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Quantale algebras as a generalization of lattice-valued frames

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Abstract

Recently, I. Stubbe constructed an isomorphism between the categories of right Q-modules and cocomplete skeletal Q-categories for a given unital quantale Q. Employing his results, we obtain an isomorphism between the categories of Q-algebras and Q-quantales, where Q is additionally assumed to be commutative. As a consequence, we provide a common framework for two concepts of lattice-valued frame, which are currently available in the literature. Moreover, we obtain a convenient setting for lattice-valued extensions of the famous equivalence between the categories of sober topological spaces and spatial locales, as well as for answering the question on its relationships to the notion of stratification of lattice-valued topological spaces.

Keywords: (Cocomplete) (skeletal) Q-category, lattice-valued frame, lattice-valued partially ordered set, quantale, quantale algebra, quantale module, sober topological space, spatial locale, stratification degree, stratified topological space.

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1. Introduction

1.1. Lattice-valued frames. The well-known equivalence between the categories of sober topological spaces and spatial locales, initiated by D. Papert and S. Papert [42], and developed in a rigid way by J. R. Isbell [31] and P. T. Johnstone [32], opened an important relationship between general topology and universal algebra. In particular, it provided a convenient framework for the famous topological representation theorems of M. Stone for Boolean algebras [71] and distributive lattices [72], which in their turn (backed by the celebrated representation of distributive lattices of H. Priestley [45] and the plethora of its induced results) started the theory of natural dualities [8], presenting a general machinery (based in some elements of category theory, but for the most part in universal algebra) for obtaining topological representations of algebraic structures. The success of the evolving theory is mostly due to the fact that it translates algebraic problems, usually stated in an abstract symbolic language, into dual, topological problems, where geometric intuition comes to our help.

No wonder then that the beginning of the fuzzy era of L. A. Zadeh [82] and J. A. Goguen [17], together with the almost immediate fuzzification of the concept of topological space by C. L. Chang [7], R. Lowen [38] and S. E. Rodabaugh [51], turned the attention of the newly appearing fuzzy researchers to the fuzzification of the above-mentioned sobriety-spatiality equivalence. One of the first and the most successful attempt was made by S. E. Rodabaugh [52], who presented its both fixed- and variable-basis extensions, bringing the theory to its completion in [54, 56], thereby streamlining the initial machinery of P. T. Johnstone.

It soon appeared, however, that to develop properly lattice-valued pointless topology, one needs the corresponding lattice-valued generalization of locales, which should not be a direct fuzzification of the corresponding algebraic structure in the sense of fuzzy groups of A. Rosenfeld [59] and J. M. Anthony and H. Sherwood [4], or, more generally, lattice-valued algebras of A. Di Nola and G. Gerla [10], but should be capable of restoring the point-theoretic structure from a given extended locale of a lattice-valued topological space. One of the pioneering endeavors in this respect is due to D. Zhang and Y.-M. Liu [84], who introduced the concept of L-fuzzy locale as a frame homomorphism $L \xrightarrow{i_A} A$ and provided a lattice-valued sobriety-spatiality equivalence for the respective category of these structures (the comma category $(\mathbf{Loc} \downarrow L)$). A similar viewpoint was taken by W. Yao [79], who introduced L-frames through the notion of L-partially ordered set of L. Fan [12]. Moreover, in [78, 80], he developed the theory of lattice-valued domains, based in his newly established framework of L-order. Later on, W. Yao [81] constructed an isomorphism between his category of L-frames and the category of L-fuzzy frames of D. Zhang and Y.-M. Liu. On the other hand, there exists another and more sophisticated notion of lattice-valued frame, introduced by A. Pultr and S. E. Rodabaugh [48] and induced by the Lowen-Kubiák ι_L (fibre map) functor [37, 38], the latter providing a way of obtaining a crisp topological space from a lattice-valued one (it is important to notice that there exists another approach to the just mentioned fuzzy-crisp topological space passage, suggested by the notion of attachment of C. Guido [19] (see also [13, 14, 15, 20]), which extends the hypergraph functor of the fuzzy community [26]; whether the notion of attachment has its corresponding concept of lattice-valued frame is still an open and challenging question). The theory was given its maturity in [49, 50], which presented a new presheaf motivation for the concept as well as studied categorical properties of lattice-valued frames and deepened their relationships to lattice-valued topology.

1.2. Lattice-valued quantales. Motivated by the above-mentioned fuzzifications of the sobriety-spatiality equivalence, we extended the obtained theory in several ways [63, 65, 66, 67], thereby initiating categorically-algebraic topology [62], introduced as a common framework for the majority of modern approaches to lattice-valued topology, in order to provide convenient means of interaction between different theories. In particular, in [67] (see also [68]), we considered the notion of algebra over a given unital commutative quantale as a generalization of the concept of quantale module [36, 43, 61], whose theory has already been established as an important part of universal algebra, extending the classical theory of modules over a ring [3]. After a brief consideration, it became clear to us that the above-mentioned result of W. Yao on categorical equivalence between two concepts of lattice-valued frame is a direct consequence of a more general correspondence between quantale modules and lattice-valued $\bigvee$-semilattices, established recently by I. Stubbe [76], which, in its turn, extends the well-known isomorphism between the categories of $\mathbf{2}$-modules and $\bigvee$-semilattices [61] (cf. the similar result for $\mathbb{Z}$-modules and abelian groups [3]). More precisely, having the just mentioned correspondence in hand, one easily obtains an isomorphism between the categories of quantale algebras and lattice-valued quantales, a particular instance of the latter providing the category of L-frames of W. Yao. Moreover, an analogue of the standard representation of unital algebras over a commutative ring with identity through central ring homomorphisms [18, 30] provides an isomorphism between the categories of L-frames of W. Yao and L-fuzzy frames of D. Zhang and Y.-M. Liu (obtained in a way different from [81]). The employed machinery clearly shows the strong dependance of this isomorphism on the existence of the unit in the considered algebras, the condition, which holds trivially in the frame case. In other words, the passage from frames to quantales makes the concepts of W. Yao as well as D. Zhang and Y.-M. Liu different. In view of the above-mentioned importance of lattice-valued frames in fuzzification of the sobriety-spatiality equivalence, as an additional consequence, quantale algebras give a convenient universally algebraic framework for developing lattice-valued analogues of the latter as well as for answering the long-standing question on its relationships to the notion of stratification of lattice-valued topology [58].

1.3. Skeletal Q-categories versus lattice-valued partial orders. The developments of this paper are highly dependant on the isomorphism between the categories $\mathbf{RMod}(Q)$ of right Q-modules and $\mathbf{CSCat}(Q)$ of cocomplete skeletal Q-categories, constructed by I. Stubbe [76] for every unital quantale Q (in fact, for a small quantaloid $\mathcal{Q}$). The result extends the classical representation of the category $\mathbf{Sup}$ of $\bigvee$-semilattices in terms of Eilenberg-Moore categories of two monads.

On the one hand, there exists the well-known powerset monad $\mathbb{P} = (\mathcal{P}, \eta, \mu)$ on the category $\mathbf{Set}$ of sets and maps, which is given by the following data:

(1) $\mathcal{P}(X_1 \xrightarrow{f} X_2) = \mathcal{P}X_1 \xrightarrow{\mathcal{P}f} \mathcal{P}X_2$, where $\mathcal{P}X_i = \{S \,|\, S \subseteq X_i\}$ and $\mathcal{P}f(S) = \{f(s) \,|\, s \in S\}$;
(2) $X \xrightarrow{\eta_X} \mathcal{P}X$ is defined by $\eta_X(x) = \{x\}$;
(3) $\mathcal{P}\mathcal{P}X \xrightarrow{\mu_X} \mathcal{P}X$ is defined by $\mu_X(\mathcal{S}) = \bigcup \mathcal{S}$.

The Eilenberg-Moore category $\mathbf{Set}^{\mathbb{P}}$ of the monad $\mathbb{P}$ is then precisely the above-mentioned category $\mathbf{Sup}$.

On the other hand, there exists the down-set monad $\mathbb{D} = (\mathcal{D}, \zeta, \nu)$ on the category $\mathbf{Prost}$ of preordered sets (no anti-symmetry of partial order) and order-preserving maps, which is given by the following items:

(1) $\mathcal{D}(A_1 \xrightarrow{f} A_2) = \mathcal{D}A_1 \xrightarrow{\mathcal{D}f} \mathcal{D}A_2$, where $\mathcal{D}A_i = \{S \,|\, S \subseteq A_i \text{ and } S = \downarrow S\}$ and $\mathcal{D}f(S) = \downarrow \{f(s) \,|\, s \in S\}$;
(2) $A \xrightarrow{\zeta_A} \mathcal{D}A$ is defined by $\zeta_A(a) = \downarrow a$;

(3) $\mathcal{D}\mathcal{D}A \xrightarrow{\nu_A} \mathcal{D}A$ is defined by $\nu_A(\mathcal{S}) = \bigcup \mathcal{S}$.

The monad in question is easily seen to restrict to the full subcategory **Pos** of **Prost** of partially ordered sets (posets). The Eilenberg-Moore category $\mathbf{Pos}^{\mathbb{D}}$ of the monad $\mathbb{D}$ (whose objects have a simplified description due to the fact that the monad $\mathbb{D}$ is of Kock-Zöberlein type [35]) is again the category **Sup**. Moreover, the latter monad is induced by the reflective embedding $\mathbf{Sup} \xhookrightarrow{|-|} \mathbf{Pos}$ (which is precisely the forgetful functor), the left adjoint of which is given by the particular example of completion of posets, namely, by the above-mentioned functor $\mathcal{D}$, whose codomain is easily seen to be **Sup**, since the set $\mathcal{D}A$ is closed in $\mathcal{P}A$ under arbitrary set-theoretic unions (cf. Item (3) in the definition of the monad $\mathbb{D}$). Even more, since the forgetful functor $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ (which is no more an embedding) has a left adjoint $\mathbf{Set} \xrightarrow{K} \mathbf{Pos}$, which is given by $K(X_1 \xrightarrow{f} X_2) = (X_1, =) \xrightarrow{f} (X_2, =)$, one easily gets that the composition of the just mentioned adjoint situations gives the one, which induces the powerset monad $\mathbb{P}$ on the category **Set**.

It is well-known that given a unital quantale Q, the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{P}_Q}$ of the Q-powerset monad $\mathbb{P}_Q$ on the category **Set** provides the category $\mathbf{RMod}(Q)$ of right modules over Q, which essentially is a fuzzification of the above-mentioned isomorphism $\mathbf{Set}^{\mathbb{P}} \cong \mathbf{Sup}$, taking into consideration the simple fact that $\mathbf{Sup} \cong \mathbf{RMod}(\mathbf{2})$. Moreover, I. Stubbe [73] provided a lattice-valued analogue of both preordered and partially ordered set (Q-category and skeletal Q-category, respectively), the down-set monad $\mathbb{D}$ (the so-called contravariant presheaf monad on the category of (skeletal) Q-categories), and showed [76] that its Eilenberg-Moore category is precisely the category $\mathbf{CSCat}(Q)$ of cocomplete skeletal Q-categories, studying the properties of the latter structures in both stand-alone and category context. Additionally, he obtained that the category $\mathbf{CSCat}(Q)$ is isomorphic to the above category $\mathbf{RMod}(Q)$. Viewing the objects of the former category as a fuzzification of $\bigvee$-semilattices, we see that similar to the crisp case, where the categories $\mathbf{RMod}(\mathbf{2})$ and **Sup** are isomorphic, the categories $\mathbf{RMod}(Q)$ and $\mathbf{CSCat}(Q)$ are isomorphic as well.

The original results of I. Stubbe are more general than the above-mentioned ones, employing a (small) quantaloid $\mathcal{Q}$ instead of a quantale Q, and, therefore, using the language of enriched categories [34, 39]. As follows from the above discussion, however, their simplified Q-versions are closely related to lattice-valued mathematics. More precisely, Q-categories are nothing else than lattice-valued preorders of L. A. Zadeh [83] and S. V. Ovchinnikov [41] (see, e.g., [5] for a thorough discussion on the topic), whereas the assumption on being skeletal makes lattice-valued preorders into lattice-valued partial orders (see the above-mentioned references). Further, a contravariant Q-enriched presheaf is nothing else than a lattice-valued down-set (a covariant Q-enriched presheaf is then precisely a lattice-valued up-set), and the free cocompletion of a skeletal Q-category is a lattice-valued analogue of the above-mentioned completion of partially ordered sets (already studied elsewhere). Lastly, the assumption on cocompleteness of a skeletal Q-category provides the existence of a lattice-valued $\bigvee$-operation. As a consequence, one gets a convenient representation of lattice-valued $\bigvee$-semilattices through quantale modules (and vice versa), much relied upon in this paper.

When looking at the results of I. Stubbe though, ones notices that he neither uses the language of many-valued mathematics (even in the restricted Q-valued case), nor provides a proper (in fact, any, apart from [77], up to the knowledge of the author) placement of his achievements in that context. On the other hand, the theory of lattice-valued sets, going back up to 1965, can contribute a lot to the theory of Q-categories through the notion of lattice-valued preorder. More precisely, the theory of the latter structures is already well-developed, and, moreover, makes a significant part of lattice-valued mathematics. Since this paper targets the fuzzy community, we restate the above-mentioned isomorphism $\mathbf{RMod}(Q) \cong \mathbf{CSCat}(Q)$ of I. Stubbe in lattice-valued terms, and use it, later on, as an important tool in obtaining a characterization of lattice-valued frames. Our main point here is to contribute to the study of lattice-valued posets and not to the theory of Q-categories, the properties of which lie off the scope of this paper.

In the developments below, we rely on category theory and universal algebra. The necessary categorical background can be found in [2, 24, 39]. For algebraic notions we recommend [3, 9, 36, 43, 61]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of category and functor.

2. Quantale modules and algebras

In this section, we briefly recall the notions of quantale module and algebra (notice that these structures are closely related to many-valued mathematics [67, 68]). Both concepts rely on the notion of quantale (introduced by C. J. Mulvey [40] as an attempt to provide a possible setting for constructive foundations of quantum mechanics, and to study the spectra of non-commutative C^*-algebras, which are locales in the commutative case), whose theory has found numerous applications in both universal algebra and category theory [36, 73, 74, 75, 76] as well as in lattice-valued mathematics [25, 27, 29, 57].

1. Definition. A $\bigvee$-*semilattice* is a partially ordered set (poset, for short), which has arbitrary joins (denoted $\bigvee$). A $\bigvee$-*semilattice homomorphism* $(A, \bigvee) \xrightarrow{\varphi} (B, \bigvee)$ is a $\bigvee$-preserving map $A \xrightarrow{\varphi} B$. **Sup** is the construct of $\bigvee$-semilattices and their homomorphisms. ∎

Notice that in this article, we use the term "$\bigvee$-semilattice" instead of the more usual term "sup-lattice" as in, e.g., [16, 36, 73, 76], or the term "join-semilattice" as in, e.g., [57]. Moreover, to be in line with the overall categorical notation of this paper, we use "**Sup**" instead of "$s\ell$" [33], or "$\mathcal{SL}$" [60], or "$\mathcal{S}up$" [36].

2. Definition. A *quantale* is a triple $(Q, \bigvee, \otimes)$ such that

(1) $(Q, \bigvee)$ is a $\bigvee$-semilattice;
(2) $(Q, \otimes)$ is a semigroup, i.e., $q_1 \otimes (q_2 \otimes q_3) = (q_1 \otimes q_2) \otimes q_3$ for every $q_1, q_2, q_3 \in Q$;
(3) $q \otimes (\bigvee S) = \bigvee_{s \in S}(q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S}(s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

A *quantale homomorphism* $(P, \bigvee, \otimes) \xrightarrow{\varphi} (Q, \bigvee, \otimes)$ is a map $P \xrightarrow{\varphi} Q$, which preserves $\otimes$ and $\bigvee$. **Quant** is the category of quantales and their homomorphisms, concrete over the categories **Sup** of $\bigvee$-semilattices and **SGrp** of semigroups. ∎

Since the main results of the paper are much dependant on algebraic structures with ever growing signature (cf., e.g., the passage from $\bigvee$-semilattices to quantales), we will sometimes shorten the notion to just A (for $\bigvee$-semilattices) or Q (for quantales), making explicit just the algebraic structure which we need at the moment (cf., e.g., the notation for quantale modules of Definition 9).

The category **Quant** has been studied thoroughly in [36, 60], K. I. Rosenthal giving a coherent statement to the quantale theory. Throughout this paper, we will consider two specific types of quantales, which are mentioned below.

3. Definition. A quantale Q is said to be *unital* provided that there exists an element $1 \in Q$ such that $(Q, \otimes, 1)$ is a monoid. A *unital quantale homomorphism* should additionally preserve the unit. **UQuant** denotes the respective (non-full) subcategory of **Quant**. ∎

4. Definition. A quantale Q is said to be *commutative* provided that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every q_1, $q_2 \in Q$. **CQuant** is the respective full subcategory of **Quant**. ∎

Every quantale, being a complete lattice, has the largest element $\top$ and the smallest element $\bot$. The following examples provide more intuition for the concept.

5. Example. Every *frame*, i.e., a complete lattice L such that $a \wedge (\bigvee S) = \bigvee_{s \in S}(a \wedge s)$ for every $a \in L$ and every $S \subseteq L$ [32], is a commutative unital quantale, where $\otimes = \wedge$ and $1 = \top$. In particular, the two-element chain $\mathbf{2} = \{\bot, \top\}$ is a commutative unital quantale. ∎

6. Example. Let $(A, \cdot)$ be a semigroup. The powerset $\mathcal{P}(A)$ is a quantale, where $\bigvee$ are unions and $S \otimes T = \{s \cdot t \mid s \in S,\ t \in T\}$. If $(A, \cdot, 1)$ is a monoid, then $\mathcal{P}(A)$ is unital, with the unit $\{1\}$. If $(A, \cdot)$ is commutative, then so is $\mathcal{P}(A)$. ∎

Example 6 provides the free quantale over a given semigroup [60] (the result is extended in [68]).

7. Example. Let X be a set and let $\mathcal{R}(X)$ be the set of all binary relations on X. $\mathcal{R}(X)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{(x, y) \in X \times X \mid (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in X\}$ (standard composition of relations). $\mathcal{R}(X)$ is unital, with the diagonal relation $\triangle = \{(x, x) \mid x \in X\}$ being the unit, but not commutative. ∎

It is shown in [6] that every unital quantale is isomorphic to a *relational quantale*, namely, a subset of $\mathcal{R}(X)$, which contains $\triangle$ and is closed under composition of relations, with $\bigvee$ being (in general) different from unions (see [22] for a more general result).

8. Example. Given a $\bigvee$-semilattice A, let $\mathcal{Q}(A)$ be the set $\mathbf{Sup}(A, A)$ of all $\bigvee$-preserving maps $A \xrightarrow{\varphi} A$. Equipped with the point-wise order, the set becomes a $\bigvee$-semilattice. Moreover, $\mathcal{Q}(A)$ is a unital quantale, where multiplication is given by the map composition and the unit is the identity map $A \xrightarrow{1_A} A$. ∎

It is shown in [44] that every quantale Q has a *faithful representation*, i.e., an embedding $Q \overset{\mu}{\hookrightarrow} \mathcal{Q}(A)$ for some $\bigvee$-semilattice A (which is actually Q itself).

On the next step, we recall the category $\mathbf{Mod}(Q)$ of unital left modules over a given unital quantale Q [36, 43, 61, 67, 68]. Its definition is very similar to the (well-known to algebraists) category $\mathbf{Mod}(R)$ of unital left modules over a unital ring R [3, 18, 30].

9. Definition. Given a unital quantale Q, a *unital left Q-module* is a pair $(A, *)$, where A is a $\bigvee$-semilattice and $Q \times A \xrightarrow{*} A$ is a map (the *action* of Q on A) such that

(1) $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$ and every $S \subseteq A$;
(2) $(\bigvee T) * a = \bigvee_{t \in T}(t * a)$ for every $T \subseteq Q$ and every $a \in A$;
(3) $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every q_1, $q_2 \in Q$ and every $a \in A$;
(4) $1_Q * a = a$ for every $a \in A$.

A *unital left Q-module homomorphism* $(A, *) \xrightarrow{\varphi} (B, *)$ is a map $A \xrightarrow{\varphi} B$, which preserves $\bigvee$ and satisfies the condition $\varphi(q * a) = q * \varphi(a)$ for every $a \in A$ and every $q \in Q$. $\mathbf{Mod}(Q)$ is the category of left unital Q-modules and their homomorphisms, concrete over the category **Sup**. ∎

Notice the possibility to define the category of modules over an arbitrary quantale, omitting Item (4) of Definition 9. Recently, however, we showed [69] that every category of modules over a non-unital quantale is equivalent to the category of unital modules over a unital extension of this quantale.

For the sake of shortness, from now on, “Q-module" means “unital left Q-module". It is easy to see that the category $\mathbf{Mod}(\mathbf{2})$ (recall the two-element quantale of Example 5) is isomorphic to the category **Sup** (cf. the well-known isomorphism between the categories of modules over the ring of integers $\mathbb{Z}$ and abelian groups [30]). Also notice that every unital quantale can be considered as a module over itself (with action given by quantale multiplication).

The concept of Q-module goes back to (at least) A. Joyal and M. Tierney [33]. More precisely, since **Sup** is a monoidal closed category (a convenient description of tensor products of $\bigvee$-semilattices is presented in [23]), unital (commutative) quantales are precisely the (commutative) monoids in **Sup**. Then Q-modules of Definition 9 are Q-modules in the sense of [33] (which essentially are just the Q-actions (in the sense of monoidal categories) on the objects of the monoidal category **Sup**, with Q-action morphisms (in the sense of monoidal categories again) serving as Q-module homomorphisms), provided that one notices that most of the results of [33], which deals with the commutative setting, are valid in the non-commutative case as well. Modules over a unital quantale form the central idea in the unified treatment of process semantics developed by S. Abramsky and S. Vickers in [1].

On the last step, we define the category $\mathbf{Alg}(Q)$ of algebras over a given unital commutative quantale Q. The definition was motivated by the category $\mathbf{Alg}(K)$ of algebras over a commutative ring K with identity [3, 18, 30]. Being started rather recently, the theory is less developed than that of quantale modules.

10. Definition. Given a unital commutative quantale Q, a Q-*algebra* is a triple $(A, \otimes, *)$ such that

(1) A is a $\bigvee$-semilattice;
(2) $(A, *)$ is a Q-module;
(3) $(A, \otimes)$ is a quantale;
(4) $q * (a_1 \otimes a_2) = (q * a_1) \otimes a_2 = a_1 \otimes (q * a_2)$ for every $a_1, a_2 \in A$, $q \in Q$.

A Q-*algebra homomorphism* $(A, \otimes, *) \xrightarrow{\varphi} (B, \otimes, *)$ is a map $A \xrightarrow{\varphi} B$, which is both a quantale homomorphism and a Q-module homomorphism. $\mathbf{Alg}(Q)$ is the category of Q-algebras and their homomorphisms, concrete over both the category $\mathbf{Mod}(Q)$ and the category **Quant**. ∎

It is not difficult to see that the category $\mathbf{Alg}(\mathbf{2})$ is isomorphic to the category **Quant** (cf. the isomorphism between the categories of algebras over the ring of integers $\mathbb{Z}$ and rings [30]). Notice as well that every unital commutative quantale is an algebra over itself (with action given by quantale multiplication).

Similar to the case of quantale modules, one can see that quantale algebras also go back to (at least) the already mentioned paper of A. Joyal and M. Tierney [33]. Given a unital commutative quantale Q, $\mathbf{Mod}(Q)$ is a monoidal closed category (see, e.g., [64] for the description of its monoidal structure, namely, tensor products of quantale modules). Then Q-algebras are precisely the monoids in $\mathbf{Mod}(Q)$.

It appears that the concept of quantale algebra provides a common framework for two concepts of lattice-valued frame, currently available in the fuzzy literature.

3. Quantale algebras as comma categories

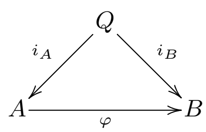
In [84] D. Zhang and Y.-M. Liu introduced the concept of L-fuzzy frame as an object of the comma category $(L \downarrow \mathbf{Frm})$, where **Frm** is the category of frames. This section extends the notion to quantales and shows its categorical equivalence to a particular instance of quantale algebras.

There exists the well-known representation of unital algebras over a commutative ring with identity through central ring homomorphisms [18, Proposition 1.1 of Chapter XIII], [30, Exercise 3 of Section IV.7]. In the following, we extend the result to quantale algebras. It should be noticed immediately that a similar achievement has been already attempted by W. Yao [81]. Due to its rather chaotic presentation, a significant flaw and the lack of proper universally algebraic background, we provide a more rigorous proof below. For convenience of the reader, we begin with certain algebraic and categorical preliminaries.

11. Definition. Given a unital commutative quantale Q, $\mathbf{UAlg}(Q)$ is the (non-full) subcategory of $\mathbf{Alg}(Q)$, whose objects additionally are unital quantales and whose morphisms additionally preserve the unit. ▪

12. Definition. The *center* of a Q-algebra A is the set $Z(A) = \{a \in A \mid a \otimes a' = a' \otimes a \text{ for every } a' \in A\}$. ▪

13. Definition. Given a unital commutative quantale Q, $(Q \downarrow \mathbf{UQuant})_z$ is the category, whose objects are the $\mathbf{UQuant}$-morphisms $Q \xrightarrow{i_A} A$ (i.e., from Q to any $\mathbf{UQuant}$-object) such that the image of i_A lies in the center of A. The morphisms of the category $(Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)$ are the $\mathbf{UQuant}$-morphisms $A \xrightarrow{\varphi} B$, which make the triangle



commute. ▪

Notice that $(Q \downarrow \mathbf{UQuant})_z$ is a full subcategory of the comma category $(Q \downarrow \mathbf{UQuant})$, whose definition is written explicitly for convenience of the reader. Moreover, to be in line with the main goal of this article, we use the notation for comma categories of [84]. The preliminaries in hand, we proceed to the main result of this section, which makes use of the following two propositions.

14. Proposition. *There exists a functor* $\mathbf{UAlg}(Q) \xrightarrow{F} (Q \downarrow \mathbf{UQuant})_z$ *defined by* $F((A, *) \xrightarrow{\varphi} (B, *)) = (Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)$, *where* $i_A(q) = q * 1_A$.

Proof. To show that the functor is correct on objects, we start by checking that the map $Q \xrightarrow{i_A} A$ is a unital quantale homomorphism. Given $S \subseteq Q$, $i_A(\bigvee S) = (\bigvee S) * 1_A = \bigvee_{s \in S}(s * 1_A) = \bigvee_{s \in S} i_A(s)$. Given q_1, $q_2 \in Q$, $i_A(q_1 \otimes q_2) = (q_1 \otimes q_2) * 1_A = q_1 * (q_2 * 1_A) = q_1 * (1_A \otimes (q_2 * 1_A)) = (q_1 * 1_A) \otimes (q_2 * 1_A) = i_A(q_1) \otimes i_A(q_2)$. Moreover, $i_A(1_Q) = 1_Q * 1_A = 1_A$.

To show that the image of i_A lies in the center of A, notice that given $q \in Q$ and $a \in A$, $i_A(q) \otimes a = (q * 1_A) \otimes a = q * (1_A \otimes a) = q * (a \otimes 1_A) = a \otimes (q * 1_A) = a \otimes i_A(q)$.

To verify that the functor is correct on morphisms, use the fact that given $q \in Q$, $\varphi \circ i_A(q) = \varphi(q * 1_A) = q * \varphi(1_A) = q * 1_B = i_B(q)$. □

An attentive reader will see that Proposition 14 makes no use of the centrality property of the objects of the category $(Q \downarrow \mathbf{UQuant})_z$. It is the functor in the opposite direction which employs the requirement.

15. Proposition. *There exists a functor* $(Q \downarrow \mathbf{UQuant})_z \xrightarrow{G} \mathbf{UAlg}(Q)$ *defined by* $G((Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)) = (A, *) \xrightarrow{\varphi} (B, *)$, *where* $q * a = i_A(q) \otimes a$.

Proof. To check the correctness of the functor on objects, we show that $(A, *)$ is a unital Q-algebra. Given $q \in Q$ and $S \subseteq A$, $q * (\bigvee S) = i_A(q) \otimes (\bigvee S) = \bigvee_{s \in S}(i_A(q) \otimes s) = \bigvee_{s \in S}(q * s)$. Given $S \subseteq Q$ and $a \in A$, $(\bigvee S) * a = i_A(\bigvee S) \otimes a = (\bigvee_{s \in S} i_A(s)) \otimes a = \bigvee_{s \in S}(i_A(s) \otimes a) = \bigvee_{s \in S}(s * a)$. Given q_1, $q_2 \in Q$ and $a \in A$, $q_1 * (q_2 * a) = q_1 * (i_A(q_2) \otimes a) =$

$i_A(q_1)\otimes(i_A(q_2)\otimes a) = (i_A(q_1)\otimes i_A(q_2))\otimes a = i_A(q_1\otimes q_2)\otimes a = (q_1\otimes q_2)*a$. Given $a\in A$, $1_Q * a = i_A(1_Q)\otimes a = 1_A\otimes a = a$. Lastly, given $q\in Q$ and $a_1, a_2\in A$, $q*(a_1\otimes a_2) = i_A(q)\otimes(a_1\otimes a_2) = (i_A(q)\otimes a_1)\otimes a_2 = (q*a_1)\otimes a_2$. Moreover, the centrality property of $Q\xrightarrow{i_A} A$ gives $(i_A(q)\otimes a_1)\otimes a_2 = (a_1\otimes i_A(q))\otimes a_2 = a_1\otimes(i_A(q)\otimes a_2) = a_1\otimes(q*a_2)$.

For correctness of the functor on morphisms, use the fact that for $q\in Q$ and $a\in A$, $\varphi(q*a) = \varphi(i_A(q)\otimes a) = (\varphi\circ i_A(q))\otimes\varphi(a) = i_B(q)\otimes\varphi(a) = q*\varphi(a)$. □

It is important to underline that W. Yao [81] erroneously used the whole category $(Q\downarrow \mathbf{UQuant})$ as the domain of the functor G of Proposition 15.

16. Theorem. *$G\circ F = 1_{\mathbf{UAlg}(Q)}$ and $F\circ G = (Q\downarrow\mathbf{UQuant})_z$, i.e., the two categories $\mathbf{UAlg}(Q)$ and $(Q\downarrow\mathbf{UQuant})_z$ are isomorphic.*

Proof. Given an $\mathbf{UAlg}(Q)$-object $(A,*)$, it follows that $G\circ F(A,*) = G(Q\xrightarrow{i_A} A) = (A,*')$, where $q*'a = i_A(q)\otimes a = (q*1_A)\otimes a = q*(1_A\otimes a) = q*a$. Given a $(Q\downarrow\mathbf{UQuant})_z$-object $Q\xrightarrow{i_A} A$, it follows that $F\circ G(Q\xrightarrow{i_A} A) = F(A,*) = Q\xrightarrow{i'_A} A$, where $i'_A(q) = q*1_A = i_A(q)\otimes 1_A = i_A(q)$. □

One should pay attention to the fact that the existence of the functor F of Proposition 14 depends on the availability of the unit in the objects of $\mathbf{UAlg}(Q)$.

4. Quantale algebras as lattice-valued quantales

In [79, 81] W. Yao developed the theory of lattice-valued frames, based in the concept of lattice-valued order of L. Fan [12]. In this section, we extend the notion to lattice-valued quantale and show its categorical equivalence to the concept of quantale algebra.

4.1. Quantale modules as lattice-valued $\bigvee$-semilattices. In view of the discussion in Subsection 1.3 on the isomorphism $\mathbf{RMod}(Q)\cong\mathbf{CSCat}(Q)$ of I. Stubbe [76], in this subsection, we restate his result in lattice-valued terms. More precisely, we consider the category $\mathbf{Sup}(Q)$ of Q-$\bigvee$-semilattices and show that it is isomorphic to the above-mentioned category $\mathbf{Mod}(Q)$. As a consequence, one obtains a particular (and very simple) case of [76, Corollary 4.13].

Before moving forward, we have to recall several basic properties of quantales and Q-modules. Given a quantale Q, there exist two residuations induced by its multiplication $\otimes$ and defined by $q_1\rightarrow_r q_2 = \bigvee\{q\in Q \mid q_1\otimes q\leqslant q_2\}$ and $q_1\rightarrow_l q_2 = \bigvee\{q\in Q\mid q\otimes q_1\leqslant q_2\}$, providing a single residuation $\cdot\rightarrow\cdot$ in case of Q being commutative. The operations enjoy the standard properties of Galois connections [11], i.e., $q_2\leqslant q_1\rightarrow_r q_3$ if and only if $q_1\otimes q_2\leqslant q_3$ iff $q_1\leqslant q_2\rightarrow_l q_3$. On the other hand, given a Q-module $(A,*)$, there exist residuations $a_1\twoheadrightarrow a_2 = \bigvee\{q\in Q\mid q*a_1\leqslant a_2\}$ and $q\rightsquigarrow a = \bigvee\{a'\in A\mid q*a'\leqslant a\}$. The operations satisfy the Galois connection property $q\leqslant a_1\twoheadrightarrow a_2$ iff $q*a_1\leqslant a_2$ iff $a_1\leqslant q\rightsquigarrow a_2$. Moreover, the subsequent two lemmas recall a number of other standard properties of the above-mentioned residuations (for their simple proofs, the reader is referred to [36, 43, 60], or any other comprehensive reference on quantales).

17. Lemma. *Given a quantale Q, the following hold:*

(1) *$q_1\rightarrow_r(q_2\rightarrow_r q_3) = (q_2\otimes q_1)\rightarrow_r q_3$ and $q_1\rightarrow_l(q_2\rightarrow_l q_3) = (q_1\otimes q_2)\rightarrow_l q_3$ for every $q_1,q_2,q_3\in Q$;*
(2) *$q\rightarrow_r(\bigwedge S) = \bigwedge_{s\in S}(q\rightarrow_r s)$ and $q\rightarrow_l(\bigwedge S) = \bigwedge_{s\in S}(q\rightarrow_l s)$ for every $q\in Q$ and every $S\subseteq Q$;*
(3) *$(\bigvee S)\rightarrow_r q = \bigwedge_{s\in S}(s\rightarrow_r q)$ and $(\bigvee S)\rightarrow_l q = \bigwedge_{s\in S}(s\rightarrow_l q)$ for every $q\in Q$ and every $S\subseteq Q$.*

If Q is unital, then, additionally,

(4) $1_Q \to_r q = q$ *and* $1_Q \to_l q = q$ *for every* $q \in Q$.

18. Lemma. *Given a* Q*-module* $(A, *)$*, the following hold:*

(1) $q \to_l (a_1 \twoheadrightarrow a_2) = (q * a_1) \twoheadrightarrow a_2$ *for every* $q \in Q$ *and every* $a_1, a_2 \in A$*;*
(2) $a \twoheadrightarrow (\bigwedge S) = \bigwedge_{s \in S}(a \twoheadrightarrow s)$ *for every* $a \in A$ *and every* $S \subseteq A$*;*
(3) $(\bigvee S) \twoheadrightarrow a = \bigwedge_{s \in S}(s \twoheadrightarrow a)$ *for every* $a \in A$ *and every* $S \subseteq A$*;*
(4) $a_1 \twoheadrightarrow (q \rightsquigarrow a_2) = q \to_r (a_1 \twoheadrightarrow a_2)$ *for every* $q \in Q$ *and every* $a_1, a_2 \in A$.

Notice that the corresponding analogue for $\cdot \to_r \cdot$ in Item (1) of Lemma 18 requires commutativity of the quantale Q. All the properties mentioned in Lemmas 17, 18 will be heavily used throughout the paper, without mentioning them explicitly on each occasion.

Some results and notation from the theory of lattice-valued powerset operators will be also used throughout the paper (we notice that our employed powerset operators were first described by L. A. Zadeh in [82]; the arrow notation and the complete development is due to S. E. Rodabaugh [53, 55, 57]). Given a map $X \xrightarrow{f} Y$, there exist *forward* $\mathcal{P}(X) \xrightarrow{f^{\to}} \mathcal{P}(Y)$ and *backward* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$ *powerset operators*, defined by $f^{\to}(S) = \{f(x) \mid x \in S\}$ and $f^{\leftarrow}(T) = \{x \in X \mid f(x) \in T\}$ respectively. Given a $\bigvee$-semilattice L, the maps can be extended to *forward* $L^X \xrightarrow{f_L^{\to}} L^Y$ and *backward* $L^Y \xrightarrow{f_L^{\leftarrow}} L^X$ *L-powerset operators*, defined accordingly by $(f_L^{\to}(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}$ and $f_L^{\leftarrow}(\beta) = \beta \circ f$.

The necessary preliminaries in hand, we can proceed to the main definition of this subsection.

19. Definition. Let Q be a unital quantale. A *Q-partially ordered set* (*Q-poset*) is a pair (A, e), where A is a set, and $A \times A \xrightarrow{e} Q$ is a map (*Q-partial order* or *Q-order* on A) such that

(1) $1_Q \leqslant e(a, a)$ for every $a \in A$ (*Q-reflexivity*);
(2) $e(a_1, a_2) \otimes e(a_2, a_3) \leqslant e(a_1, a_3)$ for every $a_1, a_2, a_3 \in A$ (*Q-transitivity*);
(3) $1_Q \leqslant e(a_1, a_2)$ and $1_Q \leqslant e(a_2, a_1)$ imply $a_1 = a_2$, for every $a_1, a_2 \in A$ (*Q-antisymmetry*).

A *Q_r-$\bigvee$-semilattice* is a triple $(A, e, \sqcup)$, where (A, e) is a Q-poset, and $Q^A \xrightarrow{\sqcup} A$ is a map (*Q_r-join operation* on A) such that $e(\sqcup \alpha, a) = \bigwedge_{a' \in A}(\alpha(a') \to_r e(a', a))$ for every $\alpha \in Q^A$ and every $a \in A$. A *Q_r-$\bigvee$-semilattice homomorphism* $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi(\sqcup \alpha) = \sqcup \varphi_Q^{\to}(\alpha)$ for every $\alpha \in Q^A$ (*Q_r-join-preserving map*). $\mathbf{Sup}_r(Q)$ is the construct of Q_r-$\bigvee$-semilattices and their homomorphisms. ∎

Replacing $\cdot \to_r \cdot$ with $\cdot \to_l \cdot$, one obtains the concept of *Q_l-$\bigvee$-semilattice*. Since they both share the same notion of lattice-valued order, we employ neither "r" nor "l" in the notation for this lattice-valued order. Moreover, the term "Q-$\bigvee$-semilattice" will suppose commutativity of the quantale Q. Given a Q-poset (A, e), there exists at most one Q-(r,l)-join operation $\sqcup$ on (A, e), since the condition $e(\sqcup \alpha, a) = \bigwedge_{a' \in A}(\alpha(a') \to_{r,l} e(a', a)) = e(\sqcup' \alpha, a)$ for every $a \in A$, implies $\sqcup \alpha = \sqcup' \alpha$ by Items (1) and (3) of Definition 19. One should also underline at once that Definition 19 uses the concepts of W. Yao [78, 79, 80, 81] developed for frames. An important difference though is the distinguishing between the two cases "r" and "l", the use of the unit 1_Q instead of the top element $\top$ and the inequality "$1_Q \leqslant \ldots$" instead of the equality "$1_Q = \ldots$". However, the case of Q being a frame, makes the two concepts coincide with that of W. Yao. To give the reader more intuition for the new notion, below, we provide its simple example based in non-commutative quantales.

20. Lemma. *Every unital quantale* Q *provides the* Q_r*-*$\bigvee$*-semilattice* $(Q, e, \sqcup)$*, where* $e(q_1, q_2) = q_1 \to_r q_2$ *and* $\sqcup \alpha = \bigvee_{q \in Q}(q \otimes \alpha(q))$.

Proof. To show that (Q,e) is a Q-poset, notice that given $q \in Q$, $q \otimes 1_Q \leqslant q$ provides $1_Q \leqslant q \rightarrow_r q = e(q,q)$. On the other hand, given $q_1, q_2, q_3 \in Q$, $e(q_1,q_2) \otimes e(q_2,q_3) = (q_1 \rightarrow_r q_2) \otimes (q_2 \rightarrow_r q_3) = \bigvee\{q \otimes q' \mid q_1 \otimes q \leqslant q_2 \text{ and } q_2 \otimes q' \leqslant q_3\}$ and then, $q_1 \otimes (q \otimes q') = (q_1 \otimes q) \otimes q' \leqslant q_2 \otimes q' \leqslant q_3$ gives $q \otimes q' \leqslant q_1 \rightarrow_r q_3 = e(q_1,q_3)$. As a result, $e(q_1,q_2) \otimes e(q_2,q_3) \leqslant e(q_1,q_3)$. Lastly, if $1_Q \leqslant e(q_1,q_2)$ and $1_Q \leqslant e(q_2,q_1)$, then $q_1 = q_1 \otimes 1_Q \leqslant q_2$ and $q_2 = q_2 \otimes 1_Q \leqslant q_1$ give $q_1 = q_2$.

To show that $\sqcup$ is the Q_r-join operation w.r.t. (Q,e), notice that for $\alpha \in Q^Q$ and $q \in Q$, it follows that

$$\bigwedge_{q' \in Q} (\alpha(q') \rightarrow_r e(q',q)) = \bigwedge_{q' \in Q} (\alpha(q') \rightarrow_r (q' \rightarrow_r q)) = \bigwedge_{q' \in Q} ((q' \otimes \alpha(q')) \rightarrow_r q) = (\bigvee_{q' \in Q} (q' \otimes \alpha(q'))) \rightarrow_r q = e(\bigvee_{q' \in Q} (q' \otimes \alpha(q')), q) = e(\sqcup \alpha, q),$$

which provides then the result in question. $\square$

Notice that the machinery of Lemma 20 is not applicable to the residuation $\cdot \rightarrow_l \cdot$. Indeed, to show Item (2) of Definition 19, one starts with $e(q_1,q_2) \otimes e(q_2,q_3) = (q_1 \rightarrow_l q_2) \otimes (q_2 \rightarrow_l q_3) = \bigvee\{q \otimes q' \mid q \otimes q_1 \leqslant q_2 \text{ and } q' \otimes q_2 \leqslant q_3\}$ and has to show that $(q \otimes q') \otimes q_1 \leqslant q_3$, which is generally not true, unless Q is commutative. An analogue of this deficiency is the main reason for our using commutative quantales in the subsequent developments. We should notice, however, immediately that the above-mentioned machinery of Q-categories of I. Stubbe [76] does not depend on commutativity of its underlying quantale (indeed, its general version relies on a quantaloid $\mathcal{Q}$ instead of a quantale Q).

21. Proposition. *Given a unital commutative quantale Q, there exists a functor* $\mathbf{Mod}(Q) \xrightarrow{F} \mathbf{Sup}(Q)$ *defined by* $F((A,*) \xrightarrow{\varphi} (B,*)) = (A,e,\sqcup) \xrightarrow{\varphi} (B,e,\sqcup)$*, where* $e(a_1,a_2) = a_1 \twoheadrightarrow a_2$ *and* $\sqcup \alpha = \bigvee_{a \in A}(\alpha(a) * a)$.

Proof. To show that the functor is correct on objects, we begin by checking that (A,e) is a Q-poset. Given $a \in A$, $1_Q * a = a$ implies $1_Q \leqslant a \twoheadrightarrow a = e(a,a)$. Given $a_1, a_2, a_3 \in A$, $e(a_1,a_2) \otimes e(a_2,a_3) = (a_1 \twoheadrightarrow a_2) \otimes (a_2 \twoheadrightarrow a_3) = \bigvee\{q \otimes q' \mid q * a_1 \leqslant a_2 \text{ and } q' * a_2 \leqslant a_3\}$ and then, $(q \otimes q') * a_1 \overset{(\dagger)}{=} (q' \otimes q) * a_1 = q' * (q * a_1) \leqslant q' * a_2 \leqslant a_3$ provides $q \otimes q' \leqslant a_1 \twoheadrightarrow a_3 = e(a_1,a_3)$, where $(\dagger)$ uses commutativity of the quantale Q. As a result, $e(a_1,a_2) \otimes e(a_2,a_3) \leqslant e(a_1,a_3)$. Lastly, if $1_Q \leqslant e(a_1,a_2)$ and $1_Q \leqslant e(a_2,a_1)$, then $a_1 = 1_Q * a_1 \leqslant a_2$ and $a_2 = 1_Q * a_2 \leqslant a_1$ give $a_1 = a_2$.

To show that $\sqcup$ provides the Q-join operation w.r.t. (A,e), use the fact that given $\alpha \in Q^A$ and $a \in A$,

$$\bigwedge_{a' \in A} (\alpha(a') \rightarrow e(a',a)) = \bigwedge_{a' \in A} (\alpha(a') \rightarrow (a' \twoheadrightarrow a)) = \bigwedge_{a' \in A} ((\alpha(a') * a') \twoheadrightarrow a) = (\bigvee_{a' \in A} (\alpha(a') * a')) \twoheadrightarrow a = e(\bigvee_{a' \in A} (\alpha(a') * a'), a) = e(\sqcup \alpha, a).$$

To show that the functor F is correct on morphisms, notice that given $\alpha \in Q^A$ and $b \in B$,

$$e(\varphi(\sqcup \alpha), b) = \varphi(\sqcup \alpha) \twoheadrightarrow b = \varphi(\bigvee_{a \in A} (\alpha(a) * a)) \twoheadrightarrow b =$$
$$(\bigvee_{a \in A} (\alpha(a) * \varphi(a))) \twoheadrightarrow b = \bigwedge_{a \in A} ((\alpha(a) * \varphi(a)) \twoheadrightarrow b) =$$
$$\bigwedge_{a \in A} (\alpha(a) \to (\varphi(a) \twoheadrightarrow b)) = \bigwedge_{a \in A} (\alpha(a) \to e(\varphi(a), b)) =$$
$$\bigwedge_{b' \in B} \bigwedge_{\varphi(a) = b'} (\alpha(a) \to e(b', b)) = \bigwedge_{b' \in B} ((\bigvee_{\varphi(a) = b'} \alpha(a)) \to e(b', b)) =$$
$$\bigwedge_{b' \in B} ((\varphi_Q^{\to}(\alpha))(b') \to e(b', b)) = e(\sqcup \varphi_Q^{\to}(\alpha), b).$$

As a result, one obtains that $\varphi(\sqcup \alpha) = \sqcup \varphi_Q^{\to}(\alpha)$. □

The functor in the opposite direction requires the following specific notation. Given a $\bigvee$-semilattice L and a set X, for every $S \subseteq X$ and every $b \in L$, there exists a map $X \xrightarrow{\alpha_S^b} L$ defined by

$$\alpha_S^b(x) = \begin{cases} b, & x \in S \\ \bot, & \text{otherwise.} \end{cases}$$

In particular, if S is a singleton $\{s\}$, then we use the notation α_s^b. An important property of such maps is contained in the next "folklore" lemma.

22. Lemma. *Given a map $X \xrightarrow{f} Y$ and a $\bigvee$-semilattice L, for every map $\alpha_S^b \in L^X$, $f_L^{\to}(\alpha_S^b) = \alpha_{f^{\to}(S)}^b$.*

23. Proposition. *Given a unital commutative quantale Q, there exists a functor $\mathbf{Sup}(Q) \xrightarrow{G} \mathbf{Mod}(Q)$ defined by $G((A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)) = (A, \leqslant, \bigvee, *) \xrightarrow{\varphi} (B, \leqslant, \bigvee, *)$, where*

1. *$a_1 \leqslant a_2$ iff $1_Q \leqslant e(a_1, a_2)$, for every $a_1, a_2 \in A$;*
2. *$\bigvee S = \sqcup \alpha_S^{1_Q}$ for every $S \subseteq A$;*
3. *$q * a = \sqcup \alpha_a^q$ for every $q \in Q$ and every $a \in A$.*

Proof. To check that G is well-defined on objects, we show that $(A, \leqslant, \bigvee, *)$ is a Q-module. The properties of Q-order of Definition 19 imply that $(A, \leqslant)$ is a poset (notice that reflexivity and antisymmetry can be obtained replacing 1_Q in Definition 19 by an arbitrary element of the quantale Q, whereas transitivity relies on the identity $1_Q = 1_Q \otimes 1_Q$).

To show that $\bigvee$ is the join operation on $(A, \leqslant)$, notice that given $S \subseteq A$, for every $s \in S$, it follows that

$$1_Q \leqslant e(\sqcup \alpha_S^{1_Q}, \sqcup \alpha_S^{1_Q}) = \bigwedge_{a \in A} (\alpha_S^{1_Q}(a) \to e(a, \sqcup \alpha_S^{1_Q})) =$$
$$\bigwedge_{s' \in S} (1_Q \to e(s', \sqcup \alpha_S^{1_Q})) = \bigwedge_{s' \in S} e(s', \sqcup \alpha_S^{1_Q}) \leqslant e(s, \sqcup \alpha_S^{1_Q})$$

and, therefore, $s \leqslant \sqcup \alpha_S^{1_Q}$. On the other hand, given $a \in A$ such that $s \leqslant a$ for every $s \in S$, it follows that

$$1_Q \leqslant \bigwedge_{s \in S} e(s, a) = \bigwedge_{a' \in A} (\alpha_S^{1_Q}(a') \to e(a', a)) = e(\sqcup \alpha_S^{1_Q}, a)$$

and, therefore, $\sqcup\alpha_S^{1_Q} \leqslant a$.

To show that $*$ is a module action on $(A, \bigvee)$, we verify the required conditions of Definition 9 in a row.

Item (1): For $q \in Q$ and $S \subseteq A$, it follows that $q * (\bigvee S) = \sqcup\alpha^q_{\bigvee S}$ and $\bigvee_{s\in S}(q * s) = \bigvee_{s\in S} \sqcup\alpha^q_s = \sqcup\alpha_T^{1_Q}$, where T is the shorthand for $\{\sqcup\alpha^q_s \mid s \in S\}$. To continue, we notice that

$$e(\sqcup\alpha^q_{\bigvee S}, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a\in A}(\alpha^q_{\bigvee S}(a) \to e(a, \sqcup\alpha_T^{1_Q})) =$$
$$q \to e(\bigvee S, \sqcup\alpha_T^{1_Q}) = q \to e(\sqcup\alpha_S^{1_Q}, \sqcup\alpha_T^{1_Q}) =$$
$$q \to (\bigwedge_{a\in A}(\alpha_S^{1_Q}(a) \to e(a, \sqcup\alpha_T^{1_Q}))) =$$
$$q \to (\bigwedge_{s\in S} e(s, \sqcup\alpha_T^{1_Q})) = \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha_T^{1_Q})).$$

For every $s \in S$, it follows that $\sqcup\alpha^q_s \leqslant \bigvee\{\sqcup\alpha^q_{s'} \mid s' \in S\} = \sqcup\alpha_T^{1_Q}$ and thus,

$$1_Q \leqslant e(\sqcup\alpha^q_s, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a\in A}(\alpha^q_s(a) \to e(a, \sqcup\alpha_T^{1_Q})) = q \to e(s, \sqcup\alpha_T^{1_Q}).$$

As a consequence, one obtains that $1_Q \leqslant \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha_T^{1_Q}))$ and, therefore, $\sqcup\alpha^q_{\bigvee S} \leqslant \sqcup\alpha_T^{1_Q}$.

For the converse inequality, one starts with the following:

$$e(\sqcup\alpha_T^{1_Q}, \sqcup\alpha^q_{\bigvee S}) = \bigwedge_{a\in A}(\alpha_T^{1_Q}(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) =$$
$$\bigwedge_{s\in S}(1_Q \to e(\sqcup\alpha^q_s, \sqcup\alpha^q_{\bigvee S})) = \bigwedge_{s\in S} e(\sqcup\alpha^q_s, \sqcup\alpha^q_{\bigvee S}) =$$
$$\bigwedge_{s\in S}\bigwedge_{a\in A}(\alpha^q_s(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) = \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha^q_{\bigvee S})).$$

To continue, we notice that

$$1_Q \leqslant e(\sqcup\alpha^q_{\bigvee S}, \sqcup\alpha^q_{\bigvee S}) = \bigwedge_{a\in A}(\alpha^q_{\bigvee S}(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) =$$
$$q \to e(\bigvee S, \sqcup\alpha^q_{\bigvee S})$$

and, therefore, $q \leqslant e(\bigvee S, \sqcup\alpha^q_{\bigvee S})$. For every $s \in S$, it follows that $q = 1_Q \otimes q \leqslant e(s, \bigvee S) \otimes e(\bigvee S, \sqcup\alpha^q_{\bigvee S}) \leqslant e(s, \sqcup\alpha^q_{\bigvee S})$ and, therefore, $1_Q \leqslant q \to e(s, \sqcup\alpha^q_{\bigvee S})$. As a consequence, one immediately obtains that $1_Q \leqslant \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha^q_{\bigvee S}))$, which then yields the desired $\sqcup\alpha_T^{1_Q} \leqslant \sqcup\alpha^q_{\bigvee S}$.

Item (2): For $S \subseteq Q$ and $a \in A$, it follows that $(\bigvee S) * a = \sqcup\alpha_a^{\bigvee S}$ and $\bigvee_{s\in S}(s * a) = \bigvee_{s\in S} \sqcup\alpha^s_a = \sqcup\alpha_T^{1_Q}$, where T is a shorthand for $\{\sqcup\alpha^s_a \mid s \in S\}$. To continue, we notice that

$$e(\sqcup\alpha_a^{\bigvee S}, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a'\in A}(\alpha_a^{\bigvee S}(a') \to e(a', \sqcup\alpha_T^{1_Q})) =$$
$$(\bigvee S) \to e(a, \sqcup\alpha_T^{1_Q}) = \bigwedge_{s\in S}(s \to e(a, \sqcup\alpha_T^{1_Q})).$$

For every $s \in S$, it follows that $\sqcup\alpha^s_a \leqslant \bigvee\{\sqcup\alpha_a^{s'} \mid s' \in S\} = \sqcup\alpha_T^{1_Q}$, which yields,

$$1_Q \leqslant e(\sqcup\alpha^s_a, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a'\in A}(\alpha^s_a(a') \to e(a', \sqcup\alpha_T^{1_Q})) = s \to e(a, \sqcup\alpha_T^{1_Q}).$$

As a result, one gets, $1_Q \leqslant \bigwedge_{s\in S}(s \to e(a, \sqcup \alpha_T^{1_Q}))$ and, therefore, the desired $\sqcup \alpha_a^{\bigvee S} \leqslant \sqcup \alpha_T^{1_Q}$ follows.

For the converse inequality, use the fact that

$$e(\sqcup \alpha_T^{1_Q}, \sqcup \alpha_a^{\bigvee S}) = \bigwedge_{a'\in A} (\alpha_T^{1_Q}(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) =$$
$$\bigwedge_{s\in S} (1_Q \to e(\sqcup \alpha_a^s, \sqcup \alpha_a^{\bigvee S})) = \bigwedge_{s\in S} e(\sqcup \alpha_a^s, \sqcup \alpha_a^{\bigvee S}) =$$
$$\bigwedge_{s\in S} \bigwedge_{a'\in A} (\alpha_a^s(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) = \bigwedge_{s\in S} (s \to e(a, \sqcup \alpha_a^{\bigvee S})) =$$
$$(\bigvee S) \to e(a, \sqcup \alpha_a^{\bigvee S}) = \bigwedge_{a'\in A} (\alpha_a^{\bigvee S}(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) =$$
$$e(\sqcup \alpha_a^{\bigvee S}, \sqcup \alpha_a^{\bigvee S}) \geqslant 1_Q.$$

Item (3): For $q_1, q_2 \in Q$ and $a \in A$, it follows that $q_1 * (q_2 * a) = q_1 * (\sqcup \alpha_a^{q_2}) = \sqcup \alpha_t^{q_1}$, where t is a shorthand for $\sqcup \alpha_a^{q_2}$, and $(q_1 \otimes q_2) * a = \sqcup \alpha_a^{q_1\otimes q_2}$. To continue, we notice that

$$e(\sqcup \alpha_t^{q_1}, \sqcup \alpha_a^{q_1\otimes q_2}) = \bigwedge_{a'\in A} (\alpha_t^{q_1}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2})) =$$
$$q_1 \to e(\sqcup \alpha_a^{q_2}, \sqcup \alpha_a^{q_1\otimes q_2}) = q_1 \to (\bigwedge_{a'\in A} (\alpha_a^{q_2}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2}))) =$$
$$q_1 \to (q_2 \to e(a, \sqcup \alpha_a^{q_1\otimes q_2})) = (q_1 \otimes q_2) \to e(a, \sqcup \alpha_a^{q_1\otimes q_2}) =$$
$$\bigwedge_{a'\in A} (\alpha_a^{q_1\otimes q_2}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2})) = e(\sqcup \alpha_a^{q_1\otimes q_2}, \sqcup \alpha_a^{q_1\otimes q_2}) \geqslant 1_Q$$

and, therefore, $\sqcup \alpha_t^{q_1} \leqslant \sqcup \alpha_a^{q_1\otimes q_2}$.

For the converse inequality, we notice that

$$e(\sqcup \alpha_a^{q_1\otimes q_2}, \sqcup \alpha_t^{q_1}) = \bigwedge_{a'\in A} (\alpha_a^{q_1\otimes q_2}(a') \to e(a', \sqcup \alpha_t^{q_1})) =$$
$$(q_1 \otimes q_2) \to e(a, \sqcup \alpha_t^{q_1}) = q_1 \to (q_2 \to e(a, \sqcup \alpha_t^{q_1})) =$$
$$q_1 \to (\bigwedge_{a'\in A} (\alpha_a^{q_2}(a') \to e(a', \sqcup \alpha_t^{q_1}))) = q_1 \to e(\sqcup \alpha_a^{q_2}, \sqcup \alpha_t^{q_1}) =$$
$$\bigwedge_{a'\in A} (\alpha_t^{q_1}(a') \to e(a', \sqcup \alpha_t^{q_1})) = e(\sqcup \alpha_t^{q_1}, \sqcup \alpha_t^{q_1}) \geqslant 1_Q.$$

Item (4): Given $a \in A$, it follows that $1_Q * a = \sqcup \alpha_a^{1_Q} = \bigvee\{a\} = a$.

To show that the functor is correct on morphisms, notice that given $S \subseteq A$, we get, $\varphi(\bigvee S) = \varphi(\sqcup \alpha_S^{1_Q}) = \sqcup \varphi_Q^{\rightarrow}(\alpha_S^{1_Q}) \stackrel{(\dagger)}{=} \sqcup \alpha_{\varphi^{\rightarrow}(S)}^{1_Q} = \bigvee \varphi^{\rightarrow}(S)$, where (†) uses Lemma 22. Moreover, given $q \in Q$ and $a \in A$, it follows that $\varphi(q * a) = \varphi(\sqcup \alpha_a^q) = \sqcup \varphi_Q^{\rightarrow}(\alpha_a^q) \stackrel{(\dagger)}{=} \sqcup \alpha_{\varphi(a)}^q = q * \varphi(a)$, where (†) again relies on Lemma 22. □

Having constructed the two functors, we can prove the main result of this subsection and one of the main (and most interesting) results of this paper. More precisely, the following theorem provides a relation between lattice-valued $\bigvee$-semilattices of Definition 19, which are expressed through fuzzy concepts (e.g., fuzzy sets and fuzzy order) and quantale modules of Definition 9, which is a notion expressed in terms of universal algebra. As a consequence, one gets an additional tool for dealing with many-valued partial orders. In particular, the tool in question (i.e., the theory of quantale modules) is already rather well developed (see, e.g., [36, 43, 64]), which opens the possibility to bring an unsolved problem from the theory of lattice-valued partial orders to the theory of quantale modules, solve it in the new framework, and get the answer back to the initiating one. We obtain thus an analogue of the results of the theory of "natural dualities" [8], which allows an easy interchange between algebraic problems, usually stated in an abstract symbolic language, and their dual, topological problems, where geometric intuition comes to our help.

24. Theorem. *Given a unital commutative quantale Q, $G \circ F = 1_{\mathbf{Mod}(Q)}$ and $F \circ G = 1_{\mathbf{Sup}(Q)}$, i.e., the two categories $\mathbf{Mod}(Q)$ and $\mathbf{Sup}(Q)$ are isomorphic.*

Proof. Given a Q-module $(A, *)$, $G \circ F(A, *) = G(A, e, \sqcup) = (A, \leqslant', \bigvee', *')$. On the other hand, given $a_1, a_2 \in A$, $a_1 \leqslant' a_2$ iff $1_Q \leqslant e(a_1, a_2) = a_1 \twoheadrightarrow a_2$ iff $a_1 = 1_Q * a_1 \leqslant a_2$. Then $\bigvee = \bigvee'$, which can be verified directly, since given $S \subseteq A$, $\bigvee' S = \sqcup\, \alpha_S^{1_Q} = \bigvee_{a \in A} (\alpha_S^{1_Q}(a) * a) = \bigvee_{s \in S} (1_Q * s) = \bigvee S$. Moreover, given $q \in Q$ and $a \in A$, $q *' a = \sqcup\, \alpha_a^q = \bigvee_{a' \in A} (\alpha_a^q(a') * a') = q * a$. Altogether, it follows that $(A, \leqslant', \bigvee', *') = (A, \leqslant, \bigvee, *)$.

Given a Q-$\bigvee$-semilattice $(A, e, \sqcup)$, $F \circ G(A, e, \sqcup) = F(A, \leqslant, \bigvee, *) = (A, e', \sqcup')$. On the other hand, given $a_1, a_2 \in A$, it follows that

$$e'(a_1, a_2) = a_1 \twoheadrightarrow a_2 = \bigvee \{q \in Q \mid q * a_1 \leqslant a_2\} = \bigvee \{q \in Q \mid 1_Q \leqslant e(\sqcup\, \alpha_{a_1}^q, a_2)\} = (\dagger).$$

Since $e(\sqcup\, \alpha_{a_1}^q, a_2) = \bigwedge_{a \in A} (\alpha_{a_1}^q(a) \to e(a, a_2)) = q \to e(a_1, a_2)$, we get that

$$(\dagger) = \bigvee \{q \in Q \mid 1_Q \leqslant q \to e(a_1, a_2)\} = \bigvee \{q \in Q \mid q \leqslant e(a_1, a_2)\} = e(a_1, a_2)$$

and, therefore, $e'(a_1, a_2) = e(a_1, a_2)$. Given $\alpha \in Q^A$, $\sqcup' \alpha = \bigvee_{a \in A} (\alpha(a) * a) = \bigvee_{a \in A} \sqcup\, \alpha_a^{\alpha(a)} = \sqcup\, \alpha_T^{1_Q}$, where T is a shorthand for $\{\sqcup\, \alpha_a^{\alpha(a)} \mid a \in A\}$. Given $a' \in A$,

$$e(\sqcup\, \alpha_T^{1_Q}, a') = \bigwedge_{a'' \in A} (\alpha_T^{1_Q}(a'') \to e(a'', a')) = \bigwedge_{a \in A} (1_Q \to e(\sqcup\, \alpha_a^{\alpha(a)}, a')) = \bigwedge_{a \in A} e(\sqcup\, \alpha_a^{\alpha(a)}, a') = \bigwedge_{a \in A} \bigwedge_{a'' \in A} (\alpha_a^{\alpha(a)}(a'') \to e(a'', a')) = \bigwedge_{a \in A} (\alpha(a) \to e(a, a')) = e(\sqcup\, \alpha, a')$$

and thus, $\sqcup' \alpha = \sqcup\, \alpha$. Taken together, it follows that $(A, e', \sqcup') = (A, e, \sqcup)$. □

Notice that Theorem 24 essentially provides two descriptions of the same concept. In the current paper, we are inclined to favor the category $\mathbf{Mod}(Q)$, whose many properties are already known, and (which is more important) whose definition enjoys an easy and straightforward universally algebraic presentation. The subsequent results of this paper will provide additional reasons for our viewpoint.

4.2. Some properties of lattice-valued $\bigvee$-semilattices. Looking closely at the category $\mathbf{Sup}(Q)$ of lattice-valued $\bigvee$-semilattices from the previous subsection, an experienced reader could ask whether its properties resemble those of the well-known and much studied category **Sup**. A more general question on the overall fruitfulness of such an extension is ultimately looming in the background. It is the main purpose of this subsection, to remove the possible doubts of that kind through considering several simple (but important) properties of the category $\mathbf{Sup}(Q)$. More precisely, we restate several of the properties of skeletal Q-categories (already obtained by, e.g., I. Stubbe [73, 74, 75, 76]) in lattice-valued terms (cf., e.g., Lemma 30 and Proposition 31). Such a restatement is required for a better development of the theory of lattice-valued partial orders, whose tools are different from the already mentioned theory of skeletal Q-categories, based in the technique of enriched categories.

The first feature we extend is the trivial fact that every $\bigvee$-preserving map is automatically monotone. Our intuition suggests that the statement should be valid in the framework of the category $\mathbf{Sup}(Q)$ as well. Strikingly enough, however, the papers of W. Yao [78, 79, 80, 81] keep silence on the topic, strictly distinguishing between lattice-valued monotonicity and preservation of lattice-valued $\bigvee$. With the help of Theorem 24 from the previous subsection, we can clarify the matter. We begin with the extension of crisp monotonicity, modifying the respective many-valued concept of W. Yao [78, 79, 80, 81] developed for frames.

25. Definition. Given two Q-ordered sets (A, e) and (B, e), a map $A \xrightarrow{f} B$ is said to be *Q-monotone* provided that $e(a_1, a_2) \leqslant e(f(a_1), f(a_2))$ for every $a_1, a_2 \in A$. ■

Notice that we do not require the quantale Q to be commutative. On the other hand, if this is really the case, one easily obtains the following result.

26. Proposition. *Given a unital commutative quantale Q, every $\mathbf{Sup}(Q)$-morphism is Q-monotone.*

Proof. Given a $\mathbf{Sup}(Q)$-morphism $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$, there exists a $\mathbf{Mod}(Q)$-morphism $(A, *) \xrightarrow{\varphi} (B, *)$ such that $F((A, *) \xrightarrow{\varphi} (B, *)) = (A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ and, therefore, we can assume that the maps e, $\sqcup$ are induced by the action $*$. Given $a_1, a_2 \in A$, $q \leqslant e(a_1, a_2) = a_1 \twoheadrightarrow a_2$ implies $q * a_1 \leqslant a_2$ implies $q * \varphi(a_1) \leqslant \varphi(a_2)$ implies $q \leqslant \varphi(a_1) \twoheadrightarrow \varphi(a_2) = e(\varphi(a_1), \varphi(a_2))$. Altogether, $e(a_1, a_2) \leqslant e(\varphi(a_1), \varphi(a_2))$. □

Proposition 26 illustrates the technique, which will be used throughout this subsection, i.e., replacing the abstract maps e and $\sqcup$ of a Q-$\bigvee$-semilattice with their concrete realizations through a module action. Simple as it looks, the machinery is capable of providing several useful results.

Our next property extends another well-known result that every $\bigvee$-semilattice is actually a complete lattice, i.e., has additionally a $\bigwedge$-operation. This fact was heavily employed in the definition of Q-$\bigvee$-semilattices in the previous subsection and also in the most important results of the latter and, therefore, the simple property should be most welcome in the extended framework. In the following, we show that this really is the case. Start with the extension of the crisp $\bigwedge$-operation to our new framework (notice that we still follow the frame path of W. Yao [78, 79, 80, 81]).

27. Definition. Given a Q-poset (A, e), the map $Q^A \xrightarrow{\sqcap} A$ is called a *Q_r-meet operation* on A provided that $e(a, \sqcap \alpha) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow_r e(a, a'))$ for every $\alpha \in Q^A$ and every $a \in A$. ■

Replacing $\cdot \rightarrow_r \cdot$ with $\cdot \rightarrow_l \cdot$, one obtains the concept of *Q_l-meet operation.* The case of a commutative quantale Q provides a nice property of these notions.

28. Proposition. *Given a unital commutative quantale Q, every $\mathbf{Sup}(Q)$-object has Q-meets.*

Proof. Given a Q-$\bigvee$-semilattice $(A, e, \sqcup)$, we know that both e and $\sqcup$ are induced by a module action $*$ on A. Define a map $Q^A \xrightarrow{\sqcap} A$ by $\sqcap \alpha = \bigwedge_{a \in A}(\alpha(a) \rightsquigarrow a)$ (recall the notation, stated before Lemma 17). To show that the map is the desired Q-meet operation on A, notice that given $\alpha \in Q^A$ and $a \in A$, it follows that $e(a, \sqcap \alpha) = a \twoheadrightarrow \sqcap \alpha = a \twoheadrightarrow (\bigwedge_{a' \in A}(\alpha(a') \rightsquigarrow a')) = \bigwedge_{a' \in A}(a \twoheadrightarrow (\alpha(a') \rightsquigarrow a')) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow (a \twoheadrightarrow a')) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow e(a, a'))$. □

It should be underlined that in case of lattice-valued frames, W. Yao [78, 79] provides a stronger result, namely, that the conditions of the existence of L-join- or L-meet operation for a frame L are equivalent. We will not pursue, however, the topic any further, which would lead us off the goal of the paper.

The last property concerns the concept of Galois connection on $\bigvee$-semilattices. The standard result (see, e.g., [11] or [16, Section 0-3]) says that every **Sup**-morphism $(A,\bigvee)\xrightarrow{\varphi}(B,\bigvee)$ has an upper adjoint map $B\xrightarrow{\psi}A$ characterized uniquely by the condition $\varphi(a)\leqslant b$ iff $a\leqslant\psi(b)$, for every $a\in A$ and every $b\in B$. The explicit formula for the map is then given by $\psi(b)=\bigvee\{a\in A\,|\,\varphi(a)\leqslant b\}=\bigvee\varphi^{\leftarrow}(\downarrow b)$, where $\downarrow b=\{b'\in B\,|\,b'\leqslant b\}$. Moreover, one can show that ψ is $\bigwedge$-preserving. Since the above machinery was much used in the previous subsection, its analogue in the extended setting seems to be highly desirable. In the following, we provide its generalization, employing the frame notions of W. Yao [78, 79, 81].

29. Definition. Given Q-posets (A,e) and (B,e), a pair (g,f) of maps $A\xrightarrow{f}B$ and $B\xrightarrow{g}A$ is a *Q-Galois connection* or a *Q-adjunction* between (A,e) and (B,e) provided that $e(f(a),b)=e(a,g(b))$ for every $a\in A$ and every $b\in B$. The map f (resp. g) is called *Q-lower* (resp. *Q-upper*) *adjoint*. ∎

The following lemma provides the extension of two well-known properties of Galois connections.

30. Lemma. *Given a Q-Galois connection (g,f) between (A,e) and (B,e), the following hold:*

(1) *both g and f are Q-monotone;*
(2) *g (resp. f) preserves the existing Q-(r,l)-$\bigwedge$ (resp. Q-(r,l)-$\bigvee$).*

Proof. To show Item (1), notice that given $a_1,a_2\in A$, $e(f(a_1),f(a_2))=e(a_1,g\circ f(a_2))\geqslant e(a_1,a_2)\otimes e(a_2,g\circ f(a_2))=e(a_1,a_2)\otimes e(f(a_2),f(a_2))\geqslant e(a_1,a_2)\otimes 1_Q=e(a_1,a_2)$. On the other hand, given $b_1,b_2\in B$, it follows that $e(g(b_1),g(b_2))=e(f\circ g(b_1),b_2)\geqslant e(f\circ g(b_1),b_1)\otimes e(b_1,b_2)=e(g(b_1),g(b_1))\otimes e(b_1,b_2)\geqslant 1_Q\otimes e(b_1,b_2)=e(b_1,b_2)$.

For Item (2), use the fact that given $\alpha\in Q^A$ such that $\sqcap\alpha$ exists and $a\in A$,

$$\begin{aligned}e(a,g(\sqcap\alpha))=e(f(a),\sqcap\alpha)=\bigwedge_{b\in B}(\alpha(b)\rightarrow_{r,l}e(f(a),b))=\\ \bigwedge_{b\in B}(\alpha(b)\rightarrow_{r,l}e(a,g(b)))=\bigwedge_{a'\in A}\bigwedge_{g(b)=a'}(\alpha(b)\rightarrow_{r,l}e(a,g(b)))=\\ \bigwedge_{a'\in A}\bigwedge_{g(b)=a'}(\alpha(b)\rightarrow_{r,l}e(a,a'))=\bigwedge_{a'\in A}((\bigvee_{g(b)=a'}\alpha(b))\rightarrow_{r,l}e(a,a'))=\\ \bigwedge_{a'\in A}((g_Q^{\rightarrow}(\alpha))(a')\rightarrow_{r,l}e(a,a')).\end{aligned}$$

It follows that $\sqcap g_Q^{\rightarrow}(\alpha)$ exists and equals $g(\sqcap\alpha)$.

Given $\alpha \in Q^A$ such that $\sqcup\,\alpha$ exists and $b \in B$,

$$e(f(\sqcup\,\alpha), b) = e(\sqcup\,\alpha, g(b)) = \bigwedge_{a\in A}(\alpha(a) \to_{r,l} e(a, g(b))) =$$
$$\bigwedge_{a\in A}(\alpha(a) \to_{r,l} e(f(a), b)) = \bigwedge_{b'\in B}\bigwedge_{f(a)=b'}(\alpha(a) \to_{r,l} e(f(a), b)) =$$
$$\bigwedge_{b'\in B}\bigwedge_{f(a)=b'}(\alpha(a) \to_{r,l} e(b', b)) = \bigwedge_{b'\in B}((\bigvee_{f(a)=b'}\alpha(a)) \to_{r,l} e(b', b)) =$$
$$\bigwedge_{b'\in B}((f_Q^{\rightarrow}(\alpha))(b') \to_{r,l} e(b', b)).$$

It follows that $\sqcup\, f_Q^{\rightarrow}(\alpha)$ exists and equals $f(\sqcup\,\alpha)$. □

Notice that in order to illustrate the extension of the classical duality machinery to the fuzzy setting, Lemma 30 provides the proofs, which usually are replaced with something like "follows through duality".

Turning back to quantale modules, to employ the standard machinery, we introduce a simple notation. Given a Q-poset (A, e), every $a \in A$ provides a map $A \xrightarrow{\downarrow_e a} Q$ defined by $(\downarrow_e a)(b) = e(b, a)$ (notice the fuzzification of the above-mentioned lower set $\downarrow a$).

31. Proposition. *Given a unital commutative quantale Q, every* $\mathbf{Sup}(Q)$*-morphism* $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ *has a Q-upper adjoint.*

Proof. We again assume that the maps e, $\sqcup$ are induced by their respective module actions on A. Define a map $B \xrightarrow{\psi} A$ by $\psi(b) = \sqcup\,\varphi_Q^{\leftarrow}(\downarrow_e b)$. To check the adjunction property, notice that given $a \in A$ and $b \in B$, $e(\varphi(a), b) = \varphi(a) \twoheadrightarrow b$, whereas $e(a, \psi(b)) = a \twoheadrightarrow \psi(b)$, where

$$\psi(b) = \sqcup\,\varphi_Q^{\leftarrow}(\downarrow_e b) = \bigvee_{a'\in A}((\varphi_Q^{\leftarrow}(\downarrow_e b))(a') * a') = \bigvee_{a'\in A}((\downarrow_e b)(\varphi(a')) * a') =$$
$$\bigvee_{a'\in A}(e(\varphi(a'), b) * a') = \bigvee_{a'\in A}((\varphi(a') \twoheadrightarrow b) * a') = \bigvee_{a'\in A}((\bigvee_{q*\varphi(a')\leqslant b} q) * a') =$$
$$\bigvee_{a'\in A}\bigvee_{q*\varphi(a')\leqslant b}(q * a') = \bigvee_{q*\varphi(a')\leqslant b}(q * a')$$

and, therefore, $e(a, \psi(b)) = a \twoheadrightarrow (\bigvee_{q*\varphi(a')\leqslant b}(q * a')) = a \twoheadrightarrow (\bigvee S)$. Given $q \in Q$, $q \leqslant \varphi(a) \twoheadrightarrow b$ implies $q * \varphi(a) \leqslant b$ implies $q * a \in S$ implies $q * a \leqslant \bigvee S$ implies $q \leqslant a \twoheadrightarrow (\bigvee S)$. On the other hand, $q \leqslant a \twoheadrightarrow (\bigvee S)$ implies $q * a \leqslant \bigvee S$ implies $\varphi(q * a) \leqslant \varphi(\bigvee S)$ implies $q * \varphi(a) \leqslant \bigvee_{q*\varphi(a')\leqslant b}(q * \varphi(a')) \leqslant b$ implies $q \leqslant \varphi(a) \twoheadrightarrow b$. Altogether, one obtains, $e(\varphi(a), b) = e(a, \psi(b))$. □

The challenging task of generalizing other important results to the new setting will be left to the subsequent developments of the topic, whereas here, we will extend Q-$\bigvee$-semilattices to lattice-valued quantales.

4.3. Quantale algebras as lattice-valued quantales. This subsection provides the main result of the section, namely, a representation of quantale algebras as lattice-valued quantales. With the concept of lattice-valued frame of W. Yao [79, 81] in mind, we introduce the latter notion in the following way (cf. the crisp case of Definition 2).

32. Definition. Given a unital quantale Q, a Q_r*-quantale* is a tuple $(A, e, \sqcup, \otimes)$, where $(A, e, \sqcup)$ is a Q_r-$\bigvee$-semilattice and $A \times A \xrightarrow{\otimes} A$ is a map (Q*-multiplication* on A) such that

(1) $(A, \otimes)$ is a semigroup;
(2) $a \otimes (\sqcup \alpha) = \sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha)$ and $(\sqcup \alpha) \otimes a = \sqcup (\cdot \otimes a)^{\rightarrow}_{Q}(\alpha)$ for every $a \in A$ and every $\alpha \in Q^{A}$.

A Q_r*-quantale homomorphism* $(A, e, \sqcup, \otimes) \xrightarrow{\varphi} (B, e, \sqcup, \otimes)$ is a map $A \xrightarrow{\varphi} B$, which is a Q_r-$\bigvee$-semilattice homomorphism $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ such that $\varphi(a_1 \otimes a_2) = \varphi(a_1) \otimes \varphi(a_2)$ for every $a_1, a_2 \in A$. $\mathbf{Quant}_r(Q)$ is the category of Q_r-quantales and their homomorphisms, concrete over both $\mathbf{Sup}_r(Q)$ and $\mathbf{SGrp}$. ∎

Similarly, one gets the category $\mathbf{Quant}_l(Q)$. Below, we generalize the fact mentioned after Definition 10 that the category $\mathbf{Alg}(\mathbf{2})$ is isomorphic to the category $\mathbf{Quant}$ (cf. the isomorphism between the category $\mathbf{Alg}(\mathbb{Z})$ of algebras over the ring of integers $\mathbb{Z}$ and the category $\mathbf{Rng}$ of rings [30]), namely, we show that given a unital commutative quantale Q, the categories $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$ are isomorphic (notice that $\mathbf{Quant}_l(Q) = \mathbf{Quant}_r(Q) = \mathbf{Quant}(Q)$ for a commutative quantale Q). The underlying machinery will rely on the isomorphism between the categories $\mathbf{Mod}(Q)$ and $\mathbf{Sup}(Q)$ of Theorem 24.

33. Proposition. *Given a unital commutative quantale* Q*, there exists a functor* $\mathbf{Alg}(Q) \xrightarrow{F} \mathbf{Quant}(Q)$ *defined by* $F((A, \otimes, *) \xrightarrow{\varphi} (B, \otimes, *)) = (A, e, \sqcup, \otimes) \xrightarrow{\varphi} (B, e, \sqcup, \otimes)$*, where the maps* e *and* $\sqcup$ *are obtained as in Proposition 21.*

Proof. In view of Proposition 21, it will be enough to check the correctness of the functor on objects and that will follow from verification of Item (2) of Definition 32. Given $a \in A$ and $\alpha \in Q^{A}$, for every $\bar{a} \in A$,

$$\begin{aligned} e(\sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha), \bar{a}) &= \bigwedge_{a' \in A} (((a \otimes \cdot)^{\rightarrow}_{Q}(\alpha))(a') \rightarrow e(a', \bar{a})) = \\ \bigwedge_{a' \in A} ((\bigvee_{a \otimes a'' = a'} \alpha(a'')) \rightarrow (a' \twoheadrightarrow \bar{a})) &= \bigwedge_{a' \in A} \bigwedge_{a \otimes a'' = a'} (\alpha(a'') \rightarrow (a' \twoheadrightarrow \bar{a})) = \\ \bigwedge_{a' \in A} \bigwedge_{a \otimes a'' = a'} (\alpha(a'') \rightarrow ((a \otimes a'') \twoheadrightarrow \bar{a})) &= \bigwedge_{a'' \in A} (\alpha(a'') \rightarrow ((a \otimes a'') \twoheadrightarrow \bar{a})) = \\ \bigwedge_{a'' \in A} ((\alpha(a'') * (a \otimes a'')) \twoheadrightarrow \bar{a}) &= \bigwedge_{a'' \in A} ((a \otimes (\alpha(a'') * a'')) \twoheadrightarrow \bar{a}) = \\ (\bigvee_{a'' \in A} (a \otimes (\alpha(a'') * a''))) \twoheadrightarrow \bar{a} &= (a \otimes (\bigvee_{a'' \in A} (\alpha(a'') * a''))) \twoheadrightarrow \bar{a} = \\ & (a \otimes (\sqcup \alpha)) \twoheadrightarrow \bar{a} = e(a \otimes (\sqcup \alpha), \bar{a}). \end{aligned}$$

As a result, one obtains that $a \otimes (\sqcup \alpha) = \sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha)$.

On the other hand,

$$\begin{aligned} e(\sqcup (\cdot \otimes a)^{\rightarrow}_{Q}(\alpha), \bar{a}) &= \bigwedge_{a' \in A} (((\cdot \otimes a)^{\rightarrow}_{Q}(\alpha))(a') \rightarrow e(a', \bar{a})) = \\ \bigwedge_{a' \in A} ((\bigvee_{a'' \otimes a = a'} \alpha(a'')) \rightarrow (a' \twoheadrightarrow \bar{a})) &= \bigwedge_{a' \in A} \bigwedge_{a'' \otimes a = a'} (\alpha(a'') \rightarrow (a' \twoheadrightarrow \bar{a})) = \\ \bigwedge_{a' \in A} \bigwedge_{a'' \otimes a = a'} (\alpha(a'') \rightarrow ((a'' \otimes a) \twoheadrightarrow \bar{a})) &= \bigwedge_{a'' \in A} (\alpha(a'') \rightarrow ((a'' \otimes a) \twoheadrightarrow \bar{a})) = \\ \bigwedge_{a'' \in A} ((\alpha(a'') * (a'' \otimes a)) \twoheadrightarrow \bar{a}) &= \bigwedge_{a'' \in A} (((\alpha(a'') * a'') \otimes a) \twoheadrightarrow \bar{a}) = \\ (\bigvee_{a'' \in A} ((\alpha(a'') * a'') \otimes a)) \twoheadrightarrow \bar{a} &= ((\bigvee_{a'' \in A} (\alpha(a'') * a'')) \otimes a) \twoheadrightarrow \bar{a} = \\ & ((\sqcup \alpha) \otimes a) \twoheadrightarrow \bar{a} = e((\sqcup \alpha) \otimes a, \bar{a}). \end{aligned}$$

As a result, we get that $(\sqcup\alpha)\otimes a=\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha)$. $\square$

Notice that to illustrate the use of the properties of quantale algebras, we provided the full proof for both the right and the left distributivity laws.

34. Proposition. *Given a unital commutative quantale Q, there exists a functor* $\mathbf{Quant}(Q)\xrightarrow{G}\mathbf{Alg}(Q)$, $G((A,e,\sqcup,\otimes)\xrightarrow{\varphi}(B,e,\sqcup,\otimes))=(A,\leqslant,\bigvee,*,\otimes)\xrightarrow{\varphi}(B,\leqslant,\bigvee,*,\otimes)$, *where $\leqslant$, $\bigvee$ and $*$ are obtained as in Proposition 23.*

Proof. In view of Proposition 23, it will be enough to show that the functor is correct on objects and that will follow from verification of Item (3) of Definition 2 and Item (3) of Definition 10.

For the first item, notice that given $S\subseteq A$ and $a\in A$, $a\otimes(\bigvee S)=a\otimes(\sqcup\alpha_S^{1_Q})=\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q})$ and $\bigvee_{s\in S}(a\otimes s)=\sqcup\alpha_T^{1_Q}$, where T is a shorthand for $\{a\otimes s\,|\,s\in S\}$. For every $\bar{a}\in A$, it follows that

$$e(\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a})=\bigwedge_{a'\in A}(((a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}))(a')\rightarrow e(a',\bar{a}))=$$
$$\bigwedge_{a'\in A}((\bigvee_{a\otimes a''=a'}\alpha_S^{1_Q}(a''))\rightarrow e(a',\bar{a}))=\bigwedge_{a'\in A}\bigwedge_{a\otimes a''=a'}(\alpha_S^{1_Q}(a'')\rightarrow e(a',\bar{a}))=$$
$$\bigwedge_{s\in S}(1_Q\rightarrow e(a\otimes s,\bar{a}))=\bigwedge_{s\in S}e(a\otimes s,\bar{a}).$$

On the other hand,

$$e(\sqcup\alpha_T^{1_Q},\bar{a})=\bigwedge_{a'\in A}(\alpha_T^{1_Q}(a')\rightarrow e(a',\bar{a}))=\bigwedge_{s\in S}(1_Q\rightarrow e(a\otimes s,\bar{a}))=$$
$$\bigwedge_{s\in S}e(a\otimes s,\bar{a}).$$

Altogether, $e(\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a})=e(\sqcup\alpha_T^{1_Q},\bar{a})$ and, therefore, $\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q})=\sqcup\alpha_T^{1_Q}$, which yields then the desired $a\otimes(\bigvee S)=\bigvee_{s\in S}(a\otimes s)$.

To show the second distributivity law, notice that $(\bigvee S)\otimes a=(\sqcup\alpha_S^{1_Q})\otimes a=\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q})$ and $\bigvee_{s\in S}(s\otimes a)=\sqcup\alpha_T^{1_Q}$, where T is a shorthand for $\{s\otimes a\,|\,s\in S\}$. For every $\bar{a}\in A$, it follows that

$$e(\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a})=\bigwedge_{a'\in A}((((\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}))(a')\rightarrow e(a',\bar{a}))=$$
$$\bigwedge_{a'\in A}((\bigvee_{a''\otimes a=a'}\alpha_S^{1_Q}(a''))\rightarrow e(a',\bar{a}))=\bigwedge_{a'\in A}\bigwedge_{a''\otimes a=a'}(\alpha_S^{1_Q}(a'')\rightarrow e(a',\bar{a}))=$$
$$\bigwedge_{s\in S}(1_Q\rightarrow e(s\otimes a,\bar{a}))=\bigwedge_{s\in S}e(s\otimes a,\bar{a}).$$

Moreover,

$$e(\sqcup\alpha_T^{1_Q},\bar{a})=\bigwedge_{a'\in A}(\alpha_T^{1_Q}(a')\rightarrow e(a',\bar{a}))=\bigwedge_{s\in S}(1_Q\rightarrow e(s\otimes a,\bar{a}))=$$
$$\bigwedge_{s\in S}e(s\otimes a,\bar{a}).$$

As a result, we obtain that $e(\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a})=e(\sqcup\alpha_T^{1_Q},\bar{a})$, namely, $\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q})=\sqcup\alpha_T^{1_Q}$, which provides then the desired $(\bigvee S)\otimes s=\bigvee_{s\in S}(s\otimes a)$.

For the second item, notice that given $q\in Q$ and $a_1,a_2\in A$, $q*(a_1\otimes a_2)=\sqcup\alpha^q_{a_1\otimes a_2}$, $(q*a_1)\otimes a_2=(\sqcup\alpha^q_{a_1})\otimes a_2=\sqcup(\cdot\otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1})$ and $a_1\otimes(q*a_2)=a_1\otimes(\sqcup\alpha^q_{a_2})=$

$\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^{q}_{a_2})$. For every $\bar{a} \in A$, it follows that

$$e(\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^{q}_{a_1}), \bar{a}) = \bigwedge_{a \in A}(((\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^{q}_{a_1}))(a) \rightarrow e(a, \bar{a})) =$$
$$\bigwedge_{a \in A}((\bigvee_{a' \otimes a_2 = a} \alpha^{q}_{a_1}(a')) \rightarrow e(a, \bar{a})) = \bigwedge_{a \in A} \bigwedge_{a' \otimes a_2 = a} (\alpha^{q}_{a_1}(a') \rightarrow e(a, \bar{a})) =$$
$$q \rightarrow e(a_1 \otimes a_2, \bar{a})$$

as well as

$$e(\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^{q}_{a_2}), \bar{a}) = \bigwedge_{a \in A}(((a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^{q}_{a_2}))(a) \rightarrow e(a, \bar{a})) =$$
$$\bigwedge_{a \in A}((\bigvee_{a_1 \otimes a' = a} \alpha^{q}_{a_2}(a')) \rightarrow e(a, \bar{a})) = \bigwedge_{a \in A} \bigwedge_{a_1 \otimes a' = a} (\alpha^{q}_{a_2}(a') \rightarrow e(a, \bar{a})) =$$
$$q \rightarrow e(a_1 \otimes a_2, \bar{a}).$$

On the other hand, we obtain that

$$e(\sqcup \alpha^{q}_{a_1 \otimes a_2}, \bar{a}) = \bigwedge_{a \in A}(\alpha^{q}_{a_1 \otimes a_2}(a) \rightarrow e(a, \bar{a})) = q \rightarrow e(a_1 \otimes a_2, \bar{a}).$$

As a consequence, one gets that

$$e(\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^{q}_{a_1}), \bar{a}) = e(\sqcup \alpha^{q}_{a_1 \otimes a_2}, \bar{a}) = e(\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^{q}_{a_2}), \bar{a}).$$

It immediately follows that $\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^{q}_{a_1}) = \sqcup \alpha^{q}_{a_1 \otimes a_2} = \sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^{q}_{a_2})$, which then gives the desired equality $(q * a_1) \otimes a_2 = q * (a_1 \otimes a_2) = a_1 \otimes (q * a_2)$. □

The two propositions in hand, we can prove the main result of this section.

35. Theorem. *Given a unital commutative quantale Q, $G \circ F = 1_{\mathbf{Alg}(Q)}$ and $F \circ G = 1_{\mathbf{Quant}(Q)}$, i.e., the two categories $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$ are isomorphic.*

Proof. Follows from Theorem 24, in view of Propositions 33, 34. □

Similar to the case of Theorem 24, Theorem 35 provides two descriptions of the same concept. It is our opinion that $\mathbf{Alg}(Q)$ is better suited for applications due to its compact universally algebraic definition and a certain knowledge on its properties. The next section will give more reasons for such an opinion.

4.4. Quantale algebras as lattice-valued frames. The main results of the previous two subsections can be summarized as follows (we notice that the prefix "**U**" in the notations for categories in Theorem 36 stands for "unital", which in case of, e.g., the category $\mathbf{UQuant}(Q)$ means that Item (1) of Definition 32 employs a monoid $(A, \otimes, 1)$).

36. Theorem. *Given a unital commutative quantale Q, the categories $(Q \downarrow \mathbf{UQuant})_z$, $\mathbf{UAlg}(Q)$ and $\mathbf{UQuant}(Q)$ are isomorphic.*

Proof. Follows from Theorems 16, 35 and the construction of functors of Propositions 33, 34. □

The isomorphism between $(Q \downarrow \mathbf{UQuant})_z$ and $\mathbf{UAlg}(Q)$ is more demanding, since it requires the existence of the unit in the underlying quantales of Q-algebras, and the isomorphism between $\mathbf{UAlg}(Q)$ and $\mathbf{UQuant}(Q)$ is the restriction of a more general one between $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$. In other words, one easily gets the following result (see the construction of the functors of Propositions 33, 34).

37. Corollary. *For every unital commutative quantale Q, the category $(Q \downarrow \mathbf{UQuant})_z$ is isomorphic to a non-full subcategory of the category $\mathbf{Quant}(Q)$.*

Corollary 37 acquires more importance when one considers the concepts of lattice-valued frame of D. Zhang and Y.-M. Liu [84] as well as W. Yao [79]. To make the handling of the corresponding situation easier, below we introduce two additional categories.

38. Definition. For a frame L, $\mathbf{UAlg}_{Frm}(L)$ is the full subcategory of $\mathbf{UAlg}(L)$, whose objects have frames as their underlying quantales. $\mathbf{Frm}(L)$ is the image of the subcategory $\mathbf{UAlg}_{Frm}(L)$ under the isomorphism F of Proposition 33. ▪

It is easy to check that the category $\mathbf{Frm}(L)$ is isomorphic to the category L-$\mathbf{Frm}_Y$ of L-frames of W. Yao [81]. Also notice the double simplification of the category $\mathbf{Alg}(Q)$, not only taking frames as the underlying algebraic structures of quantale algebras, but also replacing the quantale Q with a frame L. Such a reduced case makes Corollary 37 stronger and, possibly, more interesting.

39. Corollary. *Given a frame L, the categories $(L \downarrow \mathbf{Frm})$ and $\mathbf{Frm}(L)$ are isomorphic.*

Proof. By Theorem 16, $(L \downarrow \mathbf{Frm})$ is isomorphic to the category $\mathbf{UAlg}_{Frm}(L)$. □

Since the category $(L \downarrow \mathbf{Frm})$ provides the concept of lattice-valued frame of D. Zhang and Y.-M. Liu [84], Corollary 39 says that the notions of D. Zhang and Y.-M. Liu as well as of W. Yao are categorically equivalent. It should be noticed immediately that W. Yao [81] obtained the same result. Corollary 39, however, provides a more general viewpoint on this relation and employs completely different machinery. In particular, Corollary 37 shows that the passage from frames to quantales makes the setting of D. Zhang and Y.-M. Liu different from that of W. Yao. Moreover, since both concepts of lattice-valued frame are instances of quantale algebras, by our opinion, they both are categorically redundant in mathematics. The next section elaborates our opinion in full extent.

5. Applications to lattice-valued topology

When looking closely into the papers, which introduce the concepts of lattice-valued frames, considered in this article, one sees immediately that both of them are motivated by the wish of their authors to extend the well-known equivalence between sober topological spaces and spatial locales [32] to the setting of lattice-valued topology. The crucial point here is the following. Since locales come essentially from the crisp world, e.g., are nicely and conveniently related to crisp topology, they can easily loose their efficiency in the lattice-valued framework. Indeed, one is confronted with the use of one and the same algebraic structure to encode the information on both crisp and lattice-valued topological spaces. While the passage from spaces to locales causes no difficulty, the converse transformation is liable to miss some information on its way. Despite the fact that S. E. Rodabaugh [52] successfully extended the crisp localic machinery to the lattice-valued case, later on, he himself cast certain doubts on its fruitfulness and introduced a fuzzification on the localic side as well, considering lattice-valued locales [49, 50]. The previous section gave another framework for dealing with the notion. It is the main purpose of this section to show its fruitfulness in this respect.

To begin with, we recall the concept of stratified topological space [47, 58]. Notice that given a set X and a $\bigvee$-semilattice L, the L-powerset L^A is a $\bigvee$-semilattice with the pointwise algebraic structure. The result is easily extendable to other algebraic structures, e.g., unital Q-algebras. Moreover, for every $a \in L$, we denote by $\underline{a}$ the constant map $X \xrightarrow{\underline{a}} L$ with the value a.

40. Definition. Given a unital quantale Q, a *Q-topological space* or *Q-space* is a pair (X, τ), where X is a set and τ is a unital subquantale of Q^X. Given Q-spaces (X, τ) and (Y, σ), a map $X \xrightarrow{f} Y$ is said to be *Q-continuous* provided that $(f_Q^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.

$\mathbf{Top}(Q)$ is the category of Q-topological spaces and Q-continuous maps, concrete over the category **Set**. ▪

41. Definition. Given a unital quantale Q, a Q-space (X, τ) is called *stratified* provided that $\{\underline{q} \mid q \in Q\} \subseteq \tau$. $\mathbf{STop}(Q)$ is the full subcategory of $\mathbf{Top}(Q)$ consisting of stratified Q-spaces. ▪

42. Definition. Given a unital quantale Q and a unital subquantale D of Q, a Q-space (X, τ) is called *stratified to degree* D provided that $\{\underline{q} \mid q \in D\} \subseteq \tau$. $\mathbf{STop}_D(Q)$ is the full subcategory of $\mathbf{Top}(Q)$ consisting of Q-spaces, which are stratified to degree D. ▪

Notice that the stratification idea is due to R. Lowen [38], the term itself first occurring in [46]. Stratification degree was first encountered by the author in [47]. It appears that there exists a nice relation between quantale algebras and (stratified) lattice-valued topological spaces. Start with one preliminary notion.

43. Definition. A unital Q-algebra A is said to be $*$-*divisible w.r.t.* 1_A (*divisible*, for short) provided that for every $a \in A$, there exists $q \in Q$ such that $a = q * 1_A$. ▪

Every unital quantale Q provides a unital divisible Q-algebra, since given $q \in Q$, $q = q \otimes 1_Q = q * 1_Q$. In particular, every frame L is a unital divisible L-algebra.

44. Proposition. *Let A be a unital Q-algebra and let (X, τ) be an A-space. If (X, τ) is stratified, then τ is a unital sub(Q-)algebra of A^X. If A is divisible and τ is a unital sub(Q-)algebra of A^X, then τ is stratified.*

Proof. For the first statement, notice that it is enough to check the closure of τ under the module action. Given $\alpha \in \tau$ and $q \in Q$, $(q * \alpha)(x) = q * \alpha(x) = q * (1_A \otimes \alpha(x)) = (q * 1_A) \otimes \alpha(x) = \underline{q * 1_A}(x) \otimes \alpha(x) = (\underline{q * 1_A} \otimes \alpha)(x)$ for every $x \in X$. As a result, $q * \alpha = \underline{q * 1_A} \otimes \alpha \in \tau$, since $\underline{q * 1_A} \in \tau$ by stratification.

For the second statement, notice that given $a \in A$, by the condition of the proposition, there exists some $q \in Q$ such that $a = q * 1_A$. Since $\underline{1_A} \in \tau$ and τ is a Q-module, $\underline{a} = \underline{q * 1_A} = q * \underline{1_A} \in \tau$. □

With Proposition 44 in hand, one obtains the following result.

45. Theorem. *Given a unital commutative quantale Q and a Q-algebra A, there is a functor $\mathbf{STop}(A) \xrightarrow{\Omega_A} \mathbf{UAlg}(Q)$ defined by $\Omega_A((X, \tau) \xrightarrow{f} (Y, \sigma)) = \tau \xrightarrow{(f_A^{\leftarrow})^{op}} \sigma$.*

The real power of the above result can be exploited in the framework of variety-based topology [63, 65]. In particular, one easily obtains the functor in the opposite direction as well as the related concepts of sobriety and spatiality, providing an equivalence between sober topological spaces and spatial Q-algebras (see [65], where the case $A = Q$ is considered). The resulting issue here is as follows. Since the concept of Q-algebra incorporates the above-mentioned two notions of lattice-valued frames, the respective extensions of the sobriety-spatiality equivalence of D. Zhang and Y.-M. Liu [84] and W. Yao [79] are particular instances of that for Q-algebras and, therefore, are categorically redundant in lattice-valued mathematics. Based in this observation, we strongly believe in the desirability to shift from lattice-valued frames to quantale algebras.

As a final remark, we notice that the passage from unital Q-algebras to stratified topologies in Proposition 44 requires divisibility of the respective Q-algebra A. Since, in general, the property rarely holds, it is time for stratification degree to come in play. Recall from Proposition 14 that every unital Q-algebra A provides a map $Q \xrightarrow{i_A} A$ defined by $i_A(q) = q * 1_A$ and denote by D_A the image of i_A.

46. Proposition. *Given a unital Q-algebra A, every A-space is stratified to degree D_A.*

Proof. Given an A-space (X,τ), and $a \in D_A$, there exists some $q \in Q$ such that $a = q * 1_A$. Similar to the proof of the second part of Proposition 44, one obtains that $\underline{a} \in \tau$. □

As a consequence, it follows that every category $\mathbf{Top}(A)$ over a unital Q-algebra A is essentially the category $\mathbf{STop}_{D_A}(A)$ of A-spaces stratified to degree D_A. The observation provides a convenient framework for studying the concept of stratification in lattice-valued topology.

6. Conclusion: open problems

Employing the isomorphism between the categories of right Q-modules and cocomplete skeletal Q-categories, obtained by I. Stubbe [76] for every unital quantale Q (in fact, a small quantaloid $\mathcal{Q}$), in this paper, we showed that the concept of quantale algebra, introduced recently [67] as a generalization of the well-known notion of algebra over a commutative ring with identity, has a significant merit of providing a common framework for (at least) two notions of lattice-valued frames available in the literature, namely, L-fuzzy frames of D. Zhang and Y.-M. Liu [84] and L-frames of W. Yao [79]. The obtained results suggest categorical redundancy of these concepts in mathematics in (at least) two respects. Firstly, both of them are isomorphic to particular subcategories of the category of quantale algebras and, moreover, are categorically equivalent to each other (as already observed by W. Yao [81]). Secondly, their motivating extensions of the classical equivalence of the categories of sober topological spaces and spatial locales to the lattice-valued world can be done much easier and more straightforward in the setting of quantale algebras. The quantale algebra extension in its turn follows from the results obtained in the realm of variety-based topology, providing another fruitful example of its usefulness as well as making its current generalization to categorically-algebraic (catalg) topology [62, 70] most desirable. Moreover, the isomorphism between the categories of quantale algebras and lattice-valued quantales of Theorem 35, suggest categorical redundancy of lattice-valued quantales (and, in particular, lattice-valued frames) in fuzzy mathematics. On the other hand, the results of Subsection 4.2 make the development of non-categorical properties of lattice-valued quantales highly desirable, in order to streamline and study deeper the classical properties of crisp quantales. It will be the topic of our forthcoming papers to investigate this issue in its full generality.

As it happens with every new theory, certain open problems arise in its development, some of which are worth (by our opinion) to be presented to the reader.

6.1. From lattice-valued frames to lattice-valued quantales. In Corollary 39, we showed categorical equivalence between the concepts of lattice-valued frame of D. Zhang, Y.-M. Liu [84] and W. Yao [79]. On the other hand, Corollary 37 shows that the frameworks are different in case of arbitrary quantales. In particular, it suggests that the setting of D. Zhang and Y.-M. Liu can be partly incorporated into that of W. Yao. The obtained relationships, however, are by no means complete, requiring further studies on the topic. At the moment, one can pose the following open problems.

47. Problem. Does the category $(Q \downarrow \mathbf{UQuant})_z$ provide a (co)reflective subcategory of $\mathbf{Quant}(Q)$? ■

48. Problem. Is the category $\mathbf{Quant}(Q)$ isomorphic to a subcategory of $(Q \downarrow \mathbf{Quant})$? ■

49. Problem. To what extent is it possible to lift the isomorphism of Corollary 39 to quantale setting? ■

The first problem deals with a generalization of the issue of adding a unit to a non-unital quantale considered in [69]. The last problem is ultimately the most important and, probably, the most difficult one.

6.2. Lattice-valued frames of A. Pultr and S. E. Rodabaugh. Having incorporated two concepts of lattice-valued frame in the setting of quantale algebras, we have exhausted the topic by no means. In particular, there exists another famous instance of the notion, introduced by A. Pultr and S. E. Rodabaugh [48] and studied by them further in [49, 50]. As has been mentioned in Introduction, its motivation came from the Lowen-Kubiák ι_L (fibre map) functor [37, 38]. As a result, the ultimate definition is more complicated than the respective concepts of this paper.

Start with a preliminary notation, namely, given a $\bigwedge$-semilattice L, let $L_\top$ denote the set $L\backslash\{\top\}$.

50. Definition. Given a chain L, an *L-frame* is a system of frame homomorphisms $A = (A^{\mathfrak{u}} \xrightarrow{\varphi_t^A} A^{\mathfrak{l}})_{t \in L_\top}$ such that

(1) $\varphi^A_{\bigwedge S} = \bigvee_{s\in S} \varphi^A_s$ for every non-empty $S \subseteq L_\top$;
(2) A is an extremal epi-sink;
(3) A is a mono-source. ∎

The condition of L being a chain deals mostly with the meet-irreducibles of L (as was pointed out by U. Höhle) and, therefore, its various modifications has already been considered by U. Höhle and S. E. Rodabaugh [28] as well as J. Gutiérrez García, U. Höhle and M. A. de Prada Vicente [21]. Despite these changes, the notion is still considerably out of the scope of the classical definitions of lattice-valued frames. In view of the results of this paper, the next problem springs into mind immediately.

51. Problem. Does there exist any connection between quantale algebras and lattice-valued frames of A. Pultr and S. E. Rodabaugh? ∎

Notice that while the concept of quantale algebra essentially provides an extension of partially ordered sets, employing generalization of partial order in the sense of Principle of Fuzzification of J. A. Goguen [17], the just mentioned notion of lattice-valued frame seems to be more sophisticated, the first of its conditions stemming from the realm of sheaves [49]. As a result, a quick look at Problem 51 inspired the author with nothing more than the following observations.

Every Q-algebra $(A, *)$ provides two families of maps: $\mathcal{A}_1 = (A \xrightarrow{q*\cdot} A)_{q\in Q}$ and $\mathcal{A}_2 = (Q \xrightarrow{\cdot * a} A)_{a \in A}$. Moreover, the Q-action on A can be restored from each of them. The next lemma shows several simple (but important) properties of these families.

52. Lemma. *Given a Q-algebra A, the following hold:*

(1) *every element of $\mathcal{A}_1$, $\mathcal{A}_2$ is a $\bigvee$-semilattice homomorphism;*
(2) *if every element of Q (resp. A) is idempotent w.r.t. the multiplication, then every element of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is a quantale homomorphism;*
(3) *$\mathcal{A}_1$ is both a mono-source and an epi-sink, whereas $\mathcal{A}_2$ is an epi-sink; both are extremal epi-sinks in the category* **Sup***;*
(4) *if $Q = \mathbf{2}$, then $\mathcal{A}_1 = (A \xrightarrow{\perp} A, A \xrightarrow{1_A} A)$, whereas $\mathcal{A}_2 = (\mathbf{2} \xrightarrow{\cdot * a} A)_{a\in A}$ with* $q * a = \begin{cases} a, & q = 1_{\mathbf{2}} \\ \perp, & \textit{otherwise}; \end{cases}$
(5) *if $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) satisfies Item (1) of Definition 50, then $(\bigwedge S) * a = (\bigvee S) * a$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$ (resp. A has no more than two elements).*

Proof. Item (1) follows from the properties of Q-algebras (Definition 10).

To show Item (2), notice that given $q \in Q$ and $a_1, a_2 \in A$, it follows that $q*(a_1 \otimes a_2) \overset{(\dagger)}{=} (q \otimes q) * (a_1 \otimes a_2) = q * (q * (a_1 \otimes a_2)) = q * (a_1 \otimes (q * a_2)) = (q * a_1) \otimes (q * a_2)$, where (†) uses the idempotency of Q. On the other hand, given $a \in A$ and $q_1, q_2 \in Q$, $(q_1 \otimes q_2) * a \overset{(\dagger)}{=} (q_1 \otimes q_2) * (a \otimes a) = q_1 * (q_2 * (a \otimes a)) = q_1 * (a \otimes (q_2 * a)) = (q_1 * a) \otimes (q_2 * a)$, where (†) uses the idempotency of A.

The first part of Item (3) follows from the fact that $1_Q * \cdot$ is the identity map on A. For the second part, notice that given $a \in A$, $1_Q * a = a$ and, therefore, $\bigcup_{a \in A} (\cdot * a)^{\rightarrow}(Q) = A$. For the last part, use the fact that both sinks are jointly surjective (cf. [2, Examples 10.65(1)]).

Item (4) is straightforward.

To verify Item (5), notice that in case of $\mathcal{A}_1$, the requirement provides $(\bigwedge S) * a = \bigvee_{s \in S} (s * a)$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$. With Definition 10 in mind, one obtains, $(\bigwedge S) * a = (\bigvee S) * a$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$.

The case of $\mathcal{A}_2$ gives $q * (\bigwedge S) = \bigvee_{s \in S} (q * s)$ for every $q \in Q$ and every non-empty $S \subseteq A_\top$. By Definition 10, substituting 1_Q for q, we get, $\bigwedge S = \bigvee S$ for every non-empty $S \subseteq A_\top$. Now, given $a_1, a_2 \in A_\top$, $a_1 \leqslant a_1 \vee a_2 = a_1 \wedge a_2 \leqslant a_2$ and, similarly, $a_2 \leqslant a_1$, resulting in $a_1 = a_2$. □

Taking into consideration the properties of frames (e.g., idempotency of the meet operation), Lemma 52 provides a point in favor of the above-mentioned representations of Q-algebras. However, its Item (5) eliminates the use of the representation $\mathcal{A}_2$ (also suggested by the second part of Item (4) of Lemma 52). It will be the topic of our further research to study the issue in full detail.

The above open problems will be addressed in our forthcoming papers.

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