contained in strongly γ-nilpotent radical of M.*

Proof. It is easy to see that a strongly nilpotent ideal of M is also a strongly γ-nilpotent ideal. Therefore, $\mathfrak{S}(M)$, the strongly nilpotent radical of M, is contained in $\mathfrak{S}_\gamma(M)$. □

4.6. Theorem. *The Levitzki nil radical of a Γ-ring M is contained in γ-Levitzki nil radical of M.*

Proof. It is easy to see that a locally nilpotent ideal of M is also a γ-locally nilpotent ideal. Therefore, $\mathfrak{L}(M)$, the Levitzki nil radical of M, is contained in $\mathfrak{L}_\gamma(M)$. □

4.7. Theorem. *The Jacobson radical of a Γ-ring M is contained in γ-Jacobson radical of M.*

Proof. It is easy to see that a right quasi regular element of M is also a γ-right quasi regular. Therefore, $\mathfrak{J}(M)$, the Jacobson radical of M, is contained in $\mathfrak{J}_\gamma(M)$. □

4.8. Corollary. *Let M be a Γ-ring.*

(i) *If the ring $(M, +, \cdot_\gamma)$ for any $\gamma \in \Gamma$ is semiprime, then the Γ-ring M is Γ-semiprime.*
(ii) *If the ring $(M, +, \cdot_\gamma)$ for any $\gamma \in \Gamma$ is semisimple, then the Γ-ring M is Γ-semisimple.*

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