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The fractional airy transform

Alireza Ansari *

Abstract

In this note, we introduce the fractional Airy transform using the higher order derivatives of the Airy function and the Airy polynomials. Then, we show that the new integral transform is coincided to the natural Airy transform in particular case of the scaling parameter. Some properties of this transform are also given.

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1. Introduction and Preliminaries

The general solution of the Airy differential equation

(1.1) $$y'' - xy = 0, \quad x \in \mathbb{R},$$

is given by

(1.2) $$y(x) = c_1 Ai(x) + c_2 Bi(x),$$

where $Ai(x)$ and $Bi(x)$ are the Airy functions of first and second kinds, respectively, such that [2, 10]

(1.3) $$Ai(x) = \frac{1}{\pi}\int_0^{\infty} \cos(xt + \frac{t^3}{3})dt,$$

(1.4) $$Bi(x) = \frac{1}{\pi}\int_0^{\infty} \left(e^{xt-\frac{t^3}{3}} + \sin(xt + \frac{t^3}{3})\right)dt.$$

In view of the Airy function of first kind $Ai(x)$, Widder in [12] introduced the Airy transform with scaling parameter α in terms of the convolution product of Fourier transform as follows

(1.5) $$\widehat{\mathcal{A}}_{\alpha}\{f(\xi);x\} = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(\xi)Ai(\frac{x-\xi}{\alpha})d\xi, \quad \alpha \in \mathbb{R}.$$

*Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran
Email: alireza_1038@yahoo.com

The inversion formula of the above transform is easily obtained via the orthogonality relation of the Airy function

$$\frac{1}{\alpha^2}\int_{-\infty}^{\infty} Ai(\frac{x-y}{\alpha})Ai(\frac{x-z}{\alpha})dx = \delta(y-z), \tag{1.6}$$

as $\widehat{\mathcal{A}}_\alpha^{-1} = \widehat{\mathcal{A}}_{-\alpha}$.

Later, Hunt in [6] and Bertoncini et al. in [4] used this transform in molecular physics and in evaluation of quantum transport, respectively. For more contributions of the Airy transform, for example, Babusci et al. [3] obtained the formal solution of the third order PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}, \quad u(x,0) = f(x), \quad t > 0 \tag{1.7}$$

with respect to the Airy transform of the function $f(x)$, that is

$$u(x,t) = \frac{1}{\sqrt[3]{3t}}\int_{-\infty}^{\infty} Ai(\frac{x-\xi}{\sqrt[3]{3t}})f(\xi)d\xi. \tag{1.8}$$

For more details of the above solution in the Airy diffusion equation, see [10]. As another application of this transform, Jiang et al. [5] used the two dimensional Airy transform with kernel $w_{\alpha\beta}(xy) = \frac{1}{|\alpha\beta|}Ai(\frac{x}{\alpha})Ai(\frac{y}{\beta})$ for analyzing the Airy beams in optics and Torre [9] applied the Airy transform to derive the three-variable Hermite polynomials and their generating functions. Also, Varlamov [11] obtained the Riesz fractional derivative of the product of Airy transforms and presented this product in terms of the Bessel function of zero order $J_0(x)$ and the Riesz fractional derivative of the Airy function.

Now in this paper, among the Airy transform of the elementary functions (which can be found in [10]), we concern to the function $x^n, n \in \mathbb{N}$, which its Airy transform leads us to the Airy polynomials $\mathrm{Pi}_n(x)$,

$$\widehat{\mathcal{A}}_\alpha\{\xi^n; x\} = \alpha^n \mathrm{Pi}_n(\frac{x}{\alpha}) = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \xi^n Ai(\frac{x-\xi}{\alpha})d\xi. \tag{1.9}$$

Some important properties of the Airy polynomials can be written by the following relations which leads us to the definition of fractional Airy transform in Section 2.

Property 1: The bi-orthogonality relation between the Airy polynomials and the higher order derivatives of Airy function is given by [1]

$$\int_{-\infty}^{\infty} \mathrm{Pi}_n(x)Ai^{(n)}(x)dx = (-1)^n n!. \tag{1.10}$$

Property 2: The Airy transform of the Airy polynomials is given in terms of the Airy polynomials as follows [9]

$$\widehat{\mathcal{A}}_\alpha\{\mathrm{Pi}_n(\xi); x\} = (1+\alpha^3)^{\frac{n}{3}}\,\mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right). \tag{1.11}$$

Property 3: The higher order derivatives of the Airy function in Property 1, can be simplified into the following relation [7, 8]

$$Ai^{(n)}(x) = p_n(x)Ai(x) + q_n(x)Ai'(x), \quad n \in \mathbb{N}, \tag{1.12}$$

where the polynomials p_n and q_n are given by the recurrence relations

$$p_{n+2}(x) = xp_n(x) + np_n(x), \tag{1.13}$$

$$q_{n+2}(x) = xq_n(x) + nq_n(x), \tag{1.14}$$

$$p_{n+1}(x) = p'_n(x) + xq_n(x), \tag{1.15}$$

$$q_{n+1}(x) = q'_n(x) + p_n(x), \tag{1.16}$$

with the generating functions formulas

$$\pi[Bi'(x)Ai(x+t) - Ai'(x)Bi(x+t)] = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{1.17}$$

$$\pi[Ai(x)Bi(x+t) - Bi(x)Ai(x+t)] = \sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!}. \tag{1.18}$$

2. The Fractional Airy Transform

Now in this section, using the obtained results in Section 1, we introduce the fractional Airy transform. In the following lemma, the Airy function is applied as a generating function.

2.1. Lemma. *For $a, b \in \mathbb{R}$, the following series holds for the Airy polynomials and the higher order derivatives of Airy function*

$$\sum_{n=0}^{\infty} \mathrm{Pi}_n(bx) Ai^{(n)}(y)\frac{a^n}{n!} = \frac{1}{(a^3-1)^{\frac{1}{3}}} Ai\left(\frac{-abx-y}{(a^3-1)^{\frac{1}{3}}}\right). \tag{2.1}$$

Proof: Setting the relations (1.9) and (1.12) in the left hand side of (2.1), we get the above series as

$$\begin{aligned} S &= \int_{-\infty}^{\infty} \left[Ai(y)\left(\sum_{n=0}^{\infty}\frac{(a\xi)^n}{n!}p_n(y)\right) + \left(Ai'(y)\sum_{n=0}^{\infty}\frac{(a\xi)^n}{n!}q_n(y)\right)\right] Ai(bx-\xi)d\xi \\ &= \pi Ai(y)\int_{-\infty}^{\infty}\left[Bi'(y)Ai(y+a\xi) - Ai'(y)Bi(y+a\xi)\right] Ai(bx-\xi)d\xi \\ &+ \pi Ai'(y)\int_{-\infty}^{\infty}\left[Ai(y)Bi(y+a\xi) - Bi(y)Ai(y+a\xi)\right] Ai(bx-\xi)d\xi \end{aligned}$$

$$= \pi\left(Ai(y)Bi'(y) - Ai'(y)Bi(y)\right)\int_{-\infty}^{\infty} Ai(y+a\xi)Ai(bx-\xi)d\xi. \tag{2.2}$$

Since the Wronskian of functions $Ai(y)$ and $Bi(y)$ is equal to $W(Ai(y), Bi(y)) = Ai(y)Bi'(y) - Ai'(y)Bi(y) = \frac{1}{\pi}$, the last integral in (2.2) is simplified to

$$S = \frac{1}{(a^3-1)^{\frac{1}{3}}} Ai\left(\frac{-abx-y}{(a^3-1)^{\frac{1}{3}}}\right), \tag{2.3}$$

where we used the following identity for simplification [10]

$$\int_{-\infty}^{\infty} Ai\left(\frac{\xi+a}{\alpha}\right)Ai\left(\frac{\xi+b}{\beta}\right)d\xi = \frac{|\alpha\beta|}{|\beta^3-\alpha^3|^{\frac{1}{3}}} Ai\left(\frac{b-a}{(\beta^3-\alpha^3)^{\frac{1}{3}}}\right),\ \beta \neq \alpha. \tag{2.4}$$

2.2. Theorem. *The following relation*

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi); x\} = \frac{1}{|(1+\alpha^3)^{\beta}-1|}\int_{-\infty}^{\infty} Ai\left(\frac{(1+\alpha^3)^{\frac{\beta-1}{3}}x-\xi}{(1+\alpha^3)^{\beta}-1}\right) f(\xi)d\xi, \tag{2.5}$$

is the fractional Airy transform of order $\beta \in \mathbb{R}$.

Proof: According to the relation (1.11), we intend to find the fractional Airy transform such that

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\mathrm{Pi}_n(\xi); x\} = (1+\alpha^3)^{\frac{\beta n}{3}}\mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right), \quad \beta \in \mathbb{R}. \tag{2.6}$$

For this purpose, we suppose that the function $f(x)$ can be expanded in terms of the Airy polynomials as

$$f(x)=\sum_{n=0}^{\infty}a_n\mathrm{Pi}_n(x), \tag{2.7}$$

where the coefficients a_n are found from the bi-orthogonal property of Airy polynomials (1.10)

$$a_n=\frac{(-1)^n}{n!}\int_{-\infty}^{\infty}Ai^{(n)}(y)f(y)dy. \tag{2.8}$$

Finally, on account of equation (2.6), the effect of $\widehat{\mathcal{A}}_{\alpha_\beta}$ on $f(x)$ is

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}=\sum_{n=0}^{\infty}a_n(1+\alpha^3)^{\frac{\beta n}{3}}\mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right), \tag{2.9}$$

which by substituting the coefficients (2.8) into (2.9), we get $\widehat{\mathcal{A}}_{\alpha_\beta}$ as

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}=\int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty}\mathrm{Pi}_n\left(\frac{x}{(1+\alpha^3)^{\frac{1}{3}}}\right)Ai^{(n)}(\xi)\frac{[-(1+\alpha^3)^{\frac{\beta}{3}}]^n}{n!}\right)f(\xi)d\xi. \tag{2.10}$$

Now, by using Lemma 2.1 and setting $a=|-(1+\alpha^3)^{\frac{\beta}{3}}|$, $b=|\frac{1}{(1+\alpha^3)^{\frac{1}{3}}}|$, we obtain the fractional Airy transform $\widehat{\mathcal{A}}_{\alpha_\beta}$ in the following form

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}=\frac{1}{|(1+\alpha^3)^\beta-1|}\int_{-\infty}^{\infty}Ai\left(\frac{(1+\alpha^3)^{\frac{\beta-1}{3}}x-\xi}{(1+\alpha^3)^\beta-1}\right)f(\xi)d\xi. \tag{2.11}$$

2.3. Remark. It is obvious that, by setting $\beta=1,\alpha^3=\lambda$ in (2.5), we obtain the natural Airy transform with scaling parameter λ.

2.4. Remark. The relation (2.5) shows the fractional Airy transform is a natural Airy transform with the modified scaling parameter $\frac{1}{(1+\alpha^3)^\beta-1}$, that is

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}=\widehat{\mathcal{A}}_{\frac{1}{(1+\alpha^3)^\beta-1}}\{f(\xi);(1+\alpha^3)^{\frac{\beta-1}{3}}x\}. \tag{2.12}$$

2.5. Remark. According to the relation (1.6), the inversion formula of fractional Airy transform (2.5) is presented by

$$f(\xi)=\frac{1}{|(1+\alpha^3)^\beta-1|}\int_{-\infty}^{\infty}Ai\left(\frac{(1+\alpha^3)^{\frac{\beta-1}{3}}x-\xi}{(1+\alpha^3)^\beta-1}\right)\widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}dx, \tag{2.13}$$

which implies that for the value $\gamma=\frac{1}{(1+\alpha^3)^\beta-1}$, $\widehat{\mathcal{A}}_\gamma^{-1}=\widehat{\mathcal{A}}_{-\gamma}$.

2.6. Example. Using the Airy transform of $f(\xi)=e^{ik\xi}$ for $k\in\mathbb{R}$, [10]

$$\begin{aligned}\widehat{\mathcal{A}}_{\alpha_\beta}\{e^{ik\xi};x\} &= \frac{1}{(1+\alpha^3)^\beta-1}\int_{-\infty}^{\infty}Ai\left(\frac{(1+\alpha^3)^{\frac{\beta-1}{3}}x-\xi}{(1+\alpha^3)^\beta-1}\right)e^{ik\xi}d\xi,\\ &= e^{i\left(k(1+\alpha^3)^{\frac{\beta-1}{3}}x+\frac{1}{3}\frac{1}{((1+\alpha^3)^\beta-1)^3}k^3\right)},\end{aligned} \tag{2.14}$$

the fractional Airy transform of the trigonometric functions $\sin(k\xi)$ and $\cos(k\xi)$ are given by the following relations

$$\widehat{\mathcal{A}}_{\alpha_\beta}\{\cos(k\xi);x\}=\cos\left(k(1+\alpha^3)^{\frac{\beta-1}{3}}x+\frac{1}{3}\frac{1}{((1+\alpha^3)^\beta-1)^3}k^3\right), \tag{2.15}$$

(2.16) $$\widehat{\mathcal{A}}_{\alpha_\beta}\{\sin(k\xi);x\} = \sin\left(k(1+\alpha^3)^{\frac{\beta-1}{3}}x + \frac{1}{3}\frac{1}{((1+\alpha^3)^\beta-1)^3}k^3\right).$$

2.7. Corollary. *Let $F_{\alpha_\beta}(x)$ be the fractional Airy transform of $f(\xi)$, then the fractional Airy transform of $\xi f(\xi)$ is*

(2.17) $$\widehat{\mathcal{A}}_{\alpha_\beta}\{\xi f(\xi);x\} = \delta x F_{\alpha_\beta}(x) - \frac{\gamma^3}{\delta^2}F''_{\alpha_\beta}(x),$$

where the parameters γ and δ are given by

(2.18) $$\gamma = \frac{1}{(1+\alpha^3)^\beta - 1}, \quad \delta = (1+\alpha^3)^{\frac{\beta-1}{3}}.$$

Proof: According to the definition of fractional Airy transform of $F''_{\alpha_\beta}(x)$ and relation (1.1), we easily arrive at (2.17).

2.8. Example. Setting $f(\xi) = \frac{1}{\xi}$ in Corollary 2.7 and using the fact that $\widehat{\mathcal{A}}_{\alpha_\beta}\{1;x\} = 1$, we get the fractional Airy transform of $f(\xi) = \frac{1}{\xi}$ in terms of the Scorer function [10]

(2.19) $$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin(xt + \frac{t^3}{3})dt,$$

as follows

(2.20) $$\widehat{\mathcal{A}}_{\alpha_\beta}\{\frac{1}{\xi};x\} = \frac{\pi}{\gamma}Gi(\frac{\delta}{\gamma}x).$$

2.9. Corollary. *The Parseval identity for the fractional Airy transform is*

(2.21) $$\int_{-\infty}^{\infty} f(\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} \widehat{\mathcal{A}}_{\alpha_\beta}\{f(\xi);x\}\widehat{\mathcal{A}}_{\alpha_\beta}\{g(\xi);x\}dx.$$

Proof: Using the orthogonality relation of the Airy function (1.6), the proof is completed.

2.10. Example. Setting $f(\xi) = \frac{1}{\xi}$ and $g(\xi) = \sin(k\xi)$ in Corollary 2.9, and using the relations (2.16) and (2.23), we get the value of following well-known integral in terms of the Scorer function [10]

(2.22) $$\int_{-\infty}^{\infty} Gi(\frac{\delta}{\gamma}x)\sin\left(k\delta x + \frac{k^3}{3\gamma^3}\right)dx = \gamma\ \mathrm{sgn}(k),$$

where we used the following fact for computation

(2.23) $$\int_{-\infty}^{\infty}\frac{\sin(k\xi)}{\xi}d\xi = \pi\mathrm{sgn}(k).$$

3. Concluding Remarks

This paper provides a fractionalization form of the Airy transform. On the base of the Airy polynomials, we introduced a generalized form of the natural Airy transform and named it as the fractional Airy transform. Some properties of this transform such as transformations of elementary functions and Parseval identity were also obtained. It is hope that the employed integral transform can be considered as a promising approach in a fairly wide context of applied mathematics and physics in near future.

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