

On one problem of a cusped elastic prismatic shells in case of the third model of Vekua's hierarchical model

Natalia Chinchaladze*†

Abstract

In the present paper hierarchical model for cusped, in general, elastic prismatic shells is considered, when on the face surfaces a normal to the projection of the prismatic shell component of a traction vector and parallel to the projection of the prismatic shell components of a displacement vector are known.

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1. Introduction

Investigations of cusped elastic prismatic shells actually takes its origin from the fifties of the last century, namely, in 1955 I.Vekua raised the problem of investigation of elastic cusped prismatic shells, whose thickness on the prismatic shell entire boundary or on its part vanishes (see [15], [16], [9] and references therein).

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O . We conditionally assume the x_3 -axis vertical. The elastic body is called a prismatic shell if it is bounded above and below by, respectively, the surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \omega,$$

laterally by a cylindrical surface Γ of generatrix parallel to the x_3 -axis and its vertical dimension is sufficiently small compared with other dimensions of the body. $\bar{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell on $x_3 = 0$.

The main difference between the prismatic shell of a constant thickness and the standard shell of a constant thickness is the following: the lateral boundary of the standard

*I. Vekua Institute of Applied Mathematics & Faculty of Exact and Natural Sciences of Iv. Javakhishvili Tbilisi State University, 2 University st., 0186 Tbilisi, Georgia, Email: chinchaladze@gmail.com

†Corresponding Author.

shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$.

Let the thickness of the prismatic shell be

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \begin{cases} > 0 & \text{for } (x_1, x_2) \in \omega, \\ \geq 0 & \text{for } (x_1, x_2) \in \partial\omega. \end{cases}$$

If the thickness of the prismatic shell vanishes on $\gamma_0 \subset \partial\omega$, it is called cusped one.

Below we consider symmetric prismatic shell, i.e. the case when

$$\overset{(+)}{h}(x_1, x_2) = -\overset{(-)}{h}(x_1, x_2),$$

with the thickness as follows

$$(1.1) \quad 2h := h_0 x_2^\varkappa, \quad h_0, \varkappa = \text{const}, \quad h_0, \varkappa > 0.$$

I. Vekua [15], [16] constructed hierarchical models for elastic prismatic shells, in particular, plates of variable thickness, when on the face surfaces either es (the first model) or displacements (the second model) are known. The updated survey of results concerning cusped elastic prismatic shells in the cases of the first and second models is given in [9] (see also [1], [5], [6], [10], [12], [14] and references therein). In the present paper the third hierarchical model for cusped elastic prismatic shells is analyzed. It means that on the face surfaces a normal to the projection of the prismatic shell component $Q_{\nu_3}^{(\pm)}$ of a traction vector and parallel to the projection of the prismatic shell components $u_\alpha(x_1, x_2, \overset{(\pm)}{h}, t)$ of a displacement vector are known. The third model was first suggested in [8].

In what follows the usual notations are used: X_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, F_i are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Moreover, repeated indices imply summation (Greek letters run from 1 to 2 and Latin letters run from 1 to 3).

In the fifties of the twentieth century, I.Vekua ([9], [15], [16]) introduced a new mathematical model for elastic prismatic shells which was based on expansions of the three-dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable of plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so-called N -th approximation. Each of these approximations for $N = 0, 1, \dots$ can be considered as an independent mathematical model of plates. In particular, in case of the first model the approximations for $N = 0$ and $N = 1$ correspond to the plane deformation and classical Kirchhoff-Love plate model, respectively (see [9]).

For the sake of simplicity we consider zero approximation of the hierarchical model. Basic equation system can be written as follows (see e.g. [8], [3])

$$(1.2) \quad \begin{aligned} & \mu(hv_{\alpha 0})_{,\beta\beta} + (\lambda + \mu)(hv_{\gamma 0})_{,\gamma\alpha} \\ & - (\ln h)_{,\beta} \{ \lambda \delta_{\alpha\beta} (hv_{\gamma 0})_{,\gamma} + \mu [(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}] \} + \Phi_{\alpha 0} = \rho h \ddot{v}_{\alpha 0}, \end{aligned}$$

$$(1.3) \quad \mu(hv_{30,\beta})_{,\beta} + \Phi_{30} = \rho h \ddot{v}_{30},$$

where

$$\begin{aligned}
X_{\alpha\beta 0}(x_1, x_2, t) &= \lambda \delta_{\alpha\beta} [(hv_{\gamma 0})_{,\gamma} + \Psi_{\gamma\gamma}] + \mu [(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha} + 2\Psi_{\alpha\beta}], \\
X_{3\beta 0}(x_1, x_2, t) &= \mu hv_{30,\beta}, \quad X_{330} = \lambda [(hv_{\gamma 0})_{,\gamma} + \Psi_{\gamma\gamma}], \\
e_{\alpha\beta 0} &= \frac{1}{2} [(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}] + \Psi_{\alpha\beta}, \quad e_{3\beta 0} = \frac{1}{2} hv_{30,\beta}, \quad e_{330} = 0, \\
\Phi_{\alpha 0} &:= 2\mu \Psi_{\alpha\beta,\beta} + \lambda \Psi_{\gamma\gamma,\alpha} - (\ln h)_{,\beta} [\lambda \delta_{\alpha\beta} \Psi_{\gamma\gamma} + 2\mu \Psi_{\alpha\beta}] + F_{\alpha 0}, \\
\Phi_{30} &:= Q_{(\frac{+}{-})_3} \sqrt{\binom{+}{h}_{,1}^2 + \binom{+}{h}_{,2}^2 + 1} + Q_{(\frac{-}{-})_3} \sqrt{\binom{-}{h}_{,1}^2 + \binom{-}{h}_{,2}^2 + 1} + F_{30}, \\
\Psi_{\alpha\beta} &:= \frac{1}{2} \left[u_{\beta}(x_1, x_2, \binom{-}{h}, t) \binom{-}{h}_{,\alpha} - u_{\beta}(x_1, x_2, \binom{+}{h}, t) \binom{+}{h}_{,\alpha} \right. \\
&\quad \left. + u_{\alpha}(x_1, x_2, \binom{-}{h}, t) \binom{-}{h}_{,\beta} - u_{\alpha}(x_1, x_2, \binom{+}{h}, t) \binom{+}{h}_{,\beta} \right],
\end{aligned}$$

X_{ij0} , e_{ij0} , u_{i0} and F_{i0} are the zeroth order moments of X_{ij} , e_{ij} , u_i and F_i , respectively; $v_{i0} := h^{-1}u_{i0}$ are called weighted moments of the function u_i .

The case of cylindrical bending of the plates with the thickness (1.1) is considered in [8]. In this case the system (1.2)-(1.3) can be rewritten as follows

$$\begin{aligned}
\mu(h(x_2)v_{10}(x_2))_{,22} - \mu(\ln h(x_2))_{,2} (h(x_2)v_{10}(x_2))_{,2} + \Phi_{10}(x_2) &= 0 \\
(\lambda + 2\mu)(h(x_2)v_{20}(x_2))_{,22} - (\lambda + 2\mu)(\ln h(x_2))_{,2} (h(x_2)v_{20}(x_2))_{,2} + \Phi_{20}(x_2) &= 0, \\
\mu(h(x_2)v_{30,2}(x_2))_{,2} + \Phi_{30}(x_2) &= 0.
\end{aligned}$$

In [8] it is shown that $v_{\alpha 0}$ can not be prescribed in cusped edge (i.e., Dirichlet problem are not satisfied) if $\varkappa > 0$, and v_{30} can not be prescribed in cusped edge if $\varkappa \geq 1$.

The weak setting of the homogeneous Dirichlet problem of the following system

$$\begin{aligned}
\mu(hv_{\alpha 0})_{,\beta\beta} + (\lambda + \mu)(hv_{\gamma 0})_{,\gamma\alpha} \\
- (\ln h)_{,\beta} \{ \lambda \delta_{\alpha\beta} (hv_{\gamma 0})_{,\gamma} + \mu [(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}] \} + \Phi_{\alpha 0} &= 0, \\
\mu(hv_{30,\beta})_{,\beta} + \Phi_{30} &= 0,
\end{aligned}$$

is considered in [3].

2. Vibration problem

We will consider the case of harmonic vibration

$$\begin{aligned}
v_{i0}(x, t) &:= e^{-\iota\vartheta t} v_{i0}^0(x), \quad \Phi_{i0}(x, t) := e^{-\iota\vartheta t} \Phi_{i0}^0(x), \quad \iota^2 = -1, \\
\vartheta &= \text{const} > 0, \quad x := (x_1, x_2) \in \omega, \quad i = 1, 2, 3.
\end{aligned}$$

Taking into account of (1.1), (1.2), and (1.3) for $v_{i0}^0(x)$ we get the following system (the overscript index 0 is omitted below)

$$\begin{aligned}
-\rho\vartheta^2 hv_{10} - \mu\Delta_2(hv_{10}) - (\lambda + \mu) [(hv_{10})_{,11} + (hv_{20})_{,21}] \\
+ \mu(\ln h)_{,2} [(hv_{10})_{,2} + (hv_{20})_{,1}] &= \Phi_{10}, \\
-\rho\vartheta^2 hv_{20} - \mu\Delta_2(hv_{20}) - (\lambda + \mu) [(hv_{10})_{,12} + (hv_{20})_{,22}] \\
+ (\ln h)_{,2} \{ \lambda [(hv_{10})_{,1} + (hv_{20})_{,2}] + 2\mu(hv_{20})_{,2} \} &= \Phi_{20}, \\
-\rho\vartheta^2 hv_{30} - \mu [(hv_{30,1})_{,1} + (hv_{30,2})_{,2}] &= \Phi_{30},
\end{aligned}$$

where Δ_2 is a two dimensional Laplace operator.

We can rewrite obtained system in the following vector form

$$(2.1) \quad \mathbf{A}v(x) = \Phi(x), \quad x \in \omega,$$

where

$$\mathbf{A} := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

$$A_{11} := -\rho\vartheta^2 h - (\lambda + 2\mu)h\partial_{11} - \mu [h\partial_{22} + 2h_{,2} \partial_2 + h_{,22}] + \mu(\ln h)_{,2} [h\partial_2 + h_{,2}],$$

$$A_{12} := -(\lambda + \mu) [h\partial_{12} + h_{,2} \partial_1] + \mu(\ln h)_{,2} h\partial_1,$$

$$A_{21} := -(\lambda + \mu) [h\partial_{12} + h_{,2} \partial_1] + \lambda(\ln h)_{,2} h\partial_1,$$

$$A_{22} := -\rho\vartheta^2 h - \mu h\partial_{11} - (\lambda + 2\mu) [h\partial_{22} + 2h_{,2} \partial_2 + h_{,22}] + (\lambda + 2\mu)(\ln h)_{,2} [h\partial_1 + h_{,2}],$$

$$A_{13} = A_{23} = A_{31} = A_{32} = 0, \quad A_{33} := -\rho\vartheta^2 h - \mu h(\partial_{11} + \partial_{22}) + \mu h_{,2} \partial_2,$$

$$v := (v_{10}, v_{20}, v_{30})^\top, \quad \Phi := (\Phi_{10}, \Phi_{20}, \Phi_{30}),$$

the symbol $(\cdot)^\top$ means transposition.

Let

$$v, v^* \in C^2(\omega) \cap C^1(\bar{\omega}), \quad v^* := (v_{10}^*, v_{20}^*, v_{30}^*)^\top,$$

where v and v^* are arbitrary vectors of the above class. We obtain the following Green's formula

$$(2.2) \quad \int_{\omega} \mathbf{A}v \cdot v^* d\omega = \mathbf{J}(v, v^*) - \int_{\partial\omega} X_n v \cdot v^* d\partial\omega = \int_{\omega} \Phi \cdot v^* d\omega.$$

Here $n := (n_1, n_2)$ is the inward normal to $\partial\omega$:

$$X_n := \{X_{n10}, X_{n20}, X_{n30}\},$$

with

$$X_{ni0} = X_{ij0}n_j,$$

$$\begin{aligned} \mathbf{J}(v, v^*) &:= \int_{\omega} -h\rho\vartheta^2 v_{i0}v_{i0}^* d\omega + \int_{\omega} \frac{\lambda}{h} [(hv_{10})_{,1} (hv_{10}^*)_{,1} + (hv_{20})_{,2} (hv_{20}^*)_{,2} \\ &\quad + (hv_{10})_{,1} (hv_{20}^*)_{,2} + (hv_{20})_{,2} (hv_{10}^*)_{,1}] d\omega + \int_{\omega} \frac{\mu}{h} [2f(hv_{10})_{,1} (hv_{10}^*)_{,1} \\ &\quad + (hv_{10})_{,2} (hv_{10}^*)_{,2} + (hv_{20})_{,1} (hv_{10}^*)_{,2} + (hv_{20})_{,1} (hv_{20}^*)_{,1} + (hv_{10})_{,2} (hv_{20}^*)_{,1} \\ &\quad + 2(hv_{20})_{,2} (hv_{20}^*)_{,2} + hv_{30,1} hv_{30,1}^* + hv_{30,2} hv_{30,2}^*] d\omega \\ &= \int_{\omega} -h\rho\vartheta^2 v_{i0}v_{i0}^* d\omega + \int_{\omega} \frac{\lambda}{h} (hv_{\alpha 0})_{,\alpha} (hv_{\beta 0}^*)_{,\beta} d\omega \\ &\quad + \int_{\omega} \frac{\mu}{h} \{ [(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}] [(hv_{\alpha 0}^*)_{,\beta} + (hv_{\beta 0}^*)_{,\alpha}] + (hv_{30,\alpha})(hv_{30,\alpha}^*) \} d\omega \\ &= \int_{\omega} a[-h^2\rho\vartheta^2 v_{i0}v_{i0}^* + \lambda e_{kk0}^1(v)e_{ii0}^1(v^*) + 2\mu e_{ij0}^1(v)e_{ij0}^1(v^*)] d\omega, \end{aligned}$$

where

$$a := \frac{1}{h},$$

$$e_{ij0}^1(v) := \begin{cases} \frac{1}{2} [(hv_{i0})_{,j} + (hv_{j0})_{,i}], & i, j = 1, 2, \\ \frac{1}{2} hv_{i0,j}, & i = 3, j = 1, 2, \\ 0, & i = j = 3. \end{cases}$$

If we consider BVPs for system (2.1) with homogeneous boundary conditions for which the curvilinear integral along $\partial\omega$ in (2.2) disappears, we arrive at the equation

$$\mathbf{J}(v, v^*) = \int_{\omega} \Phi \cdot v^* d\omega.$$

Denote by $\mathcal{D}(\omega)$ a space of infinity differentiable functions with compact support in ω and introduce the linear form $[\mathcal{D}(\omega)]^3$ by the formula:

$$\begin{aligned} (v, v^*)_{X_{\vartheta}^{\varkappa}} &= \int_{\omega} \left[h^2 \rho \vartheta^2 v_{i0} v_{i0}^* + e_{ij0}^1(v) e_{ij0}^1(v^*) \right] \frac{1}{h} d\omega, \\ \|v\|_{X_{\vartheta}^{\varkappa}}^2 &= \int_{\omega} \left[h \rho \vartheta^2 v_{i0} v_{i0} + \frac{1}{4h} \left(4[(hv_{10})_{,1}]^2 \right. \right. \\ &\quad \left. \left. + 4[(hv_{20})_{,2}]^2 + 2((hv_{10})_{,2} + (hv_{20})_{,1})^2 + 2(hv_{30,1})^2 + 2(hv_{30,2})^2 \right) \right] d\omega. \end{aligned}$$

$X_{\vartheta}^{\varkappa}$ is a Hilbert space.

The classical and weak setting of the homogeneous Dirichlet problem can be formulated as follows:

2.1. Problem. Find a 3-dimensional vector v in ω satisfying the system of differential equations (2.1) in ω and the homogeneous Dirichlet boundary condition

$$(2.3) \quad [v(x)]^+ = 0, \quad x \in \partial\omega.$$

2.2. Problem. Find a vector $v \in X_{\vartheta}^{\varkappa}$ satisfying the equality

$$(2.4) \quad \mathbf{J}(v, v^*) = \langle \Phi, v^* \rangle \quad \text{for all } v^* \in X_{\vartheta}^{\varkappa},$$

here, the vector Φ belongs to the adjoint space $[X_{\vartheta}^{\varkappa}]^*$, and $\langle \cdot, \cdot \rangle$ denotes duality brackets between the spaces $[X_{\vartheta}^{\varkappa}]^*$ and $X_{\vartheta}^{\varkappa}$.

Further, we construct the vectors in $\Omega := \{(x; x_3) : x \in \omega, -h(x) < x_3 < h(x)\}$:

$$\begin{aligned} w_i(x, x_3) &= \frac{1}{2} v_{i0}(x), \quad i = 1, 2, 3, \\ w_i^*(x, x_3) &= \frac{1}{2} v_{i0}^*(x), \quad i = 1, 2, 3. \end{aligned}$$

It can be shown that

$$(2.5) \quad J(w, w^*) := \int_{\Omega} [-\rho \vartheta^2 w_i w_i^* + \sigma_{ij}(w) e_{ij}(w^*)] d\Omega = \mathbf{J}(v, v^*),$$

where $w(x, x_3) := (w_1, w_2, w_3)$ and $w^*(x, x_3) := (w_1^*, w_2^*, w_3^*)$ are vectors and $J(w, w^*)$ is the bilinear form corresponding to the three-dimensional potential energy for the displacement vector w .

In view of the homogeneous Dirichlet boundary condition (2.3), if $\varkappa > 1$, the following Hardy inequality holds (see [13], p. 69; [11])

$$(2.6) \quad \int_{\varepsilon}^l x_2^{\varkappa-2} v_{\alpha 0}^2 dx_2 \leq \frac{4}{(\varkappa-1)^2} \int_{\varepsilon}^l x_2^{\varkappa} (v_{\alpha 0,2})^2 dx_2, \quad \varkappa > 1.$$

Replacing in (2.6) \varkappa by $\varkappa + 2$, we obtain

$$(2.7) \quad \int_{\varepsilon}^l x_2^{\varkappa} v_{\alpha 0}^2 dx_2 \leq \frac{4}{(\varkappa + 1)^2} \int_{\varepsilon}^l x_2^{\varkappa+2} (v_{\alpha 0,2})^2 dx_2, \quad \text{for any } \varkappa > 0.$$

Now, considering the limit procedure as $\varepsilon \rightarrow 0+$, since the limits of the integrals in (2.7) exist for $v_{\alpha 0} \in X_{\vartheta}^{\varkappa}$, we immediately get the following

$$(2.8) \quad \int_0^l x_2^{\varkappa} v_{\alpha 0}^2 dx_2 \leq \frac{4}{(\varkappa + 1)^2} \int_0^l x_2^{\varkappa+2} (v_{\alpha 0,2})^2 dx_2, \quad \text{for any } \varkappa > 0.$$

Integrating by x_1 both side of (2.8) over $]x_1^0, x_1^1[$, we get

$$(2.9) \quad \int_{\omega} x_2^{\varkappa} v_{\alpha 0}^2 d\omega \leq \frac{4}{(\varkappa + 1)^2} \int_{\omega} x_2^{\varkappa+2} (v_{\alpha 0,2})^2 d\omega, \quad \text{for any } \varkappa > 0.$$

2.3. Lemma. *The bilinear form $\mathbf{J}(\cdot, \cdot)$ is bounded and strictly coercive in the space $X_{\vartheta}^{\varkappa}(\omega)$, i.e., there are positive constant C_0 and C_1 such that*

$$(2.10) \quad |\mathbf{J}(v, v^*)| \leq C_1 \|v\|_{X_{\vartheta}^{\varkappa}} \|v^*\|_{X_{\vartheta}^{\varkappa}},$$

$$(2.11) \quad \mathbf{J}(v, v) \geq C_0 \|v\|_{X_{\vartheta}^{\varkappa}}^2$$

for all $v, v^* \in X_{\vartheta}^{\varkappa}$ and $\vartheta^2 < \frac{\mu(\varkappa+1)^2}{16\rho l^2}$.

Proof. In view of (2.5) we have

$$\begin{aligned} |\mathbf{J}(v, v^*)|^2 &= |J(w, w^*)|^2 \\ &= \left[\int_{\Omega} -\rho\vartheta^2 w_i w_i^* + (2\mu e_{ij}(w) + \lambda\delta_{ij} e_{kk}(w)) e_{ij}(w^*) d\Omega \right]^2 \\ &\leq \left| \int_{\Omega} \rho\vartheta^2 w_i w_i^* d\Omega \right|^2 + C_3 \left| \int_{\Omega} (2\mu e_{ij}(w) + \lambda\delta_{ij} e_{kk}(w)) e_{ij}(w^*) d\Omega \right|^2 \\ &\leq \left| \int_{\omega} h\rho\vartheta^2 v_{i0} v_{i0}^* d\omega \right|^2 + C_2 \sum_{i,j=1}^3 \int_{\Omega} e_{ij}^2(w) d\Omega \sum_{i,j=1}^3 \int_{\Omega} e_{ij}^2(w^*) d\Omega \\ &\leq \left| \int_{\omega} h\rho\vartheta^2 v_{i0} v_{i0}^* d\omega \right|^2 + C_2 \int_{\omega} \frac{1}{2} \sum_{i,j=1}^3 e_{ij0}^2(v) \frac{d\omega}{h} \int_{\omega} \frac{1}{2} \sum_{i,j=1}^3 e_{ij0}^2(v^*) \frac{d\omega}{h} \\ &\leq \int_{\omega} h\rho\vartheta^2 \sum_{i=1}^3 v_{i0}^2 d\omega \int_{\omega} h\rho\vartheta^2 \sum_{i=1}^3 v_{i0}^{*2} d\omega \\ &\quad + C_2 \int_{\omega} \frac{1}{2} \sum_{i,j=1}^3 e_{ij0}^2(v) \frac{d\omega}{h} \int_{\omega} \frac{1}{2} \sum_{i,j=1}^3 e_{ij0}^2(v^*) \frac{d\omega}{h} \\ &\leq C_1 \|v\|_{X_{\vartheta}^{\varkappa}}^2 \|v^*\|_{X_{\vartheta}^{\varkappa}}^2, \end{aligned}$$

where

$$C_1 := \max\{1, C_2\}.$$

Whence (2.10) follows.

Further, taking into account of (2.9) and of the fact that $2\lambda + 3\mu > 0$, $\mu > 0$ we get

$$\begin{aligned}
\|v\|_{X_\vartheta^\varkappa}^2 &\leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{\vartheta^2 \rho h_0}{\mu} \int_\omega x_2^\varkappa v_{i0,2}^2 d\omega \leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{4\vartheta^2 \rho h_0}{\mu(\varkappa + 1)^2} \int_\omega x_2^{\varkappa+2} (v_{i0,2})^2 d\omega \\
&\leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{4\vartheta^2 \rho h_0 l^2}{\mu(\varkappa + 1)^2} \int_\omega x_2^\varkappa (v_{i0,2})^2 d\omega \leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{4\vartheta^2 \rho l^2}{\mu(\varkappa + 1)^2} \int_\omega \frac{(hv_{i0,2})^2}{h} d\omega \\
&= \frac{\mathbf{J}(v, v)}{2\mu} + \frac{4\vartheta^2 \rho l^2}{\mu(\varkappa + 1)^2} \int_\omega \left[\frac{(hv_{10,2})^2}{h} + \frac{(hv_{20,2})^2}{h} + \frac{(hv_{30,2})^2}{h} \right] d\omega \\
&\leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{2\vartheta^2 \rho l^2}{\mu(\varkappa + 1)^2} \int_\omega \left[\frac{2(hv_{10,2})^2}{h} + \frac{4(hv_{20,2})^2}{h} + \frac{2(hv_{30,2})^2}{h} \right] d\omega \\
&\leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{2\vartheta^2 \rho l^2}{\mu(\varkappa + 1)^2} \int_\omega \left[\frac{2[(hv_{10})_{,2}]^2}{h} + \frac{4[(hv_{20})_{,2}]^2}{h} + \frac{2[(hv_{30,2})^2]}{h} \right] d\omega \\
&\leq \frac{\mathbf{J}(v, v)}{2\mu} + \frac{8\vartheta^2 \rho l^2}{\mu(\varkappa + 1)^2} \|v\|_{X_\vartheta^\varkappa}^2,
\end{aligned}$$

from here we have

$$(2.12) \quad \mathbf{J}(v, v) \geq \left(2\mu - \frac{16\vartheta^2 \rho l^2}{(\varkappa + 1)^2}\right) \|v\|_{X_\vartheta^\varkappa}^2.$$

If we assume $\vartheta^2 < \frac{\mu(\varkappa+1)^2}{16\rho l^2}$ inequality (2.11) immediately follows from (2.12). ■

2.4. Remark. If $\mathbf{J}(v, v) = 0$, then $v \equiv 0$ by (2.12).

2.5. Theorem. Let $F \in [X_\vartheta^\varkappa]^*$. Then the variational problem (2.4) has a unique solution $v \in X_\vartheta^\varkappa$ for an arbitrary value of the parameter \varkappa and $\|v\|_{X_\vartheta^\varkappa} \leq \frac{1}{C_0} \|F\|_{[X_\vartheta^\varkappa]^*}$.

Proof. The proof can be realized by means of Lax-Milgram theorem (see Appendix A.1). ■

It can be easily shown that if $\Phi \in [L(\omega)]^3$ and $\text{supp } \Phi \cap \bar{\gamma}_0 = \emptyset$, then $\Phi \in [X_\vartheta^\varkappa]^*$ and

$$\langle \Phi, v^* \rangle = \int_\omega \Phi(x) v^*(x) d\omega,$$

since $v^* \in [H^1(\omega_\varepsilon)]^3$, where ε is sufficiently small positive number such that $\text{supp } \Phi \subset \omega_\varepsilon = \omega \cap \{x_2 > \varepsilon\}$. Therefore,

$$\begin{aligned}
|\langle \Phi, v^* \rangle| &= \left| \int_\omega \Phi(x) v^*(x) d\omega \right| \leq \|\Phi\|_{[L_2(\omega)]^3} \|v^*\|_{[L_2(\omega_\varepsilon)]^3} \\
&\leq \|\Phi\|_{[L_2(\omega)]^3} \|v^*\|_{[H^1(\omega_\varepsilon)]^3} \leq C_\varepsilon \|\Phi\|_{[L_2(\omega)]^3} \|v^*\|_{X_\vartheta^\varkappa}.
\end{aligned}$$

In this case, we obtain the estimate

$$\|v\|_{X_\vartheta^\varkappa} \leq \frac{C_\varepsilon}{C_0} \|\Phi\|_{[L_2(\omega)]^3}.$$

For establishing a representation of the space X_0^\varkappa as a weighted Sobolev space, we introduce the following space:

$$Y_0^\varkappa := \left[W_{2,\varkappa}^1(\omega) \right]^2,$$

where $W_{2,\varkappa}^1(\omega)$ is a completion $\mathcal{D}(\omega)$ by means of the norm

$$\|f\|_{W_{2,\varkappa}^1(\omega)}^2 := \int_\omega x_2^\varkappa (|\nabla f|^2) d\omega, \quad \nabla f = (f_{,1}, f_{,2}).$$

The norm in the space Y_0^\varkappa for a vector (v_{10}, v_{20}, v_{30}) reads as

$$\|v\|_{Y_0^\varkappa}^2 := \int_\omega x_2^\varkappa \left(\sum_{\alpha=1}^2 |\nabla v_{\alpha 0}|^2 \right) d\omega.$$

Using Korn's and Hardy's inequalities (see Appendix) the following theorem can be proved (similarly, to the Theorem 5.1 of [4])

2.6. Theorem. *The linear spaces X_0^\varkappa and Y_0^\varkappa as sets of vector functions coincide and the norms $\|\cdot\|_{X_0^\varkappa}, \|\cdot\|_{Y_0^\varkappa}$ are equivalent if $\varkappa = 0$ and $\vartheta^2 < \min\{\frac{\mu(\varkappa+1)^2}{16\rho l^2}, \frac{2}{h_0\rho l^2}\}$.*

2.7. Remark. Note that if $v \in X_\vartheta^\varkappa$, then all the components of v possess the zero traces on part γ_1 of the boundary $\partial\omega$ for arbitrary \varkappa due to the well-known trace theorem in the Sobolev space W^1 . This follows, on the one hand, from the fact that the elliptic system under consideration is non-degenerated at the curve γ_1 and, on the other hand, from the construction of the space X_ϑ^\varkappa .

3. Appendix

A.1. The Lax-Milgram theorem. Let V be a real Hilbert space and let $J(w, v)$ be a bilinear form defined on $V \times V$. Let this form be continuous, i.e., let there exist a constant $K > 0$ such that

$$|J(w, v)| \leq K \|w\|_V \|v\|_V$$

holds $\forall w, v \in V$ and V -elliptic, i.e., let there exist a constant $\alpha > 0$ such that

$$J(w, w) \geq \alpha \|w\|_V^2$$

holds $\forall w \in V$. Further let F be a bounded linear functional from V^* dual of V . Then there exists one and only one element $z \in V$ such that

$$J(z, v) = \langle F, v \rangle \equiv Fv \quad \forall v \in V$$

and

$$\|z\|_V \leq \alpha^{-1} \|F\|_{V^*}.$$

Let ω be as in Section 1 and let $\mathcal{D}(\omega)$ be a space of infinitely differentiable functions with compact support in ω .

A.2. Hardy's inequality. For every $f \in \mathcal{D}(\omega)$ and $\nu \neq 1$ there holds the inequality

$$(A.1) \quad \int_\omega x_2^{\nu-2} f^2(x) d\omega \leq C_\nu \int_\omega x_2^\nu |\nabla f(x)|^2 d\omega,$$

where the positive constant C_ν is independent of f .

By completion of $\mathcal{D}(\omega)$ with the norm

$$\|f\|_{\overset{\circ}{W}_{2,\nu}^1(\omega)}^2 := \int_\omega x_2^\nu |\nabla f(x)|^2 d\omega,$$

we conclude that the inequality (A.1) holds for arbitrary $f \in \overset{\circ}{W}_{2,\nu}^1(\omega)$.

For proof see [7].

A.3. Korn's weighted inequality. Let $\varphi = (\varphi_1, \varphi_2) \in [\overset{\circ}{W}_{2,\nu}^1(\omega)]^2$ and $\nu \neq 1$. Then

$$\int_\omega x_2^\nu [|\nabla \varphi_1(x)|^2 + |\nabla \varphi_2(x)|^2] d\omega$$

$$\leq C_\nu \int_{\omega} x_2^\nu [\varphi_{1,1}^2(x) + \varphi_{2,2}^2(x) + (\varphi_{1,2}(x) + \varphi_{2,1}(x))^2] d\omega,$$

where the positive constant C_ν is independent of φ .

The proof can be found in [7], [17].

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