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On moduli of smoothness and approximation by trigonometric polynomials in weighted Lorentz spaces

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Abstract

We investigate the approximation properties of the functions by trigonometric polynomials in weighted Lorentz spaces with weights satisfying so called Muckenhoupt's A_p condition. Relations between moduli of smoothness of the derivatives of the functions and those of the functions itself are studied. In weighted Lorentz spaces we also prove a theorem on the relationship between the derivatives of a polynomial of best approximation and the best approximation of the function. Moreover, we study relationship between modulus of smoothness of the function and its de la Vallée-Poussin sums in these spaces.

Keywords: moduli of smoothness, weighted Lorentz spaces, Muckenhoupt weight, trigonometric approximation, best approximation.

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1. Introduction and the main results

Let $\mathbb{T} = [-\pi, \pi]$. A function $\omega : \mathbb{T} \to [0, \infty]$ will be called a *weight function* if ω is locally integrable and almost everywhere (a.e.) positive. The function ω generates the Borel measure

$$\omega(E) = \int_E \omega(x)dx.$$

By

$$f_\omega^*(t) = \inf\{\nu \geq 0 : \ \omega(\{x \in \mathbb{T} : |f(x)| > \nu\}) \leq t\}$$

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we denote the nondecreasing rearrangement of a function $f:\mathbb{T}\rightarrow[0,\infty]$. We denote also

$$f^{**}(t):=\frac{1}{t}\int\limits_{0}^{t}f_{\omega}^{*}(u)du.$$

Let $0<p<\infty$, $0<q<\infty$. A measurable and a.e. finite function f on $\mathbb{T}$ belongs to the Lorentz space $L_{\omega}^{pq}(\mathbb{T})$ if

$$\|f\|_{L_{\omega}^{pq}}:=\left(\int\limits_{\mathbb{T}}(t^{\frac{1}{p}}f^{**}(t))^{q}\frac{dt}{t}\right)^{1/q}<\infty.$$

Note that Lorentz spaces, introduced by G. Lorentz in the 1950 s. [24], [25]. As seen the weighted Lorentz spaces $L_{\omega}^{pq}(\mathbb{T})$ is expressed not only in terms of the parameter p, but also in terms of the second parameter q. If $p=q$, then $L_{\omega}^{pq}(\mathbb{T})$ is the weighted Lebesgue space $L_{\omega}^{p}(\mathbb{T})$ [10, p. 20]. If $q<r$, then the space $L_{\omega}^{pq}(\mathbb{T})$ is contained in $L_{\omega}^{pr}(\mathbb{T})$. Detailed information about properties of the Lorentz spaces can be found in [12], [20], [26] and [31].

Let $1<p<\infty$, $p'=\frac{p}{p-1}$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ if

$$\sup_{J}\left(\frac{1}{|J|}\int\limits_{J}\omega\left(t\right)dt\right)\left(\frac{1}{|J|}\int\limits_{J}\omega^{1-p'}\left(t\right)dt\right)^{p-1}<\infty\ ,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length. Note that the weight functions belonging to the A_p- class, introduced by Muckenhoupt [27], play a very important role in different fields of mathematical analysis.

We use $c,\ c_1,\ c_2,\ ...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest. We shall also employ the symbol $A\asymp B$, denoting that $cA\leq B\leq C$, where $c,\ C$ are constants.

Let $\alpha\in\mathbb{Z}^{+}$ and $f\in L^{1}(\mathbb{T})$. Suppose that $x,\ h$ are real, and let us take into

$$\Delta_{t}^{\alpha}f(x):=\sum_{j=0}^{\alpha}(-1)^{j}\binom{\alpha}{j}\ f\ (x+(\alpha-j)t),$$

where $\binom{\alpha}{j}:=\frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-j+1)}{j!}$, $j>1$ is the Binomial coefficients and $\binom{\alpha}{0}:=1,\binom{\alpha}{1}:=\alpha$.

Let $1<p,q<\infty$, $\omega\in A_{p}(\mathbb{T})$, $f\in L_{\omega}^{pq}(\mathbb{T})$. We put

$$\sigma_{\delta}^{\alpha}f(x):=\frac{1}{\delta}\int\limits_{0}^{\delta}\left|\Delta_{t}^{\alpha}f(x)\right|dt.$$

If $f\in L_{\omega}^{pq}(\mathbb{T})$, $\omega\in A_{p}(\mathbb{T})$ according to [6] the Hardy-Littlewood Maximal function is bounded in $L_{\omega}^{pq}(\mathbb{T})$, $\omega\in A_{p}(\mathbb{T})$. Then we have

$$\|\sigma_{\delta}^{\alpha}f\|_{L_{\omega}^{pq}}\leq c_{1}\|f\|_{L_{\omega}^{pq}.}<\infty.$$

For $1<p,q<\infty$, $\omega\in A_{p}(\mathbb{T})$, $f\in L_{\omega}^{pq}(\mathbb{T})$, $\alpha\in\mathbb{Z}^{+}$ we define the *$\alpha-th$ mean modulus of smoothness* $\omega_{\alpha}(f,\ .)_{L_{\omega}^{pq}}$ by

$$\omega_{\alpha}(f,\ h)_{L_{\omega}^{pq}}:=\sup_{|\delta|\leq h}\|\sigma_{\delta}^{\alpha}f(x\|_{L_{\omega}^{pq}}$$

Let $f \in L^{pq}_{\omega}(\mathbb{T})$, $\alpha \in \mathbb{Z}^{+}$ the modulus of smoothness $\omega_{\alpha}(f,\ .)_{L^{pq}_{\omega}}$ is a nondecreasing, nonnegative, function and

$$\begin{aligned} \omega^{p}_{\alpha}(f_1+f_2,\ .)_{\ L^{pq}_{\omega}} &\leq \omega^{p}_{\alpha}(f_1,\ .)_{\ L^{pq}_{\omega}} + \omega^{p}_{\alpha}(f_2,\ .)_{\ L^{pq}_{\omega}}, \\ \lim_{\delta\to 0^{+}} \omega_{\alpha}(f,\ \delta)_{L^{pq}_{\omega}} &= 0. \end{aligned}$$

For $f\ \in L^{pq}_{\omega}(\mathbb{T})$, we define the $\alpha-th$ derivative of f as function $g \in L^{pq}_{\omega}(\mathbb{T})$ satisfying

$$\lim_{h\to 0^{+}} \left\| \frac{\Delta^{\alpha}_{h}(f)}{h^{\alpha}} - g \right\|_{L^{pq}_{\omega}} = 0, \tag{1.1}$$

in which case we write $g = f^{\ (\alpha)}$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(f,x),\ \ A_k(f,x) := a_k(f)\cos kx + b_k(f)\sin kx \tag{1.2}$$

be the Fourier series of thefunction $L^{1}(\mathbb{T})$. The *nth partial sums*, and *de la Vallée-Poussin sum* of the series (1.2) are defined, respectively, as

$$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(f,x),$$

$$V_n(f) = \frac{1}{n} \sum_{\nu=1}^{2n-1} S_{\nu}(f).$$

We denote by $E_n(f)_{L^{pq}_{\omega}}$ $(n = 0,1,2,...)$ the best approximation of $f \in L^{pq}_{\omega}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i. e.,

$$E_n(f)_{L^{pq}_{\omega}} := \inf\left\{ \|f - T_n\|_{L^{pq}_{\omega}} : T_n \in \Pi_n \right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $W^{\alpha}_{pq,\omega}(\mathbb{T})$ $(r = 1,2,...)$ be the linear space of functions $f \in L^{pq}_{\omega}(\mathbb{T})$, $1 < p,q < \infty$, $\omega \in A_p(\mathbb{T})$, such that $f^{\ (\alpha)} \in L^{pq}_{\omega}(\mathbb{T})$. It becomes a Banach space with the norm

$$\|f\|_{W^{\alpha}_{pq,\omega}(\mathbb{T})} := \|f\|_{L^{pq}_{\omega}} + \left\| f^{(\alpha)} \right\|_{L^{pq}_{\omega}}.$$

The problems of approximation theory in the weighted and nonweighted Lorentz space have been investigated in [1], [21], [35] and [37]. The approximation problems by trigonometric polynomials in different spaces have been investigated by several authors (see, for example, [2-5], [7], [9], [11], [13-19], [22], [23], [28-30], [33] and [34]).

In this work we study the approximation problems of functions by trigonometric polynomials in the weighted Lorentz space $L^{pq}_{\omega}(\mathbb{T})$ with Muckenhoupt weights. Relations between moduli of smoothness of the derivatives of a function and those of the function itself are investigated. We also prove a theorem on the relationship between derivatives of a polynomial of best approximation and the best approximation of the function in the weighted Lorentz space $L^{pq}_{\omega}(\mathbb{T})$. In addition, in the weighted Lorentz space $L^{pq}_{\omega}(\mathbb{T})$ relationship between modulus of smoothness of the function and its de la Vallée-Poussin sums is studied. Similar problems in defferent spaces were investigated in [9], [30], [32].

Our main results are the following.

Theorem 1.1. *Let* $1 < p,\ q < \infty$, $\omega \in A_p(\mathbb{T})$, $f \in L^{pq}_{\omega}(\mathbb{T})$ *and* T_n *a trigonometric polynomial of degree* n *satisfying the following conditions:*

$$\|f - T_n\|_{L^{pq}_{\omega}} = o\left(\frac{1}{n}\right) \ \ \textit{and}\ \ \left\|g - T_n'\right\|_{L^{pq}_{\omega}} = o(1), \quad n \to \infty.$$

Then we obtain $f' = g$, that is, the function g satisfies the condition (1.1).

Using the same method as in the proof of Theorem 1.1 we have the following Corollary.

Corollary1.1. *Let $1 < p,\ q < \infty$, $\omega \in A_p(\mathbb{T})$, $f, g_1, ..., g_k \in L^{pq}_{\omega}(\mathbb{T})$ and T_n be a trigonometric polynomial satisfying, for $i = 1, ..., k$, the conditions*

$$\|f - T_n\|_{L^{pq}_{\omega}} = \mathrm{o}\left(\frac{1}{n^k}\right), \quad n \to \infty,$$
$$\left\|g_i - T_n^{(i)}\right\|_{L^{pq}_{\omega}} = \mathrm{o}\left(\frac{1}{n^{k-i}}\right), \quad n \to \infty.$$

Then we obtain $g_i = g'_{i-1}$ $(f = g_0)$ in the sense of (1.1).

Theorem 1.2. *Let $1 < p < \infty$ and $1 < q \leq 2$ or $p > 2$ and $q \geq 2$. Then, for a given $\omega \in A_p(T)$, $f \in L^{pq}_{\omega}(T)$ and integers α, r satisfying $\alpha > r$ we have*

$$\omega_{\alpha-r}\left(f^{(r)}, t\right)_{L^{pq}_{\omega}} \leq c_2 \left\{ \int_0^t \frac{\omega_\alpha\left(f,\ u\right)^s_{L^{pq}_{\omega}}}{u^{sr+1}} du \right\}^{1/s},$$

where $s = \min\ (q, 2)$.

Theorem 1.3. *Let $1 < p,\ q < \infty$, $\omega \in A_p(T)$, $f \in L^{pq}_{\omega}(T)$, $\alpha, r \in \mathbb{Z}^+$ $(\alpha > r > 0)$ and let $T_n(f) \in \Pi_n$ be the polynomial of best approximation to f in $L^{pq}_{\omega}(T)$. In order that*

$$\left\|T_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}} = O(n^{\alpha-r})$$

it is necessary and sufficient that

$$E_n(f)_{L^{pq}_{\omega}} = O(n^{-r}).$$

Theorem 1.4. *Let $1 < p,\ q < \infty$, $\omega \in A_p(T)$, $\alpha \in \mathbb{Z}^+$. If $f \in L^{pq}_{\omega}$, then*

1.

$$c_3\omega_\alpha(f, \frac{1}{n})_{L^{pq}_{\omega}} \leq \left(n^{-\alpha}\left\|V_n^{(\alpha)}(\ f)\right\|_{L^{pq}_{\omega}} + \|f(x) - V_n(\ f)\|_{L^{pq}_{\omega}}\right)$$
$$\leq c_4\omega_\alpha(f, \frac{1}{n})_{L^{pq}_{\omega}} \tag{1.3}$$

where the constants c_4 and c_5 are dependent on α, p and q.

2.

$$c_5\omega_\alpha(f, \frac{1}{n})_{L^{pq}_{\omega}} \leq \left(n^{-\alpha}\left\|S_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}} + \|f(x) - S_n(\ f)\|_{L^{pq}_{\omega}}\right)$$
$$\leq c_6\omega_\alpha(f, \frac{1}{n})_{L^{pq}_{\omega}}, \tag{1.4}$$

where the constants c_6 and c_7 are dependent on α, p and q.

2. Proofs of main results

We need the following results obtained in [35].

Lemma 2.1. *Let $\omega \in A_p(T)$, $1 < p, q < \infty$. If $f \in L^{pq}_{\omega}(T)$ and $\alpha = 1, 2, ...$, then there exists a constant $c_7 > 0$ depending α, p and q such that*

$$E_n(f)_{L^{pq}_{\omega}} \leq c_7\omega_\alpha(f, \frac{1}{n})_{L^{pq}_{\omega}}.$$

holds where $n = 0, 1, 2, ...$

Lemma 2.2. *Let $\omega \in A_p(T)$ and $\alpha \in Z^+$, $1 < p, q < \infty$. If $T_n \in \Pi_n$, $n \geq 1$, then there exists a constant $c_8 > 0$ depending only on α, p and q such that*

$$\omega_\alpha(T_n,\ h)_{L^{pq}_\omega} \leq c_8 h^\alpha \left\| T_n^{(\alpha)} \right\|_{L^{pq}_\omega},\ 0 < h \leq \pi$$

Lemma 2.3. *Let $\omega \in A_p(T)$, $1 < p, q < \infty$. If $T_n \in \Pi_n$, $n \geq 1$ and $\alpha \in Z^+$, then there exists a constant $c_9 > 0$ depending only on α, p and q such that*

$$\left\| T_n^{(\alpha)} \right\|_{L^{pq}_\omega} \leq c_9 n^\alpha \left\| T_n \right\|_{L^{pq}_\omega}.$$

Proof of Theorem 1.1. We take $\varepsilon > 0$. We choose a natural number $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$

$$\left\| f - T_n \right\|_{L^{pq}_\omega} \leq \varepsilon \frac{1}{n}, \qquad \left\| g - T_n' \right\|_{L^{pq}_\omega} \leq \varepsilon. \tag{2.1}$$

Taking account of (2.1) for h satisfying the condition $\frac{\sqrt{\varepsilon}}{n} \leq h \leq \frac{1}{n}$ we obtain

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{pq}_\omega} \leq 2^{\frac{p}{2}} \tag{2.2}$$

Considering [8] we have

$$\Delta_h^r T_n(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i T_n \left(x + \left(\frac{r}{2} - i \right) h \right) =$$

$$= \sum_{j=r}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left(\frac{r}{2} - i \right)^j \frac{h^j}{j!} T_n^{(j)}(x) =$$

$$= h^r T_n^{(r)}(x) + \sum_{j=r+1}^{\infty} \eta(r,j)^{j-r} T_n^{(j)}(x), \tag{2.3}$$

where $-\frac{r}{2} < \eta(r,j) < \frac{r}{2}$ and $\eta(r,j) = 0$ if $j - r$ is odd. Then using (2.3) and Lemma 2.3 for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we find that

$$\left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{pq}_\omega} \leq \sum_{m=2}^{\infty} \left(\frac{h^{m-1}}{m!} \right)^p \left\| T_n^{(m)} \right\|^p_{L^{pq}_\omega} \leq$$

$$\leq \sum_{m=2}^{\infty} (hn)^{(m-1)p} \left\| T_n \right\|^p_{L^{pq}_\omega} \leq 4 \frac{\varepsilon}{1 - 2^p \varepsilon^{p/2}} \left\| T_n \right\|^p_{L^{pq}_\omega} \leq c_{12} \varepsilon^p \left\| T_n \right\|^p_{L^{pq}_\omega}. \tag{2.4}$$

Using (2.2), (2.4) and (2.1) for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we reach

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - g \right\|^p_{L^{pq}_\omega} \leq \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T_n(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{pq}_\omega} +$$

$$+ \left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{pq}_\omega} +$$

$$+ \left\| T_n' - g \right\|^p_{L^{pq}_\omega} \leq c_{10} \left(\varepsilon^{p/2} + \varepsilon^p \left\| f \right\|^p_{L^{pq}_\omega} + \varepsilon^p \right)$$

From the last inequality we have $g = f'$ in the sense of (1.2). Then the proof of Theorem 1.1 is completed.

Proof of Theorem 1. 2. The function $\omega_m(F,t)_{L^{pq}_\omega}$ non-decreasing and according to reference [34] the following inequality holds:

$$\omega_\alpha(F, 2t)_{L^{pq}_\omega} \leq c_{11} \omega_\alpha(F,\ t)_{L^{pq}_\omega} \tag{2.5}$$

It is sufficient to prove theorem for $t=2^{-n}$. Then using of (2.5) we obtain

$$\left\{\int_0^{2^{-n}} \frac{\omega_\alpha(f,\ u)^s_{L^{pq}_\omega}}{u^{sr+1}}du\right\}^{1/s} \asymp \left\{\sum_{\nu=n}^{\infty} 2^{\nu sr}\omega_\alpha(f,\ 2^{-\nu})^s_{L^{pq}_\omega}\right\}^{1/s}.$$

Therefore for all n it is sufficient to prove the following inequality:

$$\omega_{\alpha-r}(f^{(r)}\ ,\ 2^{-n})_{L^{pq}_\omega} \le \left\{\sum_{\nu=n}^{\infty} 2^{\nu sr}\omega_\alpha(f,\ 2^{-\nu})^s_{L^{pq}_\omega}\right\}^{1/s}. \tag{2.6}$$

By [34] for any trigonometric polynomial Q_n of degree cn and $F\in L^{pq}_\omega(\mathbb{T})$ we obtain

$$\omega_\alpha(F\ ,\ 1/n)_{L^{pq}_\omega} \le c_{12}\left(\|F-Q_n\|_{L^{pq}_\omega} + n^{-\alpha}\left\|Q_n^{(\alpha)}\right\|_{L^{pq}_\omega}\right). \tag{2.7}$$

Therefore we aim to find Q_{2^n} of degree $c2^n$ such that both $\left\|f^{(r)}-Q_{2^n}\right\|_{L^{pq}_\omega}$ and $2^{-n(\alpha-r)}\left\|Q_{2^n}^{(\alpha-r)}\right\|_{\|f\|_{L^{pq}_\omega}}$ are bounded by the right-hand side of inequality (2.6). Let $T_n\in\Pi_n$ $(n=0,1,2,...)$ be the polynomial of best approximation to f. It is known that [34] the set of trigonometric polynomials is dense in $L^{pq}_\omega(\mathbb{T})$. Then we have $\|f-T_{2^\nu}\|_{L^{pq}_\omega}\to 0$ as $\nu\to\infty$.

Let $f\in L^{pq}_\omega(\mathbb{T})$ has the Fourier series

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=0}^{\infty}A_k(f).$$

We define trigonometric polynomial $\nu_N f$ as

$$\nu_N f \ =\sum_{k=0}^{\infty}\nu(\frac{k}{N})A_k(f),$$

where $\nu\in C^\infty\,[0,\infty)$, $\nu(x)=1$ for $x\le 1$ and $\nu(x)=0$ for $x\ge 1$. Note that trigonometric polynomial $\nu_N f$ has the following properties:

I) $\nu_N f$ is a trigonometric polynomial of degree smaller than N;

II) If g is a trigonometric polynomial of degree $[N/2]$, then $\nu_N g=g$;

III) $\|\nu_N f\|_{L^{pq}_\omega}\le c\|f\|_{L^{pq}_\omega}$.

According to reference [34] we have

$$\|\nu_N f\ -f\|_{L^{pq}_\omega}\le c_{13}E_{N/2}(f)_{L^{pq}_\omega},$$

where $E_k(f)_{L^{pq}_\omega}$ is the best approximation of $f\in L^{pq}_\omega(\mathbb{T})$ trigonometric polynomials of degree not exceeding k. We now choose the Q_n of (2.7) for $F=f^{(r)}$ to be $(\nu_n f\ )^{(r)}$. It is cleary that $\|f-\nu_n f\|_{L^{pq}_\omega}=o(1)$ as $n\to\infty$.

The following identity holds:

$$\nu_{2^k}f\ -\nu_{2^n}f\ =\sum_{m=n}^{k-1}(\nu_{2^{m+1}}f\ -\nu_{2^m}f\ )\equiv\sum_{m=n}^{k-1}\gamma_m f.$$

Then

$$(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}=\sum_{m=1}^{k-1}(\gamma_m f)^{(r)}.$$

Using the Littlewood- Paley inequality for the weighted Lorentz spaces $L^{pq}_{\omega}(\mathbb{T})$ in [21] we have

$$\begin{aligned} & c_{14}\left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L^{pq}_{\omega}} \\ \leq & \left\|\left(\sum_{m=n}^{k-1}\left\{(\gamma_m f)^{(r)}\right\}^2\right)^{1/2}\right\|_{L^{pq}_{\omega}} \\ \leq & c_{15}\left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L^{pq}_{\omega}}. \end{aligned} \tag{2.8}$$

According to [21, Lemma 4.2 and 4.3] we get

$$\left\|\left(\sum_{m=n}^{k-1}\left\{(\gamma_m f)^{(r)}\right\}^2\right)^{1/2}\right\|_{L^{pq}_{\omega}} \leq \left(\sum_{m=n}^{k-1}\left\|(\gamma_m f)^{(r)}\right\|^s_{L^{pq}_{\omega}}\right)^{1/s}, \tag{2.9}$$

where $s=\min(q,2)$.

Note that $\nu_n f$ is the near best approximation to f in L^{pq}_{ω}. Then using [35] we reach the following equivalence

$$\omega_{\alpha}(f,1/n)\asymp\|f-\nu_n f\|_{L^{pq}_{\omega}}+n^{-\alpha}\left\|(\nu_n f)^{(\alpha)}\right\|_{L^{pq}_{\omega}}. \tag{2.10}$$

From (2.8)- (2.10) and Lemma 2.3 we conclude that

$$\begin{aligned} & \left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L^{pq}_{\omega}} \\ \leq & c_{16}\left(\sum_{m=n}^{k-1}2^{mrs}\|(\gamma_m f)\|^s_{L^{pq}_{\omega}}\right)^{1/s} \\ \leq & c_{17}\left(\sum_{m=n}^{k-1}2^{mrs}\omega_{\alpha}(f,2^{-m})^s_{L^{pq}_{\omega}}\right)^{1/s}, \end{aligned}$$

where c_1 independent of m,k and f.

Use of $Q_{2^n}=\nu_{2^n}f$ and (2.10) gives us

$$\begin{aligned} 2^{-n(\alpha-r)}\left\|((\nu_{2^n}f)^{(r)})^{(\alpha-r)}\right\|_{L^{pq}_{\omega}} &= 2^{-n(\alpha-r)}\left\|(\nu_{2^n}f)^{(\alpha)}\right\|_{L^{pq}_{\omega}} \\ &\leq 2^{nr}\omega_{\alpha}(f,2^{-n})_{L^{pq}_{\omega}}\leq c_{18}\left(\sum_{m=n}^{\infty}2^{mrs}\omega_{\alpha}(f,2^{-m})^s_{L^{pq}_{\omega}}\right)^{1/s}. \end{aligned}$$

The proof of Theorem 1.2 is completed.

Proof of Theorem 1. 3. We suppose that

$$E_n(f)_{L^{pq}_{\omega}}=\|f-T_n(f)\|_{L^{pq}_{\omega}}=O(n^{-r}),\qquad (r>0). \tag{2.11}$$

Taking into account Lemma 2.3 and the relations (2.11) we obtain

$$\left\|T_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}}\leq c_{19}n^{\alpha}\|T_n(f)\|_{L^{pq}_{\omega}}\leq n^{\alpha}\|f-T_n(f)\|_{L^{pq}_{\omega}}+\|T_n(f)\|_{L^{pq}_{\omega}}\leq c_{20}n^{\alpha-r}.$$

Now we suppose that

$$\left\|T_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}}=O(n^{\alpha-r}). \tag{2.12}$$

Using Lemma 2.1, Lemma 2.2 and (2.2) we get

$$\begin{aligned} \|T_{2n}(f)-T_n(T_{2n}(f))\|_{L^{pq}_\omega} &\le E_n(T_{2n}(f))_{L^{pq}_\omega} \le c_{21}\omega_\alpha(T_{2n},\frac{1}{n})_{L^{pq}_\omega}. \\ &\le c_{22}n^{-\alpha}\left\|T^{(\alpha)}_{2n}\right\| \le c_{23}n^{-\alpha}(n^{\alpha-r}) \le c_{24}n^{-r}. \end{aligned} \tag{2.13}$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\begin{aligned} \|T_{2n}(f)-T_n(T_{2n}(f))\|_{L^{pq}_\omega} &= \|f-T_n(T_{2n}(f))-(f-T_{2n}(f))\|_{L^{pq}_\omega} \\ &\ge \|f-T_n(T_{2n}(f))\|_{L^{pq}_\omega} - \|f-T_{2n}(f))\|_{L^{pq}_\omega} \\ &\ge E_n(f)_{L^{pq}_\omega} - E_{2n}(f)_{L^{pq}_\omega} \ge 0. \end{aligned} \tag{2.14}$$

Use of (2.13) and (2.14) gives us

$$0 \le E_n(f)_{L^{pq}_\omega} - E_{2n}(f)_{L^{pq}_\omega} \le c_{25}n^{-r}. \tag{2.15}$$

Since $E_n(f)_{L^{pq}_\omega} \to 0$ from the inequality (2.15) we conclude that

$$\sum_{k=n_0}^{\infty}\left\{E_{2^k}(f)_{L^{pq}_\omega} - E_{2^{k+1}}(f)_{L^{pq}_\omega}\right\} \le c_{26}\sum_{k=n_0}^{\infty}2^{-kr}.$$

Then from the last inequality we obtain

$$E_{2^{n_0}}(f)_{L^{pq}_\omega} \le c_{27}2^{-n_0 r}. \tag{2.16}$$

It is clear that inequality (2.16) is equivalent to $E_n(f)_{L^{pq}_\omega} \le c_{28}(n^{-r})$. This completes the proof.

Proof of Theorem 1. 4. In view of Lemma 2.2 the inequality

$$\omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega} \le c_{29}n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega}, \tag{2.17}$$

holds, where T_n is a trigonometric polynomial of order n. Using the properties of smoothness $\omega_\alpha(f,.)_{L^{pq}_\omega}$ and (2.17), we reach

$$\begin{aligned} \omega_\alpha(f,\frac{1}{n})_{L^{pq}_\omega} &\le \left(\omega_\alpha(f-T_n,\frac{1}{n})_{L^{pq}_\omega} + \omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega}\right) \\ &\le c_{30}\left(\|f-T_n\|_{L^{pq}_\omega} + n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega}\right). \end{aligned} \tag{2.18}$$

Considering [34] there exists a constant $c > 0$ depending only on α, p and q such that

$$n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega} \le c_{31}\omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega}. \tag{2.19}$$

By virtue of Lemma 2.1

$$E_n(f)_{L^{pq}_\omega} \le c_{32}\omega_\alpha(f,\frac{1}{n})_{L^{pq}_\omega}. \tag{2.20}$$

It is known that [34] for the de la Vallée-Poussin mean the inequality

$$\|f-V_n(f)\|_{L^{pq}_\omega} \le c_{33}E_n(f)_{L^{pq}_\omega}. \tag{2.21}$$

holds. Use of (2.19)-(2. 21) gives us

$$\begin{aligned} & n^{-\alpha} \left\| V_n^{(\alpha)}(f) \right\|_{L_\omega^{pq}} + \|f - V_n(f)\|_{L_\omega^{pq}} \\ \leq \; & c_{34} \left(\omega_\alpha(V_n, \frac{1}{n})_{L_\omega^{pq}} + E_n(f)_{L_\omega^{pq}} \right) \\ \leq \; & c_{35} \left(\omega_\alpha(f, \frac{1}{n})_{L_\omega^{pq}} + \omega_\alpha(f - V_n, \frac{1}{n})_{L_\omega^{pq}} + E_n(f)_{L_\omega^{pq}} \right) \\ \leq \; & c_{36} \omega_\alpha(f, \frac{1}{n})_{L_\omega^{pq}}. \end{aligned}$$

The last inequality and (2.18) imply that (1.3).

According to [35] there exists a constant c_{25} such that

$$\|f - S_n(f)\|_{L_\omega^{pq}} \leq c_{37} E_n(f)_{L_\omega^{pq}}. \tag{2.22}$$

If the inequality (2.22) and the scheme of proof of the estimation (1.3) is used we obtain the estimation (1.4).

Theorem 1.4 is proved.

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