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Contact CR-warped product submanifolds in Sasakian space forms

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Abstract

In this paper we consider Contact CR-warped product submanifolds and we investigate the status of equality in the inequality which has been found by I. Hasegawa and I. Mihai in [8].

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1. Preliminaries

Recently, in [8] I. Hasegawa and I. Mihai studied contact CR-warped products in Sasakian manifolds and obtained inequalities for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products in a Sasakian space form. Afterwards, I. Mihai and K. Arslan studied warped products which are CR-submanifolds in Sasakian and Kenmotsu manifolds, respectively, and established general sharp inequalities for a CR-warped product in Sasakian and Kenmotsu space forms(see [1, 7]).

In [2], we also studied contact CR-warped product submanifolds in a cosymplectic manifold and gave a necessary and sufficient condition for a contact CR-warped product to be contact CR product. In this paper we give a necessary and sufficient condition for contact CR-warped product to be contact CR-product in a Sasakian space form.

In this section, we will give some notations used throughout this paper. We recall some necessary facts and formulas from the theory of Sasakian manifolds and their submanifolds.

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A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ and a 1-form η, satisfying;

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \ \ \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{1.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{1.2}$$

for any vector fields X, Y tangent to $\bar{M}$. Furthermore, an almost contact metric manifold is called a Sasakian manifold if ϕ and ξ satisfy;

$$(\bar{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X \ \ \textit{and} \ \ \bar{\nabla}_X \xi = \phi X, \tag{1.3}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$[5].

Now, let $\bar{M}$ be a $(2n+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g) and M be an m-dimensional isometrically immersed the submanifold in $\bar{M}$. Moreover, we denote the Levi-Civita connections by $\bar{\nabla}$ and ∇, respectively. Then the Gauss and Weingarten formulas for M in $\bar{M}$ are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{1.5}$$

for any vector fields X, Y tangent to M and vector V normal to M, where $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, h and A denote the second fundamental form and shape operator of M in $\bar{M}$, respectively. The A and h are related by

$$g(h(X,Y), V) = g(A_V X, Y). \tag{1.6}$$

We denote the Riemannian curvature tensors of $\bar{\nabla}$ and the induced connection ∇ by $\bar{R}$ and R, respectively. Then the Gauss and Codazzi equations are, respectively, given by

$$(\bar{R}(X,Y)Z)^{\top} = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \tag{1.7}$$

and

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{1.8}$$

for any vector fields X, Y, Z tangent to M, where the covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.9}$$

for any vector fields X, Y, Z tangent to M, where $(\bar{R}(X,Y)Z)^{\perp}$ and $(\bar{R}(X,Y)Z)^{\top}$ denote the normal and tangent components of $\bar{R}(X,Y)Z$, respectively, with respect to the submanifold[8].

For any vector field X tangent to M, we set

$$\phi X = fX + \omega X, \tag{1.10}$$

where fX and ωX are the tangential and normal components of ϕX, respectively. Then f is an endomorphism of the TM and ω is a normal-bundle valued 1-form of TM. For the same reason, any vector field V normal to M, we set

$$\phi V = BV + CV, \tag{1.11}$$

where BV and CV are the tangential and normal components of ϕV, respectively. Then B is an endomorphism of the normal bundle $T^{\perp}M$ to TM and C is a normal-bundle valued 1-form of $T^{\perp}M$.

A plane section π in $\Gamma(T\bar{M})$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a Sasakian space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ &+ \eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(Z,\phi Y)\phi X-g(Z,\phi X)\phi Y\\ &- 2g(\phi X,Y)\phi Z\} \qquad (1.12)\end{aligned}$$

for any vector fields X, Y, Z tangent to $\bar{M}$[6]. For more details, we refer to the references.

2. Contact CR-Warped Product Submanifolds in Sasakian Manifolds

In this section, we will define contact CR-warped product submanifolds in a Sasakian manifold, have obtained some the inequalities in a Sasakian manifold.

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive smooth function on M_1. We consider the product manifold $M_1 \times M_2$ with its projections $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\eta : M_1 \times M_2 \longrightarrow M_2$. The warped product $M = M_1 \times_f M_2$ is a manifold $M_1 \times M_2$ equipped with the Riemannian metric such that

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (fo\pi)^2 g_2(\eta_* X, \eta_* Y),$$

for any $X, Y \in \Gamma(TM)$, where $*$ stand for differential of map and $\Gamma(TM)$ denote set of the differentiable vector fields on M. Thus we have $g = g_1 \otimes f^2 g_2$. The function f is called the warping function of the warped product manifold $M = M_1 \times_f M_2$. If we denote the Levi Civita connection on M by ∇, then we have the following Proposition for the warped product manifold[3].

2.1. Proposition. *Let $M = M_1 \times_f M_2$ be a warped product manifold. For $X, Y \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have*

(1) $\nabla_X Y \in \Gamma(TM_1)$, *that is, M_1 is totally geodesic submanifold in M,*

(2) $\nabla_X V = \nabla_V X = X(\ln f)V$,

(3) $nor\nabla_Z V = -g(Z,V)grad(\ln f)$, *that is, M_2 is totally umbilical submanifold in M,*

(4) $tan\nabla_Z V = \nabla'_Z V \in \Gamma(TM_2)$ *is the lift of $\nabla'_Z V$ on M_2, where ∇' denote the Levi-Civita connection of g_2*[4].

If the warping function f is constant, then the warped product is said to be Riemannian product.

Let M be an m-dimensional Riemannian manifold with Riemannian metric g, and let $\{e_1, e_2, ..., e_m\}$ be an orthonormal basis for $\Gamma(TM)$. For a smooth function f on M, the gradient and Hessian of f are, respectively, defined by

$$X(f) = g(\text{grad}(f), X) \qquad (2.1)$$

and

$$H^f(X,Y) = X(Y(f)) - (\nabla_X Y)f = g(\nabla_X \mathrm{grad}(f), Y) \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$. The Laplacian of f is defined by

$$\triangle f = \sum_{i=1}^{m}\{(\nabla_{e_i} e_i)f - e_i(e_i(f))\} = -\sum_{i=1}^{m} g(\nabla e_i \mathrm{grad}(f), e_i). \tag{2.3}$$

From (2.2) and (2.3), note that the Laplacian is essentially the negative of the trace of the Hessian.

From the integration theory on manifolds, if M is a compact orientable Riemannian manifold without boundary, we have

$$\int_M \Delta f dV = 0, \tag{2.4}$$

where dV is the volume element of M[2].

By analogy with submanifolds in a Kenmotsu manifold, different classes of submanifolds in a Sasakian manifold were considered by many geometers(see, references).

2.2. Definition. Let M be an isometrically immersed submanifold of a Sasakian manifold $\bar{M}$.
(1) A submanifold M is tangent to ξ is called an invariant submanifold if ϕ preserves any tangent space of M, that is, $\phi(T_M(p)) \subset T_M(p)$, for every $p \in M$.
(2) A submanifold M tangent to ξ is called an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_M(p)) \subset T_M^{\perp}(p)$, for every $p \in M$.
(3) A submanifold M tangent to ξ is called a contact CR-submanifold if it admits an invariant distribution whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is, $TM = D \oplus D^{\perp}$, with $\phi(D_p) \subset D_p$ and $\phi(D_p^{\perp}) \subset T_M^{\perp}(p)$, for every $p \in M$.

In this paper, we shall consider warped product manifolds which are in the form $M = M_T \times_f M_{\perp}$ in a Sasakian manifold $\bar{M}$ such that M is tangent to ξ, where M_T is an invariant submanifold tangent to ξ and $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$. We simply call such manifolds contact CR-product submanifolds.

3. Contact CR Warped Product Submanifolds in Sasakian Space Forms

In this section, we will give the main results of this paper. Firstly, we will give the following two lemmas and a theorem for later use.

3.1. Lemma. *Let $M = M_T \times_f M_{\perp}$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then we have*

$$g(h(X,Y), \phi Y) = [\eta(X) - \phi X(\ln f)]g(Y,Y) \tag{3.1}$$

and

$$g(h(\phi X, Y), \phi Y) = \|Y\|^2 X(\ln f) \tag{3.2}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_{\perp})$.

Proof. For any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$, by using (1.3), (1.4) and considering Proposition 2.1(2), we have

$$\begin{aligned} g(h(X,Y),\phi Y) &= g(\bar{\nabla}_Y X, \phi Y) = -g(\phi\bar{\nabla}_Y X, Y) \\ &= -g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, Y) \\ &= -g(\nabla_Y \phi X, Y) + g(-g(X,Y)\xi + \eta(X)Y, Y) \\ &= -\phi X(\ln f) g(Y,Y) + \eta(X) g(Y,Y) \end{aligned}$$

and

$$\begin{aligned} g(h(\phi X,Y),\phi Y) &= g(\bar{\nabla}_Y \phi X, \phi Y) = g((\bar{\nabla}_Y \phi)X + \phi\bar{\nabla}_Y X, \phi Y) \\ &= g(-g(X,Y)\xi + \eta(X)Y, \phi Y) - g(\bar{\nabla}_Y X, \phi^2 Y) \\ &= -g(\bar{\nabla}_Y X, -Y + \eta(Y)\xi) \\ &= g(\nabla_Y X, Y) = X(\ln f) g(Y,Y), \end{aligned}$$

which proves our assertion. □

3.2. Lemma. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then we have*

$$\|h(X,Y)\|^2 = g(h(\phi X,Y), \phi h(X,Y)) + [\eta(X) - \phi X \ln f]^2 g(Y,Y), \tag{3.3}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

Proof. Making use of (1.3), (1.4) and consider Proposition 2.1 and Lemma 3.1 we have

$$\begin{aligned} g(h(\phi X,Y),\phi h(X,Y)) &= g(\bar{\nabla}_Y \phi X - \nabla_Y \phi X, \phi h(X,Y)) \\ &= g((\bar{\nabla}_Y \phi)X + \phi\bar{\nabla}_Y X - \phi X(\ln f)Y, \phi h(X,Y)) \\ &= g(-g(X,Y)\xi + \eta(X)Y, \phi h(X,Y)) \\ &+ g(\phi\bar{\nabla}_Y X, \phi h(X,Y)) + \phi X(\ln f) g(h(X,Y),\phi Y) \\ &= -\eta(X) g(h(X,Y),\phi Y) + g(h(X,Y),h(X,Y)) + \phi X(\ln f) g(h(X,Y),\phi Y) \\ &= \|h(X,Y)\|^2 + [\phi X(\ln f) - \eta(X)] g(h(X,Y),\phi Y) \\ &= \|h(X,Y)\|^2 + [\phi X(\ln f) - \eta(X)][\eta(X) - \phi X(\ln f)] g(Y,Y) \end{aligned}$$

This completes the proof of the Lemma. □

3.3. Theorem. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then we have*

$$\begin{aligned} 2\|h(X,Y)\|^2 &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) - 2\phi X(\ln f)\eta(X) \\ &+ 2(\phi X(\ln f))^2 + 2\eta^2(X) + \eta(\nabla_X X)\eta(grad \ln f) \\ &+ \left(\frac{c+3}{4}\right) g(\phi X,\phi X)\} g(Y,Y), \end{aligned} \tag{3.4}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

Proof. By using (1.8), (1.9) and making use of $\bar{\nabla}$ being Levi-Civita connection, we have

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= g((\bar{\nabla}_X h)(\phi X,Y)-(\bar{\nabla}_{\phi X}h)(X,Y),\phi Y)\\
&= g(\bar{\nabla}_X h(\phi X,Y)-h(\nabla_X\phi X,Y)-h(\phi X,\nabla_X Y),\phi Y)\\
&- g(\bar{\nabla}_{\phi X} h(X,Y)-h(\nabla_{\phi X}X,Y)-h(\nabla_{\phi X}Y,X),\phi Y)\\
&= X[g(h(\phi X,Y),\phi Y)]-g(\bar{\nabla}_X\phi Y,h(\phi X,Y))-g(h(\nabla_X\phi X,Y),\phi Y)\\
&- g(h(\nabla_X Y,\phi X),\phi Y)-\phi X[g(h(X,Y),\phi Y)]+g(\bar{\nabla}_{\phi X}\phi Y,h(X,Y))\\
&+ g(h(\nabla_{\phi X}X,Y),\phi Y)+g(h(\nabla_{\phi X}Y,X),\phi Y).
\end{aligned}$$

Taking into account (3.1), (3.2) and Proposition 2.1(2), we obtain

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= X[X(\ln f)g(Y,Y)]-g(h(\phi X,Y),(\bar{\nabla}_X\phi)Y+\phi\bar{\nabla}_X Y)\\
&- g(Y,Y)\{\eta(\nabla_X\phi X)-(\phi\nabla_X\phi X)(\ln f)\}-X(\ln f)g(h(Y,\phi X),\phi Y)\\
&- \phi X[\{\eta(X)-\phi X(\ln f)\}g(Y,Y)]+g(h(X,Y),(\bar{\nabla}_{\phi X}\phi)Y+\phi\bar{\nabla}_{\phi X}Y)\\
&+ g(Y,Y)\{\eta(\nabla_{\phi X}X)-(\phi\nabla_{\phi X}X)(\ln f)\}+\phi X(\ln f)g(h(X,Y),\phi Y)\\
&= X(X(\ln f))g(Y,Y)+2(X(\ln f))^2 g(Y,Y)-g(h(\phi X,Y),\phi\nabla_X Y+\phi h(X,Y))\\
&- \eta(\nabla_X\phi X)g(Y,Y)+g(Y,Y)(\phi\nabla_X\phi X)(\ln f)-(X(\ln f))^2 g(Y,Y)\\
&- \phi X[\eta(X)-\phi X(\ln f)]g(Y,Y)-2\phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)\\
&+ g(h(X,Y),\phi\nabla_{\phi X}Y+\phi h(\phi X,Y))+\eta(\nabla_{\phi X}X)g(Y,Y)\\
&- (\phi\nabla_{\phi X}X)(\ln f)g(Y,Y)+\phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)\\
&= X(X(\ln f))g(Y,Y)+(X(\ln f))^2 g(Y,Y)-X(\ln f)g(h(\phi X,Y),\phi Y)\\
&- g(h(\phi X,Y),\phi h(X,Y))-\eta(\nabla_X\phi X)g(Y,Y)+(\phi\nabla_X\phi X)(\ln f)g(Y,Y)\\
&- \phi X[\eta(X)]g(Y,Y)+\phi X(\phi X(\ln f))g(Y,Y)-2\phi X(\ln f)\eta(X)g(Y,Y)\\
&+ 2(\phi X(\ln f))^2 g(Y,Y)+\phi X(\ln f)g(h(X,Y),\phi Y)\\
&+ g(h(X,Y),\phi h(\phi X,Y))+\eta(\nabla_{\phi X}X)g(Y,Y)-(\phi\nabla_{\phi X}X)(\ln f)g(Y,Y)\\
&+ \phi X(\ln f)\eta(X)g(Y,Y)-(\phi X(\ln f))^2 g(Y,Y)\\
&= X(X(\ln f))g(Y,Y)-2g(h(\phi X,Y),\phi h(X,Y))-\eta(\nabla_X\phi X)g(Y,Y)\\
&+ (\phi\nabla_X\phi X)(\ln f)g(Y,Y)-\phi X[\eta(X)]g(Y,Y)+\phi X(\phi X(\ln f))g(Y,Y)\\
&- \phi X(\ln f)\eta(X)g(Y,Y)+(\phi X(\ln f))^2 g(Y,Y)\\
&+ \phi X(\ln f)\{\eta(X)-\phi X(\ln f)\}g(Y,Y)+\eta(\nabla_{\phi X}X)g(Y,Y)\\
&- (\phi\nabla_{\phi X}X)(\ln f)g(Y,Y). \qquad (3.5)
\end{aligned}$$

We know that on a Sasakian manifold

$$\begin{aligned}
\phi X[\eta(X)] &= \phi X g(X,\xi)=g(\bar{\nabla}_{\phi X}X,\xi)+g(X,\bar{\nabla}_{\phi X}\xi)\\
&= \eta(\nabla_{\phi X}X)+g(\phi^2 X,X)=\eta(\nabla_{\phi X}X)-g(\phi X,\phi X), \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\eta(\nabla_X\phi X) &= g(\bar{\nabla}_X\phi X,\xi)=g((\bar{\nabla}_X\phi)X+\phi\nabla_X X,\xi)\\
&= g(-g(X,X)\xi+\eta(X)X,\xi)=-g(X,X)+\eta^2(X)=-g(\phi X,\phi X). \qquad (3.7)
\end{aligned}$$

Furthermore, considering Proposition 2.1, M_T is totally geodesic in M and $\mathrm{grad}(\ln f)\in\Gamma(TM_T)$, by direct calculations, we obtain

$$\begin{aligned}
(\phi\nabla_{\phi X}X)(\ln f) &= g(\phi\nabla_{\phi X}X,\mathrm{grad}(\ln f))=g(\bar{\nabla}_{\phi X}\phi X-(\bar{\nabla}_{\phi X}\phi)X,\mathrm{grad}(\ln f))\\
&= g(\nabla_{\phi X}\phi X,\mathrm{grad}(\ln f))-g(-g(\phi X,X)\xi+\eta(X)\phi X,\mathrm{grad}(\ln f))\\
&= (\nabla_{\phi X}\phi X)(\ln f)-\eta(X)\phi X(\ln f) \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}(\phi\nabla_X\phi X)(\ln f) &= g(\phi\nabla_X\phi X, \mathrm{grad}(\ln f)) = -g(\bar{\nabla}_X\phi X, \phi\mathrm{grad}(\ln f))\\ &= -g((\bar{\nabla}_X\phi)X + \phi\bar{\nabla}_X X, \phi\mathrm{grad}(\ln f))\\ &= g(-g(X,X)\xi + \eta(X)X, \phi\mathrm{grad}(\ln f)) - g(\phi\nabla_X X, \phi\mathrm{grad}(\ln f))\\ &= -(\nabla_X X)(\ln f) + \eta(\nabla_X X)\eta(\phi\mathrm{grad}(\ln f)) + \eta(X)\phi X(\ln f). \qquad (3.9)\end{aligned}$$

So by substituting (3.6), (3.7), (3.8) and (3.9) into (3.5), we get

$$\begin{aligned}g(\bar{R}(X,\phi X)Y,\phi Y) &= X(X(\ln f))g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)) + g(\phi X,\phi X)g(Y,Y)\\ &+ g(Y,Y)\{\eta(X)\phi X(\ln f) - (\nabla_X X)(\ln f) + \eta(\nabla_X X)\eta(\mathrm{grad}(\ln f))\}\\ &- g(Y,Y)\{\eta(\nabla_{\phi X}X) - g(\phi X,\phi X)\} + \phi X(\phi X(\ln f))g(Y,Y)\\ &+ \eta(\nabla_{\phi X}X)g(Y,Y) - (\nabla_{\phi X}\phi X)(\ln f)g(Y,Y) + \eta(X)\phi X(\ln f)g(Y,Y)\\ &= \{X(X(\ln f)) + \phi X(\phi X(\ln f)) - (\nabla_X X)(\ln f)\\ &- (\nabla_{\phi X}\phi X)(\ln f) + 2g(\phi X,\phi X) + \eta(\nabla_X X)\eta(\mathrm{grad}(\ln f))\\ &+ 2\eta(X)\phi X(\ln f)\}g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y))\\ &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) + 2g(\phi X,\phi X) + 2\eta(X)\phi X(\ln f)\\ &+ \eta(\nabla_X X)\eta(\mathrm{grad}(\ln f))\}g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)).\end{aligned}$$

Thus, from (3.3), we conclude that

$$\begin{aligned}g(\bar{R}(X,\phi X)Y,\phi Y) &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X) + 2g(\phi X,\phi X) - 2\eta(X)\phi X(\ln f)\\ &+ 2\eta^2(X) + 2(\phi X(\ln f))^2 + \eta(\nabla_X X)\eta(\mathrm{grad}(\ln f))\}g(Y,Y)\\ &- 2\|h(X,Y)\|^2. \qquad (3.10)\end{aligned}$$

On the other hand, by using (1.12), we get

$$g(\bar{R}(X,\phi X)Y,\phi Y) = -\left(\frac{c-1}{2}\right)g(\phi X,\phi X)g(Y,Y). \qquad (3.11)$$

By corresponding (3.12) and (3.13), we reach at (3.4). □

Now, Let $\{e_o = \xi, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p, e^1, e^2, ..., e^q\}$ be orthonormal basis of $\Gamma(TM)$ such that $e_o, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p$, are tangent to $\Gamma(TM_T)$ and $e^1, e^2, ..., e^q$ are tangent to $\Gamma(TM_\perp)$. Moreover, we suppose that $\{\phi e^1, \phi e^2, ..., \phi e^q, N_1, N_2, ..., N_{2r}\}$ is an orthonormal basis of $\Gamma(TM^\perp)$ such that $\{\phi e^1, \phi e^2, ..., \phi e^q\}$ are tangent to $\Gamma(\phi TM_\perp)$ and $\{N_1, N_2, ..., N_{2r}\}$ are tangent to $\Gamma(\nu)$, where ν denote the orthogonal distribution of $\phi D^\perp$ in $T^\perp M$.

We can give the main theorem in the rest of this paper.

3.4. Theorem. *Let M be a compact orientable contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then M is a contact CR-product if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 \geq (\frac{c+3}{4})pq, \qquad (3.12)$$

where, h_2 denote the component of h in $\Gamma(\nu)$.

Proof. By using (2.3), the Laplacian of $\ln f$ is given by

$$\begin{aligned}-\Delta\ln f &= \sum_{i=1}^{p} g(\nabla_{e_i}\mathrm{grad}(\ln f), e_i) + \sum_{i=1}^{p} g(\nabla_{\phi e_i}\mathrm{grad}(\ln f), \phi e_i) + \sum_{j=1}^{q}(\nabla_{e^j}\mathrm{grad}(\ln f), e^j)\\ &+ g(\nabla_\xi\mathrm{grad}(\ln f), \xi).\end{aligned}$$

Considering ∇ being Levi-Civita connection, M_T is totally geodesic in M, $M_\perp$ is totally umbilical in M, $\mathrm{grad}(\ln f) \in \Gamma(TM_T)$ and Proposition 2.1, we have

$$\begin{aligned}
-\Delta \ln f &= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&+ \sum_{j=1}^{q}\{e^j g(\mathrm{grad}(\ln f), e^j) - g(\nabla_{e^j} e^j, \mathrm{grad}(\ln f))\} + g(\nabla_\xi \mathrm{grad}(\ln f), \xi) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&- \sum_{j=1}^{q}\{-g(e^j, e^j) g(\mathrm{grad}(\ln f), \mathrm{grad}(\ln f))\} + g(\phi \mathrm{grad}(\ln f), \xi) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} + q\|\mathrm{grad}(\ln f)\|^2.
\end{aligned}$$

Let $X = e_i$ and $Y = e^j$ be in (3.4), $1 \le i \le p$ and $1 \le j \le q$. By direct calculations, we have

$$\begin{aligned}
2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i, e^j)\|^2 &= \{\sum_{i=1}^{p}\{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
&+ 2\sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\frac{c+3}{2})p\}q \\
&= \{-\Delta \ln f - q\|\mathrm{grad}(\ln f)\|^2 + 2\sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\frac{c+3}{2})p\}q.
\end{aligned}$$

Thus we get

$$-\Delta \ln f = \frac{2}{q}\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i, e^j)\|^2 + q\|\mathrm{grad}(\ln f)\|^2 - 2\sum_{i=1}^{p}(\phi e_i(\ln f))^2 - (\frac{c+3}{2})p. \tag{3.13}$$

Furthermore, from linear algebra rules, we know that h can be written as

$$h(e_i, e^j) = \sum_{k=1}^{p} g(h(e_i, e^j), \phi e^k)\phi e^k + \sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)N_\ell.$$

Also, making use of (3.1), we have

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{q} g(h(e_i, e^j), h(e_i, e^j)) &= \sum_{i=1}^{p}\sum_{k,j=1}^{q} g(h(e_i, e^j), \phi e^k)^2 \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)^2 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\{g(e^j, e^j)(\eta(e_i) - \phi e_i(\ln f))^2\} + \sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 \\
&= q\sum_{i=1}^{p}(\phi e_i(\ln f))^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2. \qquad (3.14)
\end{aligned}$$

Finally, substituting (3.14) into (3.13), we get

$$-q\Delta \ln f = 2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 - (\frac{c+3}{2})pq + q^2\|\text{grad}(\ln f)\|^2.$$

From (2.4) we conclude that

$$\int_M \{\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 - (\frac{c+3}{4})pq + \frac{q^2}{2}\|\text{grad}(\ln f)\|^2\}dV = 0, \tag{3.15}$$

that is,

$$\int_M \{\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 + \frac{q^2}{2}\|\text{grad}(\ln f)\|^2\}dV = \mathtt{Vol}(M)(\frac{c+3}{4})pq.$$

Here, if

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 \geq (\frac{c+3}{4})pq,$$

it implies $\|\text{grad}(\ln f)\| = 0$ because $q.p \neq 0$, that is, the warping function f is constant. So contact CR-warped product becomes a contact CR-product. □

As a consequence, $c \leq -3$ is necessary so that $M(c)$ is the standard sphere S^{2n+1}. The natural isometric embedding

$$\mathbb{C}^{k+1} \times \mathbb{R}^{\ell+1} \longrightarrow \mathbb{C}^{n+1}$$

defines an isometric embedding.

$$i : M(k,\ell) = \mathbb{S}^{2k+1} \times \mathbb{S}^{\ell} \longrightarrow \mathbb{S}^{2n+1}$$

in which $\mathbb{S}^{2k+1}$ is the standard sphere with constant ϕ-sectional curvature c and the ℓ-dimensional sphere $\mathbb{S}^{\ell}$ with constant sectional curvature $(\frac{c+3}{4})$. i maps the tangent bundle $T\mathbb{S}^{\ell}$ into $T^{\perp}M(k,\ell) \subset T\mathbb{S}^{2n+1}$. Hence $M(k,\ell)$ is a $(2k+\ell+1)$-dimensional contact CR-warped product submanifold by the definition.

From the integral formula (3.15) we derive the following corollaries.

3.5. Corollary. *Let M be a compact orientable contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i,e^j)\|^2 = (\frac{c+3}{4})pq. \tag{3.16}$$

Proof. If (3.16) is satisfied, then (3.15) implies that $f = constant$, that is, M is a contact CR-product.

Conversely, M is a contact CR-product , from (3.2) we know that $h(X,Y) \in \Gamma(\nu)$, for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_{\perp})$. So the equality (3.16) is satisfied □

3.6. Corollary. *There exist no compact orientable contact CR products in a Sasakian space form $\bar{M}(c)$ such that $c < -3$.*

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