Hacettepe Journal of Mathematics and Statistics
Volume 45 (5) (2016), 1329 – 1334

Asymptotic behavior of associated primes of certain ext modules

Kamal Bahmanpour*, Reza Naghipour† and Mehdi Sehatkhah‡

Abstract

Let R be a commutative Noetherian ring, I an ideal of R and M a finitely generated R-module. It is shown that, whenever I is principal, then for each integer i the set of associated prime ideals $\mathrm{Ass}_R\mathrm{Ext}^i_R(R/I^n, M)$, $n = 1, 2, \ldots$, becomes independent of n, for large n.

Keywords: Artinian module, Associated prime, Depth, Noetherian module.

2000 AMS Classification: 13E05.

Received : 20.07.2015 *Accepted* : 24.11.2015 *Doi* : 10.15672/HJMS.20164514288

1. Introduction

Let R denote a commutative Noetherian ring (with identity), I an ideal of R, and M a finitely generated R-module. In [7] L.J. Ratliff, Jr., conjectured about the asymptotic behaviour of $\mathrm{Ass}_R R/I^n$ when R is a Noetherian domain. Subsequently, M. Brodmann [1] showed that $\mathrm{Ass}_R M/I^n M$ is ultimately constant for large n. In [6], Melkersson and Schenzel asked whether the sets $\mathrm{Ass}_R\mathrm{Ext}^i_R(R/I^n, M)$ become stable for sufficiently large n. The aim of this paper is to show that, for all $i \geq 0$, the sets of prime ideals $\mathrm{Ass}_R\mathrm{Ext}^i_R(R/I^n, M)$, $n = 1, 2, \ldots$, becomes independent of n, for large n, whenever I is principal, which is an affirmative answer to the above question in the case I is principal. Also, it is shown that, if I is generated by an R-regular sequence and $\mathrm{Ext}^i_R(R/I, M)$ is Artinian, then the set $\cup_{n=1}^{\infty}\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^n, M)$ is finite.

For any R-module L, the set $\{\mathfrak{p} \in \mathrm{Ass}_R L \mid \dim R/\mathfrak{p} = \dim L\}$ is denoted by $\mathrm{Assh}_R L$.

*Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran.
Email : bahmanpour.k@gmail.com
†Department of Mathematics, University of Tabriz, Tabriz, Iran.
Email : naghipour@ipm.ir
Email : naghipour@tabrizu.ac.ir Corresponding author.
‡Roshdiyeh Higher Education Institution, Tabriz, Iran.
Email : m_sehhat@pnu.ac.ir

2. The Results

2.1. Lemma. Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module. Then the sequence $\mathrm{Ass}_R\mathrm{Ext}^1_R(R/I^n, M)$ becomes eventually constant, for large n.

Proof. See [4, Corollary 2.3]. □

2.2. Lemma. Let x be an element of the Noetherian ring R. Let M and N be two finitely generated R-modules such that $\mathrm{pd}(N) = t < \infty$. Then for each $i \geq t+2$ and for all large k,

$$\mathrm{Ass}_R\, \mathrm{Ext}^i_R(N/x^kN, M) = \mathrm{Ass}_R\, \mathrm{Ext}^{i-1}_R(N/\Gamma_{Rx}(N), M),$$

and so the sets $\mathrm{Ass}_R\mathrm{Ext}^i_R(N/x^kN, M)$ are eventually constant.

Proof. Suppose that $i \geq t+2$. As, N is finitely generated, it follows that there is an integer n such that

$$\Gamma_{Rx}(N) := \textstyle\bigcup_{i=0}^{\infty}(0 :_M Rx^i) = (0 :_N x^n) = (0 :_N x^{n+1}) = \cdots .$$

Now we claim that for any $k \geq n$,

$$\mathrm{Ext}^i_R(N/x^kN, M) \cong \mathrm{Ext}^{i-1}_R(N/\Gamma_{Rx}(N), M).$$

To do this, as $(0 :_N x^k) = \Gamma_{Rx}(N)$, it follows that $x^kN \cong N/\Gamma_{Rx}(N)$. Therefore for all $j \geq 0$ we have

$$\mathrm{Ext}^j_R(x^kN, M) \cong \mathrm{Ext}^j_R(N/\Gamma_{Rx}(N), M),$$

for all $k \geq n$. Now the exact sequence

$$0 \longrightarrow x^kN \longrightarrow N \longrightarrow N/x^kN \longrightarrow 0,$$

implies that

$$\mathrm{Ext}^i_R(N/x^kN, M) \cong \mathrm{Ext}^{i-1}_R(x^kN, M) \cong \mathrm{Ext}^{i-1}_R(N/\Gamma_{Rx}(N), M),$$

(Note that $\mathrm{pd}(N) = t$ and $i \geq t+2$.) Hence we have

$$\mathrm{Ass}_R\, \mathrm{Ext}^i_R(N/x^kN, M) = \mathrm{Ass}_R\, \mathrm{Ext}^{i-1}_R(N/\Gamma_{Rx}(N), M),$$

for all $k \geq n$, as required. □

2.3. Theorem. Let R be a Noetherian ring and let x be an element of R. Let M be a finitely generated R-module and i a non-negative integer. Then the sequence

$$\mathrm{Ass}_R\mathrm{Ext}^i_R(R/Rx^k, M),$$

of associated primes is ultimately constant for large k, and if $i \geq 2$, then

$$\mathrm{Ass}_R\, \mathrm{Ext}^i_R(R/Rx^k, M) = \mathrm{Ass}_R\, \mathrm{Ext}^{i-1}_R(R/\Gamma_{Rx}(R), M),$$

for all large k.

Proof. The result follows from Lemmas 2.1 and 2.2. □

2.4. Proposition. Let R be a Noetherian ring and let M, N be tow finitely generated R-modules. Let x be an N-regular element of R. Then, for any given integer $j \geq 0$, the set

$$\bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^j_R(N/x^nN, M),$$

of associated prime ideals, is finite.

Proof. If $j = 0$, then

$$\mathrm{Ass}_R\mathrm{Hom}_R(N/x^nN, M) = \mathrm{Ass}_R\mathrm{Hom}_R(N, \mathrm{Hom}_R(R/Rx, M)),$$

and so

$$\bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^0_R(N/x^nN, M)$$

is a finite set. Suppose then that $j \geq 1$, and we use the exact sequence

$$0 \longrightarrow N \xrightarrow{x^n} N \longrightarrow N/x^nN \longrightarrow 0,$$

to obtain the exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{j-1}_R(N, M) \xrightarrow{x^n} \mathrm{Ext}^{j-1}_R(N, M) \longrightarrow \mathrm{Ext}^j_R(N/x^nN, M)$$
$$\longrightarrow \mathrm{Ext}^j_R(N, M) \xrightarrow{x^n} \mathrm{Ext}^j_R(N, M) \longrightarrow \cdots .$$

Hence we have the following exact sequence,

$$0 \to \mathrm{Ext}^{j-1}_R(N, M)/x^n\mathrm{Ext}^{j-1}_R(N, M) \to \mathrm{Ext}^j_R(N/x^nN, M) \to (0 :_{\mathrm{Ext}^j_R(N,M)} x^n) \to 0.$$

Consequently, it follows from Brodmann's result (see [1]) that the set

$$\bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^j_R(N/x^nN, M)$$

is finite. □

2.5. Lemma. Let R be a Noetherian ring and let M be an R-module. Let N be an Artinian submodule of M. Then

$$\mathrm{Ass}_R M/N \backslash \mathrm{Supp} N = \mathrm{Ass}_R M \backslash \mathrm{Supp} N.$$

Proof. As N is an Artinian R-module, it follows that the set $\mathrm{Supp}N \subseteq \mathrm{Max}R$ is finite. Let $\mathrm{Supp}N = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ and $J := \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Then we have

$$\mathrm{Ass}_R M \backslash \mathrm{Supp} N = \mathrm{Ass}_R M/\Gamma_J(M) = \mathrm{Ass}_R M/N \backslash \mathrm{Supp} N,$$

as required. □

Following we let $H^j_I(M)$ denote the j^{th} local cohomology module of M with respect to an ideal I of a Noetherian ring R (cf. [2] and [3]).

2.6. Theorem. Let R be a Noetherian ring and let I be an ideal of R which is generated by an R-regular sequence. Let M be a finitely generated R-module and let i be a non-negative integer such that the R-module $\mathrm{Ext}^i_R(R/I, M)$ is Artinian. Then the set

$$\bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^n, M),$$

is finite. In particular, the set $\mathrm{Ass}_R H^{i+1}_I(M)$ is finite.

Proof. For $n \geq 0$, the exact sequence

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \to 0$$

induces the exact sequence

$$\mathrm{Ext}^i_R(I^n/I^{n+1}, M) \to \mathrm{Ext}^{i+1}_R(R/I^n, M) \to \mathrm{Ext}^{i+1}_R(R/I^{n+1}, M) \to \mathrm{Ext}^{i+1}_R(I^n/I^{n+1}, M).$$

Since I is generated by an R-regular sequence, by [5, page 125] I^n/I^{n+1} is a finitely generated free R/I-module, and so the sets

$$\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(I^n/I^{n+1}, M) = \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I, M), \text{ and}$$
$$\mathrm{Supp}\mathrm{Ext}^{i}_R(I^n/I^{n+1}, M) = \mathrm{Supp}\mathrm{Ext}^{i}_R(R/I, M)$$

are finite, (note that the R-module $\mathrm{Ext}^i_R(R/I, M)$ is Artinian). Therefore in view of the above exact sequence and Lemma 2.5, the set

$$\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^{n+1}, M)\backslash\mathrm{Supp}\mathrm{Ext}^{i}_R(R/I, M)$$

is a subset of

$$(\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^{n}, M)\backslash\mathrm{Supp}\mathrm{Ext}^{i}_R(R/I, M)) \cup \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I, M),$$

and so the set $\bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^n, M)$ is finite, as required. The second assertion follows from the fact that

$$\mathrm{Ass}_R H^{i+1}_I(M) \subseteq \bigcup_{n=1}^{\infty} \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^n, M).$$

□

2.7. Corollary. Let R be a Noetherian ring and let I be an ideal of R which is generated by an R-regular sequence. Let M be a finitely generated R-module and let i be a non-negative integer such that $\mathrm{Ext}^i_R(R/I, M) = 0$. Then the sequence

$$\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^k, M),$$

of associated primes is increasing and ultimately constant for all large k.

Proof. Since I^k/I^{k+1} is a free R/I-module, it follows that $\mathrm{Ext}^i_R(I^k/I^{k+1}, M) = 0$, for all $k \geq 1$. Hence the exact sequence

$$0 \longrightarrow \mathrm{Ext}^{i+1}_R(R/I^k, M) \longrightarrow \mathrm{Ext}^{i+1}_R(R/I^{k+1}, M) \longrightarrow \mathrm{Ext}^{i+1}_R(I^k/I^{k+1}, M),$$

implies that

$$\mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^k, M) \subseteq \mathrm{Ass}_R\mathrm{Ext}^{i+1}_R(R/I^{k+1}, M).$$

Now the result follows from Theorem 2.6. □

2.8. Lemma. Let $(R, \mathfrak{m})$ be a Noetherian local ring of depth d. Let M be a finitely generated R-module and N an Artinan submodule of M. Then for all $i \leq d-1$,

$$\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^i_R(M/N, R).$$

Proof. The exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Ext}^{i-1}_R(N, R) \longrightarrow \mathrm{Ext}^{i}_R(M/N, R) \longrightarrow \mathrm{Ext}^{i}_R(M, R) \longrightarrow \mathrm{Ext}^{i}_R(N, R).$$

As N has finite length and $\mathrm{depth}\, R = d$, it follows that

$$\mathrm{Ext}^i_R(N, R) = 0 = \mathrm{Ext}^{i-1}_R(N, R).$$

Hence the result follows. □

2.9. Lemma. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d and I an ideal of R. Then for any $\mathfrak{p} \in \mathrm{Ass}_R\mathrm{Ext}^{\mathrm{grade}\, I}_R(R/I, R)$,

$$\mathrm{height}\, \mathfrak{p} = \mathrm{grade}\, I.$$

Proof. Let $\operatorname{grade} I = t$. The assertion is clear when $t = 0$. Now suppose that, $t \geq 1$. There exists an R-regular sequence $x_1, \ldots, x_t \in I$. As

$$\operatorname{Ext}_R^{\operatorname{grade} I}(R/I, R) \cong \operatorname{Hom}_{R/(x_1,\ldots,x_t)}(R/I, R/(x_1, \ldots, x_t)),$$

and $R/(x_1, \ldots, x_t)$ is a Cohen-Macaulay ring it follows that

$$\operatorname{Ass}_R \operatorname{Ext}_R^{\operatorname{grade} I}(R/I, R) \subseteq \operatorname{Assh}_R R/(x_1, \ldots, x_t),$$

that implies for any $\mathfrak{p} \in \operatorname{Ass}_R \operatorname{Ext}_R^{\operatorname{grade} I}(R/I, R)$,

$$\operatorname{height} \mathfrak{p} = \operatorname{grade} I,$$

as required. □

2.10. Theorem. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension $d \geq 3$. Let I be an ideal of R such that $1 \leq \operatorname{grade} I \leq d - 2$. Then

$$\operatorname{depth} \operatorname{Ext}_R^{\operatorname{grade} I}(R/I, R) \geq 2,$$

and if $\operatorname{grade} I \leq d-3$ then the equality holds if and only if $\mathfrak{m} \in \operatorname{Ass}_R \operatorname{Ext}_R^{1+\operatorname{grade} I}(R/I, R)$.

Proof. Set $t := \operatorname{grade} I$. Let $\Gamma_{\mathfrak{m}}(R/I) := J/I$ for some ideal J of R with $I \subseteq J$. Then it is easy to see that $\mathfrak{m} \notin \operatorname{Ass}_R R/J$ and $\dim R/I = \dim R/J$. Hence as R is a Cohen-Macaulay ring, it follows that $\operatorname{grade} I = \operatorname{grade} J$. Moreover, since J/I has finite length, it follows from Lemma 2.8 that

$$\operatorname{Ext}_R^t(R/I, R) \cong \operatorname{Ext}_R^t(R/J, R) \text{ and } \operatorname{Ext}_R^{t+1}(R/I, R) \cong \operatorname{Ext}_R^{t+1}(R/J, R).$$

Therefore, we may and do replace I with J in the following. Since $\mathfrak{m} \notin \operatorname{Ass}_R R/J$, it follows that there exists an element $x \in R$ such that x is R/J-regular sequence. Then, as $\dim R/(J + Rx) = \dim R/J - 1$ and R is a Cohen-Macaulay ring, it follows that

$$\operatorname{grade}(J + Rx) = \operatorname{grade} J + 1.$$

Now the exact sequence

$$0 \to R/J \xrightarrow{x} R/J \to R/J + Rx \to 0$$

induces the exact sequence

$$0 \to \operatorname{Ext}_R^t(R/J, R) \xrightarrow{x} \operatorname{Ext}_R^t(R/J, R) \to \operatorname{Ext}_R^{t+1}(R/J + Rx, R).$$

Hence

$$\operatorname{Ass}_R \operatorname{Ext}_R^t(R/J, R)/x\operatorname{Ext}_R^t(R/J, R) \subseteq \operatorname{Ass}_R \operatorname{Ext}_R^{t+1}(R/J + Rx, R),$$

and since $1 + \operatorname{grade} J \leq d - 1$, it follows from Lemma 2.9 that

$$\mathfrak{m} \notin \operatorname{Ass}_R \operatorname{Ext}_R^{t+1}(R/J + Rx, R).$$

Now, it easily follows that

$$\operatorname{depth} \operatorname{Ext}_R^t(R/J, R) \geq 2.$$

Now, let $\operatorname{grade} J \leq d - 3$. Then we have the following exact sequence,

$$0 \to \operatorname{Ext}_R^t(R/J, R)/x\operatorname{Ext}_R^t(R/J, R) \to \operatorname{Ext}_R^{t+1}(R/J + Rx, R) \to (0 :_{\operatorname{Ext}_R^{t+1}(R/J,R)} x) \to 0.$$

Since $\operatorname{grade}(J + Rx) = t + 1$ and $t + 1 \leq d - 2$, it follows from the first part that $\operatorname{depth} \operatorname{Ext}_R^{t+1}(R/J + Rx, R) \geq 2$. Therefore it follows from the exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}, (0 :_{\operatorname{Ext}_R^{t+1}(R/J,R)} x)) \to \operatorname{Ext}_R^1(R/\mathfrak{m}, \operatorname{Ext}_R^t(R/J, R)/x\operatorname{Ext}_R^t(R/J, R)) \to 0$$

that $\operatorname{depth} \operatorname{Ext}_R^t(R/J, R) = 2$ if and only if $\operatorname{Hom}_R(R/\mathfrak{m}, (0 :_{\operatorname{Ext}_R^{t+1}(R/J,R)} x)) \neq 0$. Consequently $\operatorname{depth} \operatorname{Ext}_R^t(R/J, R) = 2$ if and only if $\mathfrak{m} \in \operatorname{Ass}_R \operatorname{Ext}_R^{t+1}(R/J, R)$, as required. □

Acknowledgments

The authors are deeply grateful to the referee for his/her valuable suggestions on the article. Also, we would like to thank Professor Hossein Zakeri for his reading of the first draft and useful comments.

References

[1] M. Brodmann, *Asymptotic stability of* $\mathrm{Ass}(M/I^nM)$, Proc. Amer. Math. Soc. **74**(1979), 16-18.

[2] M.P. Brodmann and R.Y. Sharp, *Local cohomology; an algebraic introduction with geometric applications,* Cambridge University Press, Cambridge, 2013.

[3] A. Grothendieck, *Local cohomology,* Notes by R. Hartshorne, Lecture Notes in Math., 862 (Springer, New York, 1966).

[4] K. Khashyarmanesh and S. Salarian, *Asymptotic stability of* $\mathrm{Att}_R\,\mathrm{Tor}_1^R(R/\mathfrak{a}^n, A)$, Proc. Edin. Math. Soc. **44**(2001), 479-483.

[5] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, UK, 1986.

[6] L. Melkersson and P. Schenzel, *Asymptotic prime ideals related to derived functors*, Proc. Amer. Math. Soc.**117** (1993), 935-938.

[7] L.J. Ratliff, Jr., *On the prime divisors of* I^n, n *large*, Michigan Math. J. **23** (1976), 337-352.