

## Subalgebra analogue to H-basis for ideals

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### Abstract

The H-basis concept allows an investigation of multivariate polynomial spaces degree by degree. In this paper we present the analogue of H-bases for subalgebras in polynomial rings, we call them "SH-bases". We present their connection to the Sagbi basis concept, characterize SH-basis and show how to construct them.

**Keywords:** Gröbner basis, Sagbi basis, H-basis

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### 1. Introduction

The concept of Gröbner bases, introduced by Buchberger [3] in 1965, has become an important ingredient for the treatment of various problems in computational algebra, (see [2] for an extensive survey). This concept has also been extended to more general situations, such as Gröbner bases of modules, for example, as in [9]. However, all approaches related to Gröbner bases are fundamentally tied to monomial orderings, which lead to asymmetry among the variables of interest. On the other hand, the concept of H-bases, introduced long ago by Macaulay [7], is based solely on homogeneous terms of a polynomial. In [12], an extension of Buchberger's algorithm is presented to construct H-bases algorithmically. Some applications of H-bases are given in [10], in addition, many of the problems in applications which can be solved by the Gröbner technique can also be treated successfully with H-bases.

The concept of Gröbner basis for ideals of a polynomial ring over a field  $K$  can be adopted in a natural way to  $K$ -subalgebras of a polynomial ring. In [11] Sagbi (Subalgebra Analogue to Gröbner Basis for Ideals) basis for the  $K$ -subalgebra of  $K[x_1, \dots, x_n]$  is

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defined; this concept was also independently developed in [6]. The properties and applications of Sagbi bases are typically similar to standard Gröbner basis results (see [1] and [4] for an overview of the standard theory). Like Gröbner bases, the concept of Sagbi basis is also tied to monomial orderings. Consequently, within the concept of H-bases for ideals, it is natural to probe the concept of subalgebra bases which may be based solely on homogenous terms of a polynomial. In this paper we will present the analogue to H-bases for ideals in polynomial rings, we call them "SH-bases". Unlike H-bases, SH-bases are not finite. This is not surprising because unlike ideals in polynomial rings, subalgebras in polynomial rings are not necessarily finitely generated. The subalgebras which are not finitely generated cannot have finite SH-basis. Moreover, a finitely generated subalgebra may have an infinite SH-basis (see Example 3.8).

The paper is organized as follows. In section 2, we briefly describe the underlying concept of grading which leads to Sagbi basis and SH-basis. Then, we give the notion of  $d$ -reduction, which is one of the key ingredients for the characterization and construction of SH-basis. After setting up the necessary notation, we present the  $d$ -reduction Algorithm (see Algorithm 1). Also, here we present some properties characterizing SH-basis (Theorem 2.4). In section 3, we present a criterion through which we can check that the given system of polynomials is an SH-basis of the subalgebra it generates (Theorem 3.4) and further on the basis of this theorem we present an algorithm for the construction of SH-basis (Algorithm 2).

## 2. SH-bases and Sagbi bases

Here and in the following sections we consider polynomials in  $n$  variables  $x_1, \dots, x_n$  with coefficients from a field  $K$ . For short, we write

$$\mathcal{P} := K[x_1, \dots, x_n].$$

If  $G$  is a subset of  $K[x_1, \dots, x_n]$  (not necessarily finite), then the subalgebra of  $\mathcal{P}$  generated by  $G$  is  $K[G]$ . This notion is natural since the elements of  $K[G]$  are precisely the polynomials in the set of formal variables  $G$ , viewed as elements of  $K[G]$ .

**2.1. Definition.** A  $G$ -monomial is a finite power product of the form  $G^\alpha = g_1^{\alpha_1} \dots g_m^{\alpha_m}$  where  $g_i \in G$  for  $i = 1, \dots, m$ , and  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ .

Let  $\Gamma$  denote an ordered monoid, i.e., an abelian semigroup under an operation  $+$ , equipped with a total ordering  $>$  such that, for all  $\alpha, \beta, \gamma \in \Gamma$ ,

$$\alpha > \beta \implies \alpha + \gamma > \beta + \gamma.$$

A direct sum

$$\mathcal{P} := \bigoplus_{\gamma \in \Gamma} \mathcal{P}_\gamma^{(\Gamma)}$$

is called grading (induced by  $\Gamma$ ) or briefly a  $\Gamma$ -grading if for all  $\alpha, \beta \in \Gamma$

$$(2.1) \quad f \in \mathcal{P}_\alpha^{(\Gamma)}, g \in \mathcal{P}_\beta^{(\Gamma)} \implies f \cdot g \in \mathcal{P}_{\alpha+\beta}^{(\Gamma)}.$$

Since the decomposition above is a direct sum, each polynomial  $f \neq 0$  has a unique representation

$$f = \sum_{i=1}^s f_{\gamma_i}, \quad 0 \neq f_{\gamma_i} \in \mathcal{P}_{\gamma_i}^{(\Gamma)}.$$

Assuming that  $\gamma_1 > \gamma_2 > \dots > \gamma_s$ , the  $\Gamma$ -homogeneous term  $f_{\gamma_1}$  is called the maximal part of  $f$ , denoted by  $M^{(\Gamma)}(f) := f_{\gamma_1}$ , and  $f - M^{(\Gamma)}(f)$  is called the  $d$ -reductum of  $f$ . For  $G \subset \mathcal{P}$ ,  $M^{(\Gamma)}(G) := \{M^{(\Gamma)}(g) \mid g \in G\}$ .

There are two major examples of gradings. The first one is grading by degrees,

$$\mathcal{P}_d^{(\Gamma)} = \{p \in \mathcal{P} \mid p \text{ homogenous of degree } d\} \quad \forall d \in \mathbb{N}.$$

Here,  $\Gamma = \mathbb{N}$  with the natural total ordering. This grading is called the  $H$ -grading because of the homogeneous polynomials. Therefore we also write  $H$  in place of this  $\Gamma$ . The space of all polynomials of degree at most  $d$  can now be written as

$$\mathcal{P}_d := \bigoplus_{k=0}^d \mathcal{P}_k^{(H)}.$$

The maximal part of a polynomial  $f \neq 0$  is its homogeneous form of highest degree,  $M^{(H)}(f)$ . For simplicity, let  $M^{(H)}(0) := 0$ .

Now we introduce SH-bases and some of their properties. This concept is very similar to the concept of Sagbi bases. Therefore, we will briefly explain the underlying common structure.

**2.2. Definition.** A subset  $G$  of  $\mathcal{P}$  is called SH- basis of the subalgebra  $\mathcal{A}$  of  $\mathcal{P}$  if, for all  $0 \neq f \in \mathcal{A}$ , there exist  $G$ -monomials  $G^{\alpha_i}$  and  $c_i \in K$ ,  $i = 1, \dots, p$  such that

$$(2.2) \quad f = \sum_{i=1}^p c_i G^{\alpha_i} \quad \text{and} \quad \max_{i=1}^p \{\deg(G^{\alpha_i})\} = \deg(f).$$

The representation for  $f$  in (2.2) is also called its SH-representation with respect to  $G$ .

Note that SH-basis of a subalgebra is also a generating set of it. To obtain more insight into SH-bases, we will give some equivalent definitions. First we need a more technical notion.

**2.3. Definition.** Let  $f \in \mathcal{P}$  and  $G \subset \mathcal{P}$ . We say  $f$   $d$ -reduces to  $\tilde{f}$  with respect to  $G$  if

$$\tilde{f} = f - \sum_{i=1}^m c_i G^{\alpha_i}, \quad \deg(\tilde{f}) < \deg(f),$$

holds with  $G$ -monomials  $G^{\alpha_i}$  satisfying  $\deg(G^{\alpha_1}) \leq \deg(f)$ ,  $i = 1, \dots, m$ . In this case we write

$$f \rightarrow_G \tilde{f}.$$

By  $\rightarrow_{G,*}$  we denote the transitive closure of the binary relation  $\rightarrow_G$ <sup>§</sup>.

The concept of  $d$ -reduction plays an important role in the characterization and construction of SH-basis. For  $f \in \mathcal{P}$  and  $G \subset \mathcal{P}$ , the following algorithm computes  $h$  such that  $f \rightarrow_{G,*} h$ .

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<sup>§</sup> $f \rightarrow_{G,*} h$  if we apply  $d$ -reduction iteratively such as  $f \rightarrow_G h_1 \rightarrow_G h_2 \dots \rightarrow_G h$ , where  $h$  cannot be  $d$ -reduced any further with respect to  $G$ .

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**Algorithm 1**


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**Input:** Let  $G$  and  $f$  be subset and polynomial respectively in  $\mathcal{P}$ .

**Output:**  $h \in \mathcal{P}$  such that  $f \rightarrow_{G,*} h$ .

- 1:  $h := f$ .
  - 2: while ( $h \neq 0$  and  $G_h = \{\sum_i c_i G^{\alpha_i} \mid M^{(H)}(\sum_i c_i G^{\alpha_i}) = M^{(H)}(h)\} \neq \emptyset$ )
  - 3: (a) choose  $\sum_i c_i G^{\alpha_i} \in G_h$ .
  - 4: (b)  $h := h - \sum_i c_i G^{\alpha_i}$  and continue at 2.
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We note that when step 2(b), has been performed, then  $\deg(h)$  is strictly smaller than the  $\deg(h - \sum_i c_i G^{\alpha_i})$  (by the choice of  $\sum_i c_i G^{\alpha_i}$ ). This shows that the Algorithm 1 always terminate.

**2.4. Theorem.** Let  $G \subset \mathcal{P}$  and  $\mathcal{A}$  be a subalgebra of  $\mathcal{P}$ . Then the following conditions are equivalent:

- i)  $G$  is an SH-basis of  $\mathcal{A}$ .
- ii)  $K[\{M^{(H)}(g) \mid g \in G\}] = K[\{M^{(H)}(f) \mid f \in \mathcal{A}\}]$ .
- iii) For all  $f \in \mathcal{A}$ ,  $f \rightarrow_{G,*} 0$ .

*Proof.* (i)  $\Rightarrow$  (ii) follows by

$$M^{(H)}(f) = \sum_{j \in J} c_j M^{(H)}(G^{\alpha_j}), \quad J := \{j \mid \deg(G^{\alpha_j}) = \deg(f)\}$$

for arbitrary  $f \in \mathcal{A}$  with SH-representation  $f = \sum c_i G^{\alpha_i}$ .

(ii)  $\Rightarrow$  (iii) If  $0 \neq f \in \mathcal{A}$ , then  $M^{(H)}(f) = \sum_{j \in J} c_j M^{(H)}(G^{\alpha_j})$ . Therefore,  $\tilde{f} = f - \sum_{j \in J} c_j G^{\alpha_j}$ , where  $\tilde{f} \in \mathcal{A}$  and  $\deg(\tilde{f}) \leq \deg(f)$ . Inductively, we get  $f \rightarrow_{G,*} 0$ .

(iii)  $\Rightarrow$  (i) Let

$$g_0 = f \rightarrow_G g_1 \rightarrow_G \dots \rightarrow_G g_d = 0$$

where  $M^{(H)}(g_{i-1}) = M^{(H)}(G^{\alpha_i})$  and  $\deg(G^{\alpha_{i+1}}) < \deg(G^{\alpha_i})$ ,  $i = 1, \dots, d$ . Then

$$f = \sum_{i=1}^d c_i G^{\alpha_i} \quad \text{and} \quad \deg(f) = \deg(G^{\alpha_1}) = \max_{i=1}^d \{\deg(G^{\alpha_i})\}$$

i.e.,  $f$  has an SH-representation with respect to  $G$ . □

The second major example of gradings leads to the Sagbi basis concept. Here,  $\Gamma = \mathbb{N}^n$  with componentwise addition and equipped with a total ordering satisfying (2.1) and, in addition,  $\gamma \geq 0 \forall \gamma \in \Gamma$ . For arbitrary  $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma$ , the space  $\mathcal{P}_\gamma^{(\Gamma)}$  is a vector space of dimension 1, namely,

$$\mathcal{P}_\gamma^{(\Gamma)} = \{c \cdot x^{\gamma_1} \dots x^{\gamma_n} \mid c \in K\}.$$

The maximal part of a polynomial is now a product of a leading coefficient and a leading monomial,  $M^{(\Gamma)}(f) = LC(f) \cdot LM(f)$ ,  $LC(f) \in K$ ,  $LM(f)$  a leading monomial. The s-reduction  $f \rightarrow_G \tilde{f}$  is defined if there exists a  $G$ -monomial  $G^\alpha$  such that  $LM(G^\alpha) = LM(f)$  and then we set  $\tilde{f} := f - cG^\alpha$ ,  $c \in K$ . The relation  $\rightarrow_{G,*}$  is constructed as above.

A Sagbi basis  $G$  (with respect to a given monomial ordering and a given subalgebra  $\mathcal{A}$ ) is a set of polynomials, generating the subalgebra  $\mathcal{A}$  and satisfying one of the following equivalent conditions:

- (i) Every  $f \in \mathcal{A}$  has a representation  $f = \sum_{i=1}^s c_i G^{\alpha_i}$ ,  $LM(f) = \max_{i=1}^s \{LM(G^{\alpha_i})\}$ .

(ii)  $K[\{M^{(\Gamma)}(g) \mid g \in G\}] = K[\{M^{(\Gamma)}(f) \mid f \in \mathcal{A}\}]$ .

(iii) Every  $f \in \mathcal{A}$   $s$ -reduces to 0 with respect to  $G$ .

The proof of this equivalence and many other equivalent conditions can be found in [11]. If a monomial ordering is compatible with the semi-ordering by degrees,

$$\deg(x^\gamma) > \deg(x^\beta) \implies \gamma > \beta, \quad \gamma, \beta \in \mathbb{N}^n$$

then any Sagbi-representation as given in (i) is an SH-representation, in other words, a Sagbi basis with respect to a degree compatible ordering is an SH-basis as well. The converse is false, as the following example shows.

**2.5. Example.** Let  $f_1 = x^3 + x^2y$ ,  $f_2 = y^3$ ,  $f_3 = xy + y$  and  $\mathcal{A} = K[f_1, f_2, f_3]$ . Then  $f_1, f_2$  and  $f_3$  already constitute an SH-basis of  $\mathcal{A}$ . (This is consequence of Theorem 2.4). If we order the monomials by degree lexicographical ordering then

$$K[\{M^{(H)}(f) \mid f \in \mathcal{A}\}] = K[x^3, y^3, xy, x^2y^4].$$

Every Sagbi basis  $G$  with respect to this ordering contains at least four elements, for instance SINGULAR ([5]) computes  $G = \{g_1, g_2, g_3, g_4\}$  with

$$g_1 = x^3 + x^2y = f_1$$

$$g_2 = y^3 = f_2$$

$$g_3 = xy + y = f_3$$

$$g_4 = x^2y^4 - 3x^2y^3 - 3xy^3$$

Obviously, this Sagbi basis is an SH-basis as well.

It is possible that a subalgebra has a finite SH-basis, but no finite Sagbi basis, as the following example shows.

**2.6. Example.** Let  $G = \{f_1, f_2, f_3\} \subset K[x, y]$  where  $f_1 = x + y$ ,  $f_2 = xy$ ,  $f_3 = xy^2$  and  $\mathcal{A} = K[G]$ . It is easy to see that  $G$  is an SH-basis of  $\mathcal{A}$ . However, the set  $S = \{x + y, xy, xy^2, xy^3, xy^4, \dots\} \subset \mathcal{A}$  is an infinite Sagbi basis for  $\mathcal{A}$  with respect to a monomial ordering  $x > y$ . (see [11]).

### 3. Construction of SH-bases

In this section, we present an SH-basis criterion, through which we can construct SH-basis. For this purpose, we fix some notations which are necessary for this construction.

**3.1. Definition.** Let  $G$  be a set of polynomials in  $\mathcal{P}$  and let  $\mathcal{A} = K[G]$ . We consider  $f \in \mathcal{A}$  with the representation  $f = \sum_{i=1}^m c_i G^{\alpha_i}$ . Then the degree-height of  $f$ , written  $d\text{-ht}(f)$ , with respect to this representation is  $\max_{i=1}^m \{\deg(G^{\alpha_i})\}$ .

Let  $Y = \{y_1, \dots, y_s\}$  and  $K[Y]$  be a polynomial ring over a field  $K$  in variables  $y_1, \dots, y_s$ . Let  $P(Y) = P(y_1, \dots, y_s) \in K[Y]$  and  $Y^\alpha$  be a  $Y$ -monomial.

**3.2. Definition.** Let  $G \subseteq \mathcal{P}$ . A polynomial  $P(Y) = \sum_{i=1}^m c_i Y^{\alpha_i} \in K[Y]$  (where  $c_i \in K$ ) is called  $G$ -homogenous if  $\deg(G^{\alpha_i})$  are same for  $1 \leq i \leq m$ .

**3.3. Definition.** Let  $G = \{g_1, \dots, g_s\}$  be a subset of  $K[x_1, \dots, x_n]$ . We denote  $AR((M^{(H)}(G))$ , the ideal of algebraic relations between  $M^{(H)}(g_i), i = 1, \dots, s$  defined by:

$$AR((M^{(H)}(G)) = \{h \in K[y_1, \dots, y_s] \mid h(M^{(H)}(g_1), \dots, M^{(H)}(g_s)) = 0\}$$

$AR((M^{(H)}(G))$  is an ideal in  $K[y_1, \dots, y_s]$ .

In the case of Sagbi bases, there is as an algorithm for computing Sagbi bases by means of algebraic relations (see [8]) where algebraic relations and their connection to Sagbi bases are studied in detail.) The analogue for constructing SH-bases by means of algebraic relations is based on the following result.

**3.4. Theorem. (SH-basis criterion)** *Let  $G = \{g_1, \dots, g_s\}$  be a subset of  $K[x_1, \dots, x_n]$ . Let  $A = K[G]$  and let  $\{P_j(Y) \mid j \in J\}$  be a finite set of  $G$ -homogenous generators for  $AR((M^{(H)}(G)))$ . Then the following conditions are equivalent:*

- i)  $G$  is an SH-basis of  $A$ .*
- ii) For each  $j \in J$ ,  $P_j(G) = P_j(g_1, \dots, g_s) \rightarrow_{G,*} 0$ .*

*Proof.* (i)  $\Rightarrow$  (ii): This is trivial and follows from Theorem 2.4.

(ii)  $\Rightarrow$  (i): For every  $h \in K[G]$ , we will show that

$$h = \sum_{i=1}^m c_i G^{\alpha_i} \text{ and } \deg(h) = \max_{i=1}^m \{\deg(G^{\alpha_i})\}.$$

Let  $h \in K[G]$  and write  $h = \sum_{i=1}^m c_i G^{\alpha_i}$ ; furthermore, assume that this representation has the smallest possible degree-height  $t_0 = \max_{i=1}^m \{\deg(G^{\alpha_i})\}$  of all such representation. We know that  $\deg(h) \leq t_0$ . Suppose that  $\deg(h) < t_0$ , without loss of generality, let the first  $N$  summands be the ones for which  $\deg(M^{(H)}(G^{\alpha_i})) = t_0$ . Then the cancelation of their maximal part must occur; i.e  $\sum_{i=1}^N c_i M^{(H)}(G^{\alpha_i}) = 0$ . From this condition, we obtain a polynomial  $P(Y) = \sum_{i=1}^N c_i Y^{\alpha_i} \in AR((M^{(H)}(G)))$ . We can then write

$$(3.1) \quad \sum_{i=1}^N c_i Y^{\alpha_i} = P(Y) = \sum_{j=1}^M g_j(Y) P_j(Y)$$

where the polynomials  $P_j(Y)$  are among the stated generators of  $AR((M^{(H)}(G)))$  and the polynomials  $g_j(Y) \in K[y_1, \dots, y_s]$ . Moreover, we may assume that each  $g_j(Y)$  is  $G$ -homogenous (since  $P(Y)$  and every  $P_j(Y)$  are) and also that

$$(3.2) \quad \text{d-ht}(g_j(G)) + \text{d-ht}(P_j(G)) = \text{d-ht}(P(G)) = t_0 \quad \forall j.$$

We have assumed that each  $P_j(G) \rightarrow_{G,*} 0$ ; therefore we have  $P_j(G) = \sum_{k=1}^{n_j} c_{kj} G^{\alpha_{kj}}$  where  $c_{kj} \in K$ . By definition, these sums must have degree heights  $\max_k \{\deg(G^{\alpha_{kj}})\} = \deg(P_j(G)) < \text{d-ht}(P_j(G))$  for each  $j$ , where the last inequality holds because  $P_j(Y) \in AR((M^{(H)}(G)))$ . Then for each  $j$ ,  $1 \leq j \leq M$ ,

$$(3.3) \quad g_j(G)P_j(G) = \sum_{k=1}^{n_j} c_{kj} g_j(G) G^{\alpha_{kj}}$$

Note that

$$(3.4) \quad \deg(g_j(G)P_j(G)) = \deg(g_j(G)) + \deg(P_j(G)) < \deg(g_j(G)) + \text{d-ht}(P_j(G)).$$

From our observation and using equation (3.2), we have

$$(3.5) \quad \deg(g_j(G)) + \text{d-ht}(P_j(G)) \leq \text{d-ht}(g_j(G)) + \text{d-ht}(P_j(G)) = t_0$$

Combining equations (3.4) and (3.5) we have

$$(3.6) \quad \deg(g_j(G)P_j(G)) < t_0$$

Finally, equations (3.1) and (3.3) imply that

$$(3.7) \quad h = P(G) + \underbrace{\sum_{i=N+1}^m c_i G^{\alpha_i}}_{\sum_{j=1}^M \left( \sum_{k=1}^{n_j} c_{kj} g_j(G) G^{\alpha_{kj}} \right)} + \underbrace{\sum_{i=N+1}^m c_i G^{\alpha_i}}_{\sum_{i=N+1}^m c_i G^{\alpha_i}}.$$

$$Sum_1$$

$$Sum_2$$

If we examine the expression (3.7) closely, we see that:

- By (3.6),  $d\text{-ht}(Sum_1) = \max_{j=1}^M \{\deg(g_j(G)P_j(G))\} < t_0$ ;
- By the choice of  $N$ ,  $d\text{-ht}(Sum_2) < t_0$ ;

But this contradicts our initial assumption that we have chosen a representation of  $h$  that has the smallest possible height. Thus,  $G$  is an SH-basis of  $K[G]$ .  $\square$

On the basis of Theorem 3.4, now we present an algorithm which computes SH-basis from a given set of generators. This algorithm is not necessarily terminating but does terminate, if and only if, the considered subalgebra has a finite SH-basis.

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### Algorithm 2

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**Input:** A finite subset  $G \subset \mathcal{P}$ .

**Output:** SH-basis  $G$ .

- 1: Compute a generating set  $\mathcal{S}$  for  $AR(M^{(H)}(G))$ .
  - 2: For  $P \in \mathcal{S}$
  - 3: (a)  $h \in \mathcal{P}$ , such that  $P(G) \rightarrow_{G,*} h$ .
  - 4: (b) If  $h \neq 0$ , set  $G := G \cup \{h\}$  and continue at 1.
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**3.5. Remark.** We have implemented SH-basis construction algorithm in the computer algebra system SINGULAR [5]. Code can be download from [mathcity.org/junaid](http://mathcity.org/junaid).

Now we present some examples which show the computation of SH-basis through Algorithm 2.

**3.6. Example.** The subalgebra  $\mathcal{A} \subset \mathcal{P}$  of symmetric polynomials is well known to be finitely generated by a set  $S$  which is a set of elementary symmetric polynomials in  $\mathcal{P}$ . The set  $S$  is an SH-basis of  $\mathcal{A}$  as  $AR(M^{(H)}(S)) = \{0\}$  i.e, there is no polynomial  $0 \neq P(Y) \in K[y_1, \dots, y_n]$  such that  $P(S) = 0$ .

**3.7. Example.** Let  $G = \{x + y + 1, x^2 + y^2 - x + 2, 2xy - y\}$  and  $\mathcal{A} = \mathbb{Q}[G]$ . The ideal  $AR((M^{(H)}(G)) = AR(x + y, x^2 + y^2, xy)$  in  $\mathbb{Q}[y_1, y_2, y_3]$  is generated by  $P(Y) = y_1^2 - y_2 - y_3$ . It is easy to see that the polynomial  $P(G) = 3x + 3y - 1 \rightarrow_{G,*} 0$ . This shows that  $G$  is an SH-basis of  $\mathcal{A}$ .

The next example shows that there are finitely generated algebras which do not admit a finite SH-basis.

**3.8. Example.** Let  $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z\}$  and  $\mathcal{A} = \mathbb{Q}[G]$ . Also we have  $M^{(H)}(g_1) = xz, M^{(H)}(g_2) = xyz$  and  $M^{(H)}(g_3) = xy^2z$ .

In first step,  $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z\}$ . It is evident that the ideal of relations  $AR(M^{(H)}(G)) = AR(xz, xyz, xy^2z) \subset \mathbb{Q}[y_1, y_2, y_3]$  is generated by  $P(Y) = y_1y_3 - y_2^2$ . The polynomial  $P(G) = (xz + y)(xy^2z) - (xyz)^2 = xy^3z \rightarrow_{G,*} 0$ , so  $G := G \cup \{g_4 = xy^3z\}$ .

In second step,  $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z, g_4 = xy^3z\}$ . The polynomial  $P(Y) = y_1y_4 - y_2y_3$  is one the generators of the ideal of relations  $AR(M^{(H)}(G)) = AR(xz, xyz, xy^2z, xy^3z) \subset \mathbb{Q}[y_1, y_2, y_3, y_4]$ . Here we note that the polynomial  $P(G) = (xz + y)(xy^3z) - (xyz)(xy^2z) = xy^4z \rightarrow_{G,*} 0$ , therefore we have  $G := G \cup \{xy^4z\} = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z, g_4 = xy^3z, g_5 = xy^4z\}$ .

By induction, we get  $G = \{xz + y, xyz, xy^2z, xy^3z, xy^4z, xy^5z, \dots\}$  which implies that  $\mathcal{A}$  have an infinite SH-basis.

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