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On the rescaled Riemannian metric of Cheeger-Gromoll type on the cotangent bundle

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Abstract

Let (M, g) be an $n-$dimensional Riemannian manifold and T^*M be its cotangent bundle equipped with a Riemannian metric of Cheeger-Gromoll type which rescale the horizontal part by a positive differentiable function. The main purpose of the present paper is to discuss curvature properties of T^*M and construct almost paracomplex Norden structures on T^*M. We investigate conditions for these structures to be para-Kähler (paraholomorphic) and quasi-para-Kähler. Also, some properties of almost paracomplex Norden structures in context of almost product Riemannian manifolds are presented.

Keywords: Almost paracomplex structure, connection, cotangent bundle, paraholomorphic tensor field, Riemannian metric.

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1. Introduction

Geometric structures on bundles have been object of much study since the middle of the last century. The natural lifts of the metric g, from a Riemannian manifold (M, g) to its tangent or cotangent bundles, induce new (pseudo) Riemannian structures, with interesting geometric properties. Maybe the best known Riemannian metric ${}^S g$ on the tangent bundle over Riemannian manifold (M, g) is that introduced by Sasaki in 1958 (see [25]), but in most cases the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold. The metric ${}^S g$ is called the Sasaki metric. The Sasaki metric ${}^S g$ has been extensively studied by several authors and in many different contexts. Another Riemannian metric on the tangent bundle TM defined by E. Musso and F. Tricerri [14] is the Cheeger-Gromoll metric ${}^{CG} g$. The metric

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was defined by J. Cheeger and D. Gromoll [3]; yet, E. Musso and F. Tricerri wrote down its expression, constructed it in a more "comprehensible" way, and gave it the name. In [30], B. V. Zayatuev introduced a Riemannian metric ${}^S\overline{g}$ on the tangent bundle TM given by

$$\begin{aligned} {}^S\overline{g}\left({}^HX,{}^HY\right) &= fg\left(X,Y\right), \\ {}^S\overline{g}\left({}^HX,{}^VY\right) &= {}^S\overline{g}\left({}^VX,{}^HY\right)=0, \\ {}^S\overline{g}\left({}^VX,{}^VY\right) &= g\left(X,Y\right), \end{aligned}$$

where $f>0$, $f\in C^\infty(M)$. For $f=1$, it follows that ${}^S\overline{g}={}^Sg$. The metric ${}^S\overline{g}$ is called the rescaled Sasaki metric. The authors studied the rescaled Sasaki type metric on the cotangent bundle T^*M over Riemannian manifold (M,g) (see [8]). Also, for rescaled Riemannian metrics on orthonormal frame bundles, see [11].

Let M_{2k} be a $2k$-dimensional differentiable manifold endowed with an almost (para) complex structure φ and a pseudo-Riemannian metric g of signature (k,k) such that $g(\varphi X,Y)=g(X,\varphi Y)$ for arbitrary vector fields X and Y on M_{2k}, i.e. g is pure with respect to φ. The metric g is called Norden metric. Norden metrics are also referred to as anti-Hermitian metrics or B-metrics. They present extensive application in mathematics as well as in theoretical physics. Many authors considered almost (para)complex Norden structures on the tangent, cotangent and tensor bundles [5, 7, 16, 17, 18, 19, 20, 22, 23].

In this paper, firstly, we present curvature tensor of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. Secondly, we get the conditions under which the cotangent bundle endowed with some paracomplex structures and the rescaled Riemannian metric of Cheeger-Gromoll type ${}^{CG}g_f$ is a paraholomorphic Norden manifold. Finally, for an almost paracomplex manifold to be an specialized almost product manifold, we give some results related to Riemannian almost product structures on the cotangent bundle.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(T^*M)$ the corresponding set on the cotangent bundle T^*M. The Einstein summation convention is used, the range of the indices i,j,s being always $\{1,2,...,n\}$.

2. Preliminaries

The cotangent bundle of a smooth $n-$dimensional Riemannian manifold may be endowed with a structure of $2n-$dimensional smooth manifold, induced by the structure on the base manifold. If (M,g) is a smooth Riemannian manifold of dimension n, we denote its cotangent bundle by $\pi:T^*M\to M$. A system of local coordinates $\left(U,x^i\right),\ i=1,...,n$ in M induces on T^*M a system of local coordinates $\left(\pi^{-1}\left(U\right),\ x^i,\ x^{\overline{i}}=p_i\right),\ \overline{i}=n+i=n+1,...,2n$, where $x^{\overline{i}}=p_i$ is the components of covectors p in each cotangent space $T_x^*M,\ x\in U$ with respect to the natural coframe $\left\{dx^i\right\}$.

Let $X=X^i\frac{\partial}{\partial x^i}$ and $\omega=\omega_i dx^i$ be the local expressions in U of a vector field X and a covector (1-form) field ω on M, respectively. Then the vertical lift ${}^V\omega$ of ω and the horizontal lift ${}^HX$ of X are given, with respect to the induced coordinates, by

(2.1) $$ {}^V\omega=\omega_i\partial_{\overline{i}}, $$

and

(2.2) $$ {}^HX=X^i\partial_i+p_h\Gamma_{ij}^hX^j\partial_{\overline{i}}, $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$ and Γ_{ij}^h are the coefficients of the Levi-Civita connection ∇ of g.

Let T^*M be the cotangent bundle of a Riemannian manifold (M, g). If the local expression of the metric g is $g = g_{ij}dx^i \otimes dx^j$, then the inverse of the metric g is $g^{-1} = g^{ij}\partial_i \otimes \partial_j$, where g^{ij} are the entries of the inverse matrix of g_{ij}, i.e. $g^{ij}g_{jk} = \delta_k^i$. We define $r^2 = g^{-1}(p, p) = g^{ij}p_ip_j$ and put $\alpha = 1 + r^2$. Then the rescaled Riemannian metric of Cheeger-Gromoll type ${}^{CG}g_f$ is defined on T^*M by the following three equations at $(x, p) \in T^*M$

$$ {}^{CG}g_f\left({}^V\omega, {}^V\theta\right) = \frac{1}{\alpha}(g^{-1}(\omega, \theta) + g^{-1}(\omega, p)\, g^{-1}(\theta, p)), \tag{2.3} $$

$$ {}^{CG}g_f\left({}^V\omega, {}^HY\right) = 0, \tag{2.4} $$

$$ {}^{CG}g_f\left({}^HX, {}^HY\right) = fg(X, Y) \tag{2.5} $$

for any $X, Y \in \Im_0^1(T^*M)$ and $\omega, \theta \in \Im_1^0(T^*M)$, where $f > 0$, $f \in C^\infty(M)$, $g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$.

The Lie bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas

$$ \begin{cases} \left[{}^HX, {}^HY\right] = {}^H[X, Y] + {}^V(p \circ R(X, Y)) \\ \left[{}^HX, {}^V\omega\right] = {}^V(\nabla_X\omega) \\ \left[{}^V\theta, {}^V\omega\right] = 0 \end{cases} \tag{2.6} $$

for any X, $Y \in \Im_0^1(M)$ and θ, $\omega \in \Im_1^0(M)$, where R is the Riemannian curvature of g defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (for details, see [28], p. 238, p. 277).

With the connection ∇ of g on M, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of T^*M a frame field which allows the tensor calculus to be efficiently done. The adapted frame on $\pi^{-1}(U)$ of T^*M consist of the following $2n$ linearly independent vector fields:

$$ \begin{cases} E_j = \partial_j + p_s\Gamma_{hj}^s\partial_{\bar{h}}, \\ E_{\bar{j}} = \partial_{\bar{j}}. \end{cases} \tag{2.7} $$

We can write the adapted frame as $\{E_\alpha\} = \{E_j, E_{\bar{j}}\}$. The indices $\alpha, \beta, \gamma, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame. By the straightforward calculations, we have the lemma below.

2.1. Lemma. *The Lie brackets of the adapted frame of T^*M satisfy the following identities:*

$$ \begin{aligned} [E_i, E_j] &= p_sR_{ijl}{}^sE_{\bar{l}}, \\ \left[E_i, E_{\bar{j}}\right] &= \Gamma_{il}^jE_{\bar{l}}, \\ \left[E_{\bar{i}}, E_{\bar{j}}\right] &= 0 \end{aligned} $$

where $R_{ijl}{}^s$ denote the components of the curvature tensor R of (M, g) ([28], p. 290).

Using (2.1), (2.2) and (2.7), we have

$$ {}^V\omega = \begin{pmatrix} 0 \\ \omega_j \end{pmatrix} \tag{2.8} $$

and

$$ {}^HX = \begin{pmatrix} X^j \\ 0 \end{pmatrix} \tag{2.9} $$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [28]).

3. The curvature tensor of the rescaled Riemannian metric of Cheeger-Gromoll type

From the equations (2.3)-(2.5), by virtue of (2.8) and (2.9), the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ has components with respect to the adapted frame $\{E_\alpha\}$:

$$(3.1) \qquad {}^{CG}g_f = diag\left(fg_{ij}, \frac{1}{\alpha}(g^{ij} + g^{is}g^{tj}p_s p_t) \right).$$

For the Levi-Civita connection of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ we give the next theorem.

3.1. Theorem. *Let (M, g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. Then the corresponding Levi-Civita connection $\widetilde{\nabla}$ satisfies the followings:*

$$(3.2) \qquad \begin{cases} i) \ \widetilde{\nabla}_{E_i} E_j = \{\Gamma^l_{ij} + {}^f A^l_{ij}\} E_l + \dfrac{1}{2} p_s R_{ijl}{}^s E_{\bar{l}}, \\ ii) \ \widetilde{\nabla}_{E_i} E_{\bar{j}} = \dfrac{1}{2f\alpha} p_s R^{l}_{\cdot\ i}{}^{js}_{\cdot} E_l - \Gamma^j_{il} E_{\bar{l}}, \\ iii) \ \widetilde{\nabla}_{E_{\bar{i}}} E_j = \dfrac{1}{2f\alpha} p_s R^{l}_{\cdot\ j}{}^{is}_{\cdot} E_l, \\ iv) \ \widetilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = \{\frac{-1}{\alpha}(p^i \delta^j_l + p^j \delta^i_l) + \frac{\alpha+1}{\alpha^2} g^{ij} p_l + \frac{1}{\alpha^2} p^i p^j p_l\} E_{\bar{l}} \end{cases}$$

with respect to the adapted frame, where ${}^f A^h_{ji}$ is a tensor field of type $(1,2)$ defined by ${}^f A^h_{ji} = \frac{1}{f}(f_j \delta^h_i + f_i \delta^h_j - f^m_{\cdot} g_{ji})$ and $p^i = g^{it} p_t$, $R^{k}_{\cdot\ j}{}^{is}_{\cdot} = g^{kt} g^{im} R_{tjm}{}^s$.

Proof. The connection $\widetilde{\nabla}$ is characterized by the Koszul formula:

$$\begin{aligned} 2\,{}^{CG}g_f(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) &= \widetilde{X}({}^{CG}g_f(\widetilde{Y}, \widetilde{Z})) + \widetilde{Y}({}^{CG}g_f(\widetilde{Z}, \widetilde{X})) - \widetilde{Z}({}^{CG}g_f(\widetilde{X}, \widetilde{Y})) \\ -{}^{CG}g_f(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]) &+ {}^{CG}g_f(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) + {}^{CG}g_f(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on T^*M. One can verify the Koszul formula for pairs $\widetilde{X} = E_i, E_{\bar{i}}$ and $\widetilde{Y} = E_j, E_{\bar{j}}$ and $\widetilde{Z} = E_k, E_{\bar{k}}$. In calculations, the formulas (2.7), Lemma 2.1 and the first Bianchi identity for R should be applied. We omit standard calculations. □

Let $\tilde{X}$, $\tilde{Y} \in \Im^1_0(T^*M)$. Then the covariant derivative $\tilde{\nabla}_{\tilde{Y}} \tilde{X}$ has components

$$\tilde{\nabla}_{\tilde{Y}} \tilde{X}^\alpha = \tilde{Y}^\gamma E_\gamma \tilde{X}^\alpha + \tilde{\Gamma}^\alpha_{\gamma\beta} \tilde{X}^\beta \tilde{Y}^\gamma$$

with respect to the adapted frame $\{E_\alpha\}$. Using (2.7), (2.8), (2.9) and (3.2), we have the following proposition.

3.2. Proposition. *Let (M, g) be a Riemannian manifold and $\widetilde{\nabla}$ be the Levi-Civita connection of the cotangent bundle T^*M equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. Then*

$$\begin{aligned} &i) \ \widetilde{\nabla}_{{}^H X} {}^H Y = {}^H \left(\nabla_X Y + {}^f A(X,Y)\right) + \tfrac{1}{2} {}^V \left(p \circ R(X,Y)\right), \\ &ii) \ \widetilde{\nabla}_{{}^H X} {}^V \theta = \tfrac{1}{2f\alpha} {}^H \left(p\left(g^{-1} \circ R(\ ,X)\widetilde{\theta}\right)\right) + {}^V (\nabla_X \theta), \\ &iii) \ \widetilde{\nabla}_{{}^V \omega} {}^H Y = \tfrac{1}{2f\alpha} {}^H \left(p\left(g^{-1} \circ R(\ ,Y)\widetilde{\omega}\right)\right), \\ &iv) \ \widetilde{\nabla}_{{}^V \omega} {}^V \theta = -\tfrac{1}{\alpha}\left({}^{CG}g\left({}^V\omega, \gamma\delta\right){}^V\theta + {}^{CG}g_f\left({}^V\theta, \gamma\delta\right){}^V\omega\right) \\ &+ \tfrac{\alpha+1}{\alpha} {}^{CG}g_f\left({}^V\omega, {}^V\theta\right)\gamma\delta - \tfrac{1}{\alpha} {}^{CG}g_f\left({}^V\omega, \gamma\delta\right){}^{CG}g_f\left({}^V\theta, \gamma\delta\right)\gamma\delta \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0(M)$, $R(\ ,X)\tilde{\omega} \in \Im^1_1(M)$, $g^{-1} \circ R(\ ,X)\tilde{\omega} \in \Im^1_0(M)$, R and $\gamma\delta$ denote respectively the curvature tensor of ∇ and the canonical or Liouville vector field on T^*M with the local expression $\gamma\delta = p_i E_{\bar{i}}$ (for $f = 1$, see [1]).

The Riemannian curvature tensor $\widetilde{R}$ of T^*M with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ is obtained from the well-known formula

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z}=\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z}-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}-\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

for all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Im_0^1(T^*M)$. Then from Lemma 2.1 and Theorem 3.1, we get the following proposition.

3.3. Proposition. *The components of the curvature tensor $\widetilde{R}$ of the cotangent bundle T^*M with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ are given as follows:*

$$\begin{aligned}
\widetilde{R}(E_l,E_i)E_j &= \{R_{lij}{}^{m}-\frac{1}{2f\alpha}p_tp_aR_{lih}{}^{a}R^{m}{}_{.\,j.}{}^{ht}+\frac{1}{4f\alpha}p_tp_a(R^{m}{}_{.\,l.}{}^{ht}R_{ijh}{}^{a}-R^{m}{}_{.\,i.}{}^{ht}R_{ljh}{}^{a})\\
&\quad+\nabla_l(A^m_{ij})-\nabla_i(A^m_{lj})+A^m_{lh}A^h_{ij}-A^m_{ih}A^h_{lj})\}E_m\\
&\quad+\{\frac{1}{2f}p_t(\nabla_lR_{ijm}{}^{t}-\nabla_iR_{ljm}{}^{t})+\frac{1}{2}p_t(R_{lhm}{}^{t}A^h_{ij}-R_{ihm}{}^{t}A^h_{lj})\}E_{\bar{m}},\\
\widetilde{R}(E_{\bar{l}},E_i)E_j &= \{\frac{-1}{2f\alpha}p_a\nabla_iR^{m}{}_{.\,j.}{}^{la}+\frac{1}{2f\alpha}p_a(R^{m}{}_{.\,h.}{}^{la}A^h_{ij}-R^{h}{}_{.\,j.}{}^{la}A^m_{ih}+\frac{f_i}{f}R^{m}{}_{.\,j.}{}^{la})\}E_m\\
&\quad+\{\frac{1}{2}R_{ijm}{}^{l}-\frac{1}{4f\alpha}p_tp_aR_{ihm}{}^{t}R^{h}{}_{.\,j.}{}^{la}-\frac{1}{2\alpha}p_ap^lR_{ijm}{}^{a}-\frac{\alpha+1}{2\alpha^2}p_ap_mR_{ij.}{}^{la}\}E_{\bar{m}},\\
\widetilde{R}(E_l,E_{\bar{i}})E_j &= \{\frac{1}{2f\alpha}p_a\nabla_lR^{m}{}_{.\,j.}{}^{ia}+\frac{1}{2f\alpha}p_a(R^{h}{}_{.\,j.}{}^{ia}\ A^m_{lh}-R^{m}{}_{.\,h.}{}^{ia}A^h_{lj}-\frac{f_l}{f}R^{m}{}_{.\,j.}{}^{ia})\}E_m\\
&\quad+\{\frac{-1}{2}R_{ljm}{}^{i}-\frac{1}{4f\alpha}p_tp_aR_{lhm}{}^{a}R^{h}{}_{.\,j.}{}^{it}+\frac{1}{2\alpha}p_ap^iR_{ljm}{}^{a}-\frac{\alpha+1}{2\alpha^2}p_ap_mR_{lj.}{}^{ia}\}E_{\bar{m}},\\
\widetilde{R}(E_{\bar{l}},E_{\bar{i}})E_j &= \{\frac{1}{4f^2\alpha^2}p_tp_a(R^{m}{}_{.\,h.}{}^{la}R^{h}{}_{.\,j.}{}^{it}-R^{m}{}_{.\,h.}{}^{ia}R^{h}{}_{.\,j.}{}^{lt})+\frac{1}{f\alpha}R^{m}{}_{.\,j.}{}^{il})\\
&\quad+\frac{1}{f\alpha^2}p_a(p^iR^{m}{}_{.\,j.}{}^{la}-p^lR^{m}{}_{.\,j.}{}^{ia}\}E_m,\\
\widetilde{R}(E_l,E_i)E_{\bar{j}} &= \{\frac{1}{2f\alpha}p_a(\nabla_lR^{m}{}_{.\,i.}{}^{ja}-\nabla_iR^{m}{}_{.\,l.}{}^{ja})+\frac{1}{2f\alpha}p_a(R^{h}{}_{.\,i.}{}^{ja}A^m_{lh}-R^{h}{}_{.\,l.}{}^{ja}A^m_{ih}\\
&\quad-\frac{f_l}{f}R^{m}{}_{.\,i.}{}^{ja}+\frac{f_i}{f}R^{m}{}_{.\,l.}{}^{ja})\}E_m+\{R_{ilm}{}^{j}+\frac{1}{4f\alpha}p_tp_a(R_{lhm}{}^{t}R^{h}{}_{.\,i.}{}^{ja}\\
&\quad-R_{ihm}{}^{a}R^{h}{}_{.\,l.}{}^{jt})+\frac{1}{\alpha}p_ap^jR_{l\ im}{}^{a}-\frac{\alpha+1}{\alpha^2}p_ap_mR_{li.}{}^{ja}\}E_{\bar{m}},\\
\widetilde{R}(E_{\bar{l}},E_i)E_{\bar{j}} &= \{\frac{1}{2f\alpha}R^{m}{}_{.\,i.}{}^{jl}+\frac{1}{2f\alpha^2}p_a(p^lR^{m}{}_{.\,i.}{}^{ja}+p^iR^{m}{}_{.\,i.}{}^{la})+\frac{1}{4f^2\alpha^2}p_ap_tR^{m}{}_{.\,h.}{}^{la}R^{h}{}_{.\,i.}{}^{jt}\}E_m,\\
\widetilde{R}(E_l,E_{\bar{i}})E_{\bar{j}} &= \{\frac{-1}{2f\alpha}R^{m}{}_{.\,l.}{}^{ji}+\frac{1}{2f\alpha^2}p_a(p^iR^{m}{}_{.\,l.}{}^{ja}+p^jR^{m}{}_{.\,l.}{}^{ia})-\frac{1}{4f^2\alpha^2}p_ap_tR^{m}{}_{.\,h.}{}^{ia}R^{h}{}_{.\,l.}{}^{jt}\}E_m,\\
\widetilde{R}(E_{\bar{l}},E_{\bar{i}})E_{\bar{j}} &= \{\frac{\alpha^2+\alpha+1}{\alpha^3}(g^{ij}\delta^l_m-g^{jl}\delta^i_m)+\frac{\alpha+2}{\alpha^3}(g^{lj}p^ip_m-g^{ij}p_lp_m)\\
&\quad+\frac{\alpha-1}{\alpha^3}(\delta^i_mp^lp^j-\delta^l_mp^ip^j)\}E_{\bar{m}}
\end{aligned}$$

with respect to the adapted frame $\{E_\alpha\}$ (for $f=1$, see [1]*).*

4. Para-Kähler (or paraholomorphic) Norden structures on T^*M

An almost paracomplex manifold is an almost product manifold (M_{2k},φ), $\varphi^2=id$, $\varphi\neq\pm id$, such that the two eigenbundles T^+M_{2k} and T^-M_{2k} associated to the two eigenvalues +1 and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. An almost paracomplex Norden manifold (M_{2k},φ,g) is defined to be a real differentiable manifold M_{2k} endowed with an almost paracomplex structure φ and a Riemannian metric g satisfying Nordenian property (or purity condition)

$$g(\varphi X,Y)=g(X,\varphi Y)$$

for any $X,Y\in\Im_0^1(M_{2k})$. The almost paracomplex Norden manifold (M_{2k},φ,g) is called a paraholomorphic Norden manifold (or a para-Kähler-Norden manifold) such that $\nabla\varphi=0$, where ∇ is the Levi-Civita connection of g. Also note that $\nabla\varphi=0$ is equivalent to paraholomorphy of the metric g [21], i.e $\Phi_\varphi g=0$, where Φ_φ is the Tachibana operator [27]:

$$\begin{aligned}(\Phi_\varphi g)(X,Y,Z) &= (\varphi X)(g(Y,Z))-X(g(\varphi Y,Z))\\ &+ g((L_Y\varphi)X,Z)+g(Y,(L_Z\varphi)X)\end{aligned}$$

for any $X,Y,Z\in\Im_0^1(M_{2k})$.

V. Cruceanu defined in [4] an almost paracomplex structure on T^*M as follows:

$$\text{(4.1)}\qquad \begin{cases} J(^HX)=-^HX,\\ J(^V\omega)=^V\omega\end{cases}$$

for any $X\in\Im_0^1(M)$ and $\omega\in\Im_1^0(M)$. One can easily check that the metric ${}^{CG}g_f$ is pure with respect to the almost paracomplex structure J. Hence we state the following theorem.

4.1. Theorem. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the paracomplex structure J. Then the triplet $(T^*M,J,{}^{CG}g_f)$ is an almost paracomplex Norden manifold.*

We now give conditions for the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ to be paraholomorphic with respect to the almost paracomplex structure J. Using definition of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost paracomplex structure J and by using the fact that ${}^V\omega^V(g^{-1}(\theta,\sigma))=0$ and ${}^HX^V(fg(Y,Z))={}^V(X(fg(Y,Z)))$ we calculate

$$\begin{aligned}(\Phi_J{}^{CG}g_f)(\tilde X,\tilde Y,\tilde Z) &= (J\tilde X)({}^{CG}g_f(\tilde Y,\tilde Z))-\tilde X({}^{CG}g_f(J\tilde Y,\tilde Z))\\ &+ {}^{CG}g_f((L_{\tilde Y}J)\tilde X,\tilde Z)+{}^{CG}g_f(\tilde Y,(L_{\tilde Z}J)\tilde X)\end{aligned}$$

for all $\tilde X,\tilde Y,\tilde Z\in\Im_0^1(T^*M)$. For pairs $\tilde X={}^HX,{}^V\omega$, $\widetilde Y={}^HY,{}^V\theta$ and $\widetilde Z={}^HZ,{}^V\sigma$, we get

$$\text{(4.2)}\qquad \begin{aligned}(\Phi_J{}^{CG}g_f)({}^HX,{}^V\theta,{}^HZ) &= 2\,{}^{CG}g_f({}^V\theta,{}^V(p\circ R(X,Z)),\\ (\Phi_J{}^{CG}g_f)({}^HX,{}^HY,{}^V\sigma) &= 2\,{}^{CG}g_f({}^V(p\circ R(X,Y),{}^V\sigma)\end{aligned}$$

and the others are zero. Therefore, we have the following result.

4.2. Theorem. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the paracomplex structure J. Then the triplet $\left(T^*M,J,{}^{CG}g_f\right)$ is a para-Kähler-Norden (paraholomorphic Norden) manifold if and only if (M,g) is flat.*

4.3. Remark. Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. The diagonal lift ${}^D\gamma$ of $\gamma\in\Im_1^1(M)$ to T^*M is defined by the formulas

$$\begin{aligned}{}^D\gamma^HX &= {}^H(\gamma X),\\ {}^D\gamma^V\omega &= -^V(\omega\circ\gamma)\end{aligned}$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$. The diagonal lift ${}^D I$ of the identity tensor field $I \in \Im_1^1(M)$ has the following properties

$$\begin{aligned} {}^D I^H X &= {}^H X \\ {}^D I^V \omega &= -{}^V \omega \end{aligned}$$

and satisfies $({}^D I)^2 = I_{T^*M}$. Thus, ${}^D I$ is an almost paracomplex structure. Also, the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ is pure with respect to ${}^D I$, i.e. the triplet $\left(T^*M, {}^D I, {}^{CG}g_f\right)$ is an almost paracomplex Norden manifold. Finally, by using $\Phi-$operator, we can say that the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ is paraholomorphic with respect to ${}^D I$ if and only if (M, g) is flat.

The following remark follows directly from Proposition 3.3.

4.4. Remark. Let (M, g) be a flat Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. Then the cotangent bundle $\left(T^*M, {}^{CG}g_f\right)$ is unflat.

As is known, the almost paracomplex Norden structure is a specialized Riemannian almost product structure on a Riemannian manifold. The theory of Riemannian almost product structures was initiated by K. Yano in [29]. The classification of Riemannian almost-product structure with respect to their covariant derivatives is described by A.M. Naveira in [15]. This is the analogue of the classification of almost Hermitian structures by A. Gray and L. Hervella in [10]. Having in mind these results, M. Staikova and K. Gribachev obtained a classification of the Riemannian almost product structures, for which the trace vanishes (see [26]). There are lots of physical applications involving a Riemannian almost product manifold. Now we shall give some applications for almost paracomplex Norden structures in context of almost product Riemannian manifolds.

4.1. Let us recall almost product Riemannian manifolds. If an n-dimensional Riemannian manifold M, endowed with a Riemannian metric g, admits a non-trivial tensor field F of type (1.1) such that

$$F^2 = I$$

and

$$g(FX, Y) = g(X, FY)$$

for all $X, Y \in \Im_0^1(M)$, then F is called an almost product structure and (M, F, g) is called an almost product Riemannian manifold. An almost product Riemannian manifold with integrable almost product F is called a locally product Riemannian manifold. It is known that the integrability of an almost product structure F is equivalent to the vanishing of the Nijenhuis tensor N_F given by

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + [X, Y]$$

for all $X, Y \in \Im_0^1(M)$. If F is covariantly constant with respect to the Levi-Civita connection ∇ of g which is equivalent to $\Phi_F g = 0$, then (M, F, g) is called a locally decomposable Riemannian manifold.

Now consider the almost product structure J defined by (4.1) and the Levi-Civita connection $\widetilde{\nabla}$ given by Proposition 3.1. We define a tensor field $\widetilde{S}$ of type $(1, 2)$ on T^*M by

$$\widetilde{S}(\widetilde{X}, \widetilde{Y}) = \frac{1}{2}\{(\widetilde{\nabla}_{J\widetilde{Y}} J)\widetilde{X} + J((\widetilde{\nabla}_{\widetilde{Y}} J)\widetilde{X}) - J((\widetilde{\nabla}_{\widetilde{X}} J)\widetilde{Y})\}$$

for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(T^*M)$. Then the linear connection

$$\overline{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \widetilde{S}(\widetilde{X}, \widetilde{Y}) \tag{4.3}$$

is an almost product connection on T^*M (for almost product connection, see [12]).

4.5. Theorem. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J. Then the almost product connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}$ of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J is as follows:*

$$(4.4)\qquad \begin{cases} i)\ \overline{\nabla}_{{}^H X}{}^H Y = {}^H(\nabla_X Y) + {}^H({}^f A(X,Y)), \\ ii)\ \overline{\nabla}_{{}^H X}{}^V\theta = {}^V(\nabla_X\theta), \\ iii)\ \overline{\nabla}_{{}^V\omega}{}^H Y = \dfrac{3}{2f\alpha}\,{}^H\left(p\left(g^{-1}\circ R(\ ,Y)\tilde{\omega}\right)\right), \\ iv)\ \overline{\nabla}_{{}^V\omega}{}^V\theta = -\frac{1}{\alpha}\left({}^{CG}g\left({}^V\omega,\gamma\delta\right){}^V\theta + {}^{CG}g_f\left({}^V\theta,\gamma\delta\right){}^V\omega\right) \\ \qquad\qquad +\frac{\alpha+1}{\alpha}{}^{CG}g_f\left({}^V\omega,{}^V\theta\right)\gamma\delta - \frac{1}{\alpha}{}^{CG}g_f\left({}^V\omega,\gamma\delta\right) \\ \qquad\qquad \times{}^{CG}g_f\left({}^V\theta,\gamma\delta\right)\gamma\delta. \end{cases}$$

Denoting by $\overline{T}$ the torsion tensor of $\overline{\nabla}$, we have from (4.1), (4.3) and (4.4)

$$\begin{aligned} \overline{T}({}^V\omega,{}^V\theta) &= 0, \\ \overline{T}({}^V\omega,{}^H Y) &= \frac{3}{2f\alpha}\,{}^H\left(p\left(g^{-1}\circ R(\ ,Y)\tilde{\omega}\right)\right), \\ \overline{T}({}^H X,{}^H Y) &= -{}^V(p\circ R(X,Y)). \end{aligned}$$

Hence we have the theorem below.

4.6. Theorem. *Let (M,g) be a Riemannian manifold and let T^*M be its cotangent bundle. Then the almost product connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}$ of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J is symmetric if and only if (M,g) is flat.*

As is well-known, if there exists a symmetric almost product connection on M then the almost product structure J is integrable [12]. The converse is also true [6]. Thus we get the following conclusion.

4.7. Corollary. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J. Then the triplet $(T^*M, J, {}^{CG}g_f)$ is a locally product Riemannian manifold if and only if (M,g) is flat.*

Similarly, let us consider the almost product structure ${}^D I$ and the Levi-Civita connection $\widetilde{\nabla}$ of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. Another almost product connection can be constructed.

If J is covariantly constant with respect to the Levi-Civita connection $\widetilde{\nabla}$ of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ which is equivalent to $\Phi_J{}^{CG}g_f = 0$, then $(T^*M, J, {}^{CG}g_f)$ is called a locally decomposable Riemannian manifold. In view of Theorem 4.2, we have the following.

4.8. Corollary. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J. Then the triplet $(T^*M, J, {}^{CG}g_f)$ is a locally decomposable Riemannian manifold if and only if (M,g) is flat.*

4.2. Let (M_{2k},φ,g) be a non-integrable almost paracomplex manifold with a Norden metric. An almost paracomplex Norden manifold (M_{2k},φ,g) is a quasi-para-Kähler–Norden manifold, if $\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y,Z) = 0$, where σ is the cyclic sum by three arguments [13]. In [24], the authors proved that $\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y,Z) = 0$ is equivalent to

$\underset{X,Y,Z}{\sigma}(\Phi_{\varphi} g)(X,Y,Z)=0$. We compute

$$A(\tilde{X},\tilde{Y},\tilde{Z})=\underset{\tilde{X},\tilde{Y},\tilde{Z}}{\sigma}(\Phi_J{}^{CG}g_f)(\tilde{X},\tilde{Y},\tilde{Z})$$

for all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Im_0^1(T^*M)$. By means of (4.2), we have $A(\tilde{X},\tilde{Y},\tilde{Z})=0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Im_0^1(T^*M)$. Hence we state the following theorem.

4.9. Theorem. *Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost paracomplex structure J defined by (4.1). Then the triplet $(T^*M,J,{}^{CG}g_f)$ is a quasi-para-Kähler–Norden manifold.*

O. Gil-Medrano and A.M. Naveira proved that both distributions of the almost product structure on the Riemannian manifold (M,φ,g) are totally geodesic if and only if $\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y,Z)=0$ for any $X,Y,Z\in\Im_0^1(M)$ [9]. As a consequence of Theorem 4.9, we have the following.

4.10. Corollary. *Both distributions of the almost product Riemannian manifold $(T^*M,J,{}^{CG}g_f)$ are totally geodesic.*

4.3. Let F be an almost product structure and ∇ be a linear connection on an n-dimensional Riemannian manifold M. The product conjugate connection $\nabla^{(F)}$ of ∇ is defined by

$$\nabla_X^{(F)}Y=F(\nabla_X FY)$$

for all $X,Y\in\Im_0^1(M)$. If (M,F,g) is an almost product Riemannian manifold, then $(\nabla_X^{(F)}g)(FY,FZ)=(\nabla_X g)(Y,Z)$, i.e. ∇ is a metric connection with respect to g if and only if $\nabla^{(F)}$ is so. From this, we can say that if ∇ is the Levi-Civita connection of g, then $\nabla^{(F)}$ is a metric connection with respect to g [2].

By the almost product structure J defined by (4.1) and the Levi-Civita connection $\widetilde{\nabla}$ given by Proposition 3.1, we write the product conjugate connection $\widetilde{\nabla}^{(J)}$ of $\widetilde{\nabla}$ as follows:

$$\widetilde{\nabla}_{\widetilde{X}}^{(J)}\widetilde{Y}=J(\widetilde{\nabla}_{\widetilde{X}}J\widetilde{Y})$$

for all $\widetilde{X},\widetilde{Y}\in\Im_0^1(T^*M)$. Also note that $\widetilde{\nabla}^{(J)}$ is a metric connection of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$. The standart calculations give the following theorem.

4.11. Theorem. *Let (M,g) be a Riemannian manifold and let T^*M be its cotangent bundle equipped with the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ and the almost product structure J. Then the product conjugate connection (or metric connection) $\widetilde{\nabla}^{(J)}$ is as follows:*

$$\begin{aligned}
&i)\ \widetilde{\nabla}_{{}^HX}{}^HY={}^H\left(\nabla_XY+{}^fA(X,Y)\right)-\tfrac{1}{2}{}^V\left(p\circ R(X,Y)\right),\\
&ii)\ \widetilde{\nabla}_{{}^HX}{}^V\theta=-\tfrac{1}{2f\alpha}{}^H\left(p\left(g^{-1}\circ R(\ ,X)\widetilde{\theta}\right)\right)+{}^V\left(\nabla_X\theta\right),\\
&iii)\ \widetilde{\nabla}_{{}^V\omega}{}^HY=\tfrac{1}{2f\alpha}{}^H\left(p\left(g^{-1}\circ R(\ ,Y)\tilde{\omega}\right)\right),\\
&iv)\ \widetilde{\nabla}_{{}^V\omega}{}^V\theta=-\tfrac{1}{\alpha}\left({}^{CG}g\left({}^V\omega,\gamma\delta\right){}^V\theta+{}^{CG}g_f\left({}^V\theta,\gamma\delta\right){}^V\omega\right)\\
&+\tfrac{\alpha+1}{\alpha}{}^{CG}g_f\left({}^V\omega,{}^V\theta\right)\gamma\delta-\tfrac{1}{\alpha}{}^{CG}g_f\left({}^V\omega,\gamma\delta\right){}^{CG}g_f\left({}^V\theta,\gamma\delta\right)\gamma\delta.
\end{aligned}$$

The relationship between curvature tensors R_∇ and $R_{\nabla^{(F)}}$ of the connections ∇ and $\nabla^{(F)}$ is as follows: $R_{\nabla^{(F)}}(X,Y,Z)=F(R_\nabla(X,Y,FZ)$ for all $X,Y,Z\in\Im_0^1(M)$ [2]. By means of the almost product structure J defined by (4.1) and Proposition 3.3, from $\widetilde{R}_{\widetilde{\nabla}^{(J)}}(\widetilde{X},\widetilde{Y},\widetilde{Z})=J(\widetilde{R}_{\widetilde{\nabla}}(\widetilde{X},\widetilde{Y},J\widetilde{Z})$, components of the curvature tensor $\widetilde{R}_{\widetilde{\nabla}^{(J)}}$ of the product conjugate connection (or metric connection) $\widetilde{\nabla}^{(J)}$ can easily be computed. Finally, using the almost product structure ${}^{D}I$, another metric connection of the rescaled Cheeger-Gromoll type metric ${}^{CG}g_f$ can be constructed.

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