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Dedicated to the memory of
Prof. I. T. Mamedov (1955-2003)

Solvable time-delay differential operators for first order and their spectrums

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Abstract

In this work, firstly based on the M.I.Vishik's results and using methods of operator theory all solvable extensions of a minimal operator generated by linear delay differential-operator expression of first order in the Hilbert space of vector-functions in finite interval are described. Later on, sharp formulas for the spectrums of these solvable extensions have been found. Finally, the obtained results has been supported by applications.

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1. Introduction

Delay or time-delay differential equations or compound systems as generalization of an ordinary differential equations have been studied for at least 200 years. While some of the early investigation had its origins in certain types of geometric problems and number theory, much of the impetus for the development of the theory came from studies of viscoelasticity, population dynamics and control theory. More recent work has involved models from a wide variety of scientific fields, including nonlinear optics, economies, biology and as well population dynamics, engineering, ecology, chemistry, circadian rhythms, epidemiology, the respiratory system, tumor growth, neural networks.

Note that the fundamental theory of delay differential equations has been given in many of books. The detail analysis of this theory can be found in monographs of A. Ashyralyev and P.E. Sobolevskii [1], J.K. Hale and S.M.V. Lunel [2], O. Diekmann et al.[3], L. Edelstein-Keshet [4], L.E. El'sgol'ts and S.B. Norkin [5], T. Erneux [6], H. Smith [7] and etc.

One of the basis questions of this theory is to investigate the spectral properties of the corresponding problems.

The spectral analysis for the some delay differential equations with large delay first order with matrix coefficients has been investigated in work of M.Lichther, M.Wolfrum and S.Yanchuk [8]. Some aspects of the spectral theory have been investigated by A.Politi, G.Giacomelli, W.Huang, M.Lichther, M.Wolfrum and S.Yanchuk. In particular J.Mallet-Paret and R.D.Nussbaum [9] have studied in detail the appearance of periodic solutions for compound differential equation of first order with single delay in scalar and special cases.

Since analytical computation of solutions, eigenvalues and corresponding eigenfunctions problem is very theoretically and technically difficult, then here play significant role method of numerical analysis. Numerically computing of solutions, eigenvalues and corresponding eigenfunctions of the considered delay differential equations have been done, for example in works A. Ashyralyev with his group[10-12] and E. Jarlebring [13].

Recall that an operator $S : D(S) \subset H \to H$ in Hilbert space H is called solvable, if S is one-to-one, $SD(S) = H$ and $S^{-1} \in L(H)$.

In this work, by using methods of operator theory the all solvable extensions of minimal operator generated by delay differential operator expression for first order in the Hilbert space of vector functions at finite interval have been described in terms of boundary values. In addition, in section 3 sharp formula for the spectrum of these extensions has been given.Applications of obtained results to concrete models have been applied in section 4.

2. Description of Solvable Extensions

In the Hilbert space $L^2(H,(0,1))$ of vector-functions consider a linear delay differential-operator expression for first order in the form

$$l(u) = u'(t) + A(t)u(t-\tau), \tag{2.1}$$

where:
(1) H is a separable Hilbert space with inner product $(\cdot\,,\,\cdot)_H$ and norm $\|\cdot\|_H$;
(2) operator-function $A(\cdot) : [0,1] \to L(H)$ is continuous on the uniformly operator topology;
(3) $0< \tau <1$.

On the other hand here will be considered the following differential expression

$$m(u) = u'(t), \tag{2.2}$$

in the Hilbert space $L^2(H,(0,1))$ corresponding to (2.1). It is clear that formally adjoint expression of (2.2) is of the form

$$m^+(v) = -v'(t), \tag{2.3}$$

Now let us define operator M_0' on the dense in $L^2(H,(0,1))$ set of vector-functions D_0'

$$D_0' := \Big\{ u(t) \in L^2(H,(0,1)) : \; u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k,$$
$$\varphi_k \in C_0^\infty(0,1), \; f_k \in H, k = 1,2,\ldots,n, n \in \mathbb{N} \Big\}$$

as $M_0'u = m(u)$.

The closure of M_0' in $L^2(H,(0,1))$ is the minimal operator generated by differential-operator expression(2.2) and is denoted by M_0.

In a similar way the minimal operator M_0^+ in $L^2(H,(0,1))$ corresponding to differential expression (2.3) can be defined.

The adjoint operator of M_0^+ (M_0) in $L^2(H,(0,1))$ is called the maximal operator generated by (2.2)((2.3)) and it is denoted by $M(M^+)$. Now here define a operator S_τ, $0 < \tau < 1$ in $L^2(H,(0,1))$ in form

$$S_\tau u(t) := \begin{cases} u(t-\tau), & \text{if } \tau < t < 1, \\ 0, & \text{if } 0 < t < \tau. \end{cases}$$

From this it is obtained that

$$\begin{aligned} \|S_\tau u\|^2_{L^2(H,(0,1))} &= \int_\tau^1 (u(t-\tau), u(t-\tau))_H dt \\ &= \int_0^{1-\tau} (u(x), u(x))_H dx \\ &\leq \int_0^1 \|u(x)\|_H^2 dx \\ &= \|u\|_{L^2(H,(0,1))} \end{aligned}$$

for all $u \in L^2(H,(0,1))$.

Then $\|S_\tau\| \leq 1$, $0 < \tau < 1$. On the other words $S_\tau \in L(L^2(H,(0,1)))$ for any $\tau \in (0,1)$. In this situation the tensor product A with S_τ

$$A_\tau(t) = A(t) \otimes S_\tau, \; 0 < \tau < 1$$

is a linear bounded operator in $L^2(H,(0,1))$.

Along of this work the following defined operators

$$L_0 := M_0 + A_\tau(t),$$

$$L_0 : \overset{\circ}{W}{}_2^1 (H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

and

$$L := M + A_\tau(t),$$

$$L : W_2^1(H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

will be called the minimal and maximal operators corresponding to differential expression (2.1) in $L^2(H,(0,1))$ respectively.

Now let $U(t,s), t,s \in [0,1]$, be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} U_t'(t,s)f + A_\tau(t)U(t,s)f = 0, t,s \in (0,1) \\ U(s,s)f = f, f \in H \end{cases}$$

The operator $U(t,s), t,s \in [0,1]$ is a linear continuous boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t), s,t \in [0,1].$$

(for more detail analysis of this concept see [14]).

Let us introduce the operator

$$Uz(t) := U(t,0)z(t),\ U : L^2(H,(0,1)) \to L^2(H,(0,1)).$$

In this case it is easy to see that for the differentiable vector-function $z \in L^2(H,(0,1))$, $z : [0,1] \to H$, is valid the following relation:

$$l(Uz) = (Uz)'(t) + A(t)(Uz)(t-\tau) = U(z'(t)) + (U_t' + A_\tau(t)U)z(t) = Um(z)$$

From this $U^{-1}lU(z) = m(z)$. Hence it is clear that if the $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0 \subset \widetilde{L} \subset L$, then

$$U^{-1}L_0U = M_0,\ M_0 \subset U^{-1}\widetilde{L}U = \widetilde{M} \subset M,\ U^{-1}LU = M.$$

For example, can be easily to prove the validity of last relation. It is known that

$$D(M) = W_2^1(H,(0,1)),\ D(M_0) = \overset{o}{W}{}_2^1 (H,(0,1)).$$

If $u \in D(M)$,then $l(Uz) = Um(z) \in L^2(H,(0,1))$, that is, $Uu \in D(L)$. From last relation $M \subset U^{-1}LU$. Contrary, if a vector-function $u \in D(L)$, then

$$m(U^{-1}v) = U^{-1}l(v) \in L^2(H,(0,1)),$$

that is, $U^{-1}v \in D(M)$. From last relation $U^{-1}L \subset MU$, that is $U^{-1}LU \subset M$. Hence $U^{-1}LU = M$.

The following assertions are true.

2.1. Theorem. *$KerL_0 = \{0\}$ and $\overline{R(L_0)} \neq L^2(H,(0,1))$.*

2.2. Theorem. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,1))$ is generated by the differential-operator expression (2.1) and boundary condition*

$$(K+E)u(0) = KU(0,1)u(1), \tag{2.4}$$

where $K \in L(H)$ and E is a identity operator in H. The operator K is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_K$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfy the condition (2.4) for some bounded operator $K \in L(H)$ is a solvable extension of the minimal operator L_0 in the $L^2(H,(0,1))$.

Proof. Firstly, it is described all solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(0,1))$ in terms of boundary values.

Consider the following so-called Cauchy extension M_c

$$M_cu = u'(t),\ M_c : D(M_c) = \{u \in W_2^1(H,(0,1)) : u(0) = 0\} \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

of the minimal operator M_0. It is clear that M_c is a solvable extension of M_0 and

$$M_c^{-1}f(t) = \int_0^t f(x)dx, f \in L^2(H,(0,1)),$$

$$M_c^{-1} : L^2(H,(0,1)) \to L^2(H,(0,1)).$$

Now assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$. In this case it is known that domain of $\widetilde{M}$ can be written in direct sum in form

$$D(\widetilde{M}) = D(M_0) \oplus (M_c^{-1} + K)V,$$

where $V = KerM = H, K \in L(H)$ $(see[15])$. Therefore for each $u(t) \in D(\widetilde{M})$ it is true

$$u(t) = u_0(t) + M_c^{-1}f + Kf,\ u_0 \in D(M_0),\ f \in H.$$

That is,

$$u(t) = u_0(t) + tf + Kf,\ u_0 \in D(M_0),\ f \in H.$$

Hence

$$u(0) = Kf,\ u(1) = f + Kf = (K+E)f$$

and from these relations it is obtained that

$$(K+E)u(0) = Ku(1). \tag{2.5}$$

On the other hand uniqueness of operator $K \in L(H)$ it is clear from the work [15]. Therefore $\widetilde{M} = M_K$. This completes of necessary part of this assertion.

On the contrary, if M_K is a operator generated by differential expression (2.2) and boundary condition (2.5), then M_K is boundedly invertible and

$$M_K^{-1} : L^2(H,(0,1)) \to L^2(H,(0,1)),$$

$$M_K^{-1}f(t) = \int_0^t f(x)dx + K\int_0^1 f(x)dx,\ f \in L^2(H,(0,1)).$$

Consequently, all solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$ is generated by differential expression (2.2) and boundary condition (2.5) with any linear bounded operator K.

Now consider the general case. For the this in the $L^2(H,(0,1))$ introduce a operator in form

$$U : L^2(H,(0,1)) \to L^2(H,(0,1)),\ (Uz)(t) := U(t,0)z(t), z \in L^2(H,(0,1)).$$

From the properties of family of evolution operators $U(t,s), t,s \in [0,1]$ imply that a operator U is a linear bounded, boundedly invertible and

$$(U^{-1}z)(t) = U(0,t)z(t).$$

On the other hand from the relations

$$U^{-1}L_0U = M_0, U^{-1}\widetilde{L}U = \widetilde{M}, U^{-1}LU = M$$

it is implies that a operator U is a one-to-one between of sets of solvable extensions of minimal operators L_0 and M_0 in $L^2(H,(0,1))$.

Extension $\widetilde{L}$ of the minimal operator L_0 is solvable in $L^2(H,(0,1))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is a extension of the minimal M_0 in $L^2(H,(0,1))$. Then $u \in D(\widetilde{L})$ if and only if

$$(K+E)U(0,0)u(0) = KU(0,1)u(1),$$

that is,

$$(K+E)u(0) = KU(0,1)u(1).$$

This proves the validity of the claims in theorem. $\square$

2.3. Remark. In general case $A(t)S_\tau \neq S_\tau A(t)$ in $L^2(H,(0,1))$. Indeed, if

$$(Af)(t) = tf(t),\ f \in L^2(H,(0,1)),\ A : L^2(0,1) \to L^2(0,1),$$

then for $0 < \tau < 1$, $f \in L^2(0,1)$ we have

$$(AS_\tau)f(t) = A(S_\tau f(t)) = A(f(t-\tau)) = tf(t-\tau), 0 < t < 1$$

and

$$(S_\tau A)f(t) = S_\tau(Af(t)) = S_\tau(tf(t)) = (t-\tau)f(t-\tau), 0 < t < 1.$$

2.4. Corollary. *Assume that $A(t) = A = const$ a.e. in $(0,1)$.*

In this case all solvable extensions of minimal operator L_0 are generated by delay differential expression

$$l(u) = u'(t) + Au(t-\tau), 0 < \tau < 1$$

and boundary condition

$$\begin{aligned}(K+E)u(0) &= K[u(1) - \frac{Au(1-\tau)}{1!} + \frac{A^2u(1-2\tau)}{2!} + ...] \\ &= K\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}A^n u(1-n\tau),\ K \in L(H)\end{aligned}$$

in the Hilbert $L^2(H,(0,1))$ and vice versa.

2.5. Remark. Since for any $0 < \tau < 1$ there exists $n_0 = n_0(\tau) \in \mathbb{N}$ such that

$$0 \leq 1 - n_0\tau < 1 \text{ and } 1 - (n_0+1)\tau < 0.$$

Then

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}A^n u(1-n\tau) = \sum_{n=0}^{n_0}\frac{(-1)^n}{n!}A^n u(1-n\tau).$$

2.6. Remark. All solvable extensions of minimal operator are generated by delay differential expression

$$l(u) = u'(t) + u(t-\tau),\ 0 < \tau < 1$$

and boundary condition

$$\begin{aligned}(K+E)u(0) = K[u(1) &- \frac{u(1-\tau)}{1!} + \frac{u(1-2\tau)}{1!} + ... \\ &+ \frac{(-1)^n u(1-n\tau)}{n!} + ...],\ K \in L(H),\end{aligned}$$

in the space $L^2(H,(0,1))$ and vice versa.

In addition note that following boundary value problem

$$u'(t) = -u(t-\tau),\ \tau < t < 1,\ \tau > 0,\ \ u(t) = 1,\ \tau < t < 0$$

by changing the function $u(t)$ with $y(t) = u(t) - 1$, $\tau < t < 1$ can be reduced to problem

$$y'(t) = -y(t-\tau) - 1,\ \ y(t) = 0,\ \tau < t < 0.$$

3. Spectrum of Solvable Extension

In this section will be investigated spectrum structure of solvable extensions of minimal operator L_0 in $L^2(H,(0,1))$.
Firstly, prove the following fact.

3.1. Theorem. *If $\widetilde{L}$ is a solvable extension of a minimal operator L_0 and $\widetilde{M} = U^{-1}\widetilde{L}U$ corresponding for the solvable extension of a minimal operator M_0, then for the spectrum of these extensions is true $\sigma(\widetilde{L})=\sigma(\widetilde{M})$.*

Proof. Let us consider a problem for the spectrum for a solvable extension L_K of a minimal operator L_0 generated by delay differential-operator expression (2.1), that is,

$$L_K u = \lambda u + f, \lambda \in \mathbb{C}, f \in L^2(H,(0,1)).$$

From this it is obtained that

$$(L_K - \lambda E)u = f \quad \text{or} \quad (UM_K U^{-1} - \lambda E)u = f$$

Hence

$$U(M_K - \lambda)(U^{-1}u) = f$$

the last equation explains the validity of the theorem. □

Now prove the following result for the spectrum of solvable extension.

3.2. Theorem. *If L_K a solvable extension of the minimal operator L_0 in the space $L^2(H,(0,1))$, then spectrum of L_K has the form:*

$$\sigma(L_K) = \{\lambda \in \mathbb{C} : \lambda = \ln\left|\frac{\mu+1}{\mu}\right| + i\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi i; \\ \mu \in \sigma(K)\setminus\{0,-1\}, n \in \mathbb{Z}\}.$$

Proof. Firstly, will be investigated the spectrum of the solvable extension $M_K = U^{-1}L_K U$ of the minimal operator M_0 in $L^2(H,(0,1))$. Consider the following problem for the spectrum, $M_K u = \lambda u + f, \lambda \in \mathbb{C},\ f \in L^2(H,(0,1))$. Then

$$u' = \lambda u + f,\ (K+E)u(0) = Ku(1), \lambda \in \mathbb{C}, f \in L^2(H,(0,1)), K \in L(H).$$

It is clear that a general solution of a above differential equation in $L^2(H,(0,1))$ has the form

$$u_\lambda(t) = e^{\lambda t} f_0 + \int_0^t e^{\lambda(t-s)} f(s)ds, f_0 \in H.$$

Therefore from the boundary condition $(K+E)u_\lambda(0) = Ku_\lambda(1)$ it is obtained that

$$(E + K(1-e^\lambda))f_0 = K\int_0^1 e^{\lambda(1-s)} f(s)ds.$$

For the $\lambda_m = 2m\pi i,\ m \in \mathbb{N}$ from the last relation it is established that

$$f_0^{(m)} = K\int_0^1 e^{\lambda_m(1-s)} f(s)ds, m \in \mathbb{N}.$$

Consequently, in this case the resolvent operator of M_K is in form

$$R_{\lambda_m}(M_K)f(t) = Ke^{\lambda_m t}\int_0^1 e^{\lambda_m(1-s)} f(s)ds + \int_0^t e^{\lambda_m(t-s)} f(s)ds,\ f \in L^2(H,(0,1)), m \in \mathbb{Z}.$$

On the other hand it is clear that $R_{\lambda_m}(M_K) \in L((L^2(H,(0,1))$, $m \in \mathbb{Z}$.
If $\lambda \neq 2m\pi i, m \in \mathbb{Z}, \lambda \in \mathbb{C}$, then from boundary condition we have

$$(K - \frac{1}{e^\lambda - 1}E)f_0 = \frac{1}{1-e^\lambda}K\int_0^1 e^{\lambda(1-s)}f(s)ds, f_0 \in H,\ f \in (L^2(H,(0,1)).$$

Therefore, for $\lambda \in \sigma(M_K)$ if and only if $\mu = \frac{1}{e^\lambda - 1} \in \sigma(K)$.
In this case since $e^\lambda = \frac{\mu+1}{\mu}, \mu \in \sigma(K)$, then $\lambda_n = ln|\frac{\mu+1}{\mu}| + iarg(\frac{\mu+1}{\mu}) + 2n\pi i, n \in \mathbb{Z}$.
Later on, using the last relation and Theorem 3.1 it is proved the validity of claim in theorem. □

3.3. Corollary. *Let L_K be a solvable extension of minimal operator L_0 in $L^2(H,(0,1))$.*

(1) *If $\sigma(K) \subset \{0,1\}$, then $\sigma(L_K) = \emptyset$;*
(2) *If $\sigma(K) \setminus \{0,1\} \neq \emptyset$, then $\sigma(L_K)$ is infinite.*

Now will be proved one result on the asymptotically behaviour of eigenvalues of solvable extensions in special case.

3.4. Theorem. *If $K \in L(H)$,$K \neq 0$, $\sigma(K) = \sigma_p(K)$, there exist $\alpha, \beta > 0$ such that for any $\mu \in \sigma_p(K)$ is true*

$$|\mu| \geq \alpha > 0 \quad and \quad |\mu + 1| \geq \beta > 0,$$

then $\lambda_n(M_K) \sim 2n\pi$, as $n \to \infty$.

Proof. In this case for $n \geq 1$

$$|\lambda_n(M_K)|^2 = ln^2|\frac{\mu+1}{\mu}| + |arg(\frac{\mu+1}{\mu}) + 2n\pi|^2.$$

Since for any $\mu \in \sigma_p(K)$

$$|\frac{\mu+1}{\mu}| \geq \frac{\beta}{|\mu|} \geq \frac{\beta}{\|K\|} > 0, \quad |\frac{\mu+1}{\mu}| \leq 1 + \frac{1}{|\mu|} \leq 1 + \frac{1}{\alpha},$$

then

$$ln\frac{\beta}{\|K\|} \leq ln|\frac{\mu+1}{\mu}| \leq ln(1+\frac{1}{\alpha}).$$

Therefore for any $\mu \in \sigma_p(K)$ is true

$$min\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\} \leq |ln|\frac{\mu+1}{\mu}|| \leq max\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}.$$

On the other hand for any $n \in \mathbb{Z}$

$$(2n\pi)^2 \leq |arg(\frac{\mu+1}{\mu}) + 2n\pi|^2 \leq (2(n+1)\pi)^2.$$

Consequently, for any $n \in \mathbb{N}$

$$(2n\pi)^2\left(1 + \frac{1}{4n^2\pi^2}min^2\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}\right)$$
$$\leq |\lambda_n(M_K)|^2 \leq (2n\pi)^2\left((\frac{2(n+1)\pi}{2n\pi})^2 + \frac{1}{(2n\pi)^2}max^2\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}\right)$$

This means that $\lambda_n(M_K) \sim 2n\pi$, as $n \to \infty$. □

4. Applications

4.1. Example. Assume that

$$H = \mathbb{C}, (H, \|\cdot\|_H) = (\mathbb{C}, |\cdot|),\ A(\cdot) = a(\cdot) \in C(\mathbb{R})$$

and consider the following delay differential equation in from

$$u'(t) = a(t)u(t-\tau), 0 < \tau < 1$$

with history function $u(t) = 0, -\tau < t < 0$ in the Hilbert space $L^2(0,1)$.

Then the all solvable extension L_k of minimal operator L_0 is generated by delay differential expression

$$l(u) = u'(t) - a(t)u(t-\tau)$$

and boundary condition

$$(k+1)u(0) = k exp(\int_0^1 a(t)dt)u(1), k \in \mathbb{C}$$

in $L^2(0,1)$. In addition, spectrum of L_k is in form

$$\sigma(L_k) = \{\lambda \in \mathbb{C} : \lambda = \ln|\frac{k+1}{k}| + i\arg(\frac{k+1}{k}) + 2n\pi i, n \in \mathbb{Z}\}.$$

4.2. Example. Let us

$$(H, \|\cdot\|_H) = (\mathbb{C}, |\cdot|),\ a(\cdot), b(\cdot) \in C(\mathbb{R})$$

and consider the delay differential expression in form $l(u) = u'(t) + a(t)u(t) + b(t)u(t-\tau)$, $0 < t < 1$, $0 < \tau < 1$ with history function $u(t) = 0,\ -\tau < t < 0$. If change of function $u(\cdot)$ by $y(\cdot)$

$$y(t) = \lambda(t)u(t), \lambda(t) = exp(\int_0^t a(x)dx),$$

then

$$l(\lambda^{-1}y) = y'(t) + c(t)y(t-\tau),$$

where

$$c(t) = \frac{\lambda(t)b(t)}{\lambda(t-\tau)} = b(t)exp(\int_{t-\tau}^t a(x)dx).$$

In this case all solvable extension P_k of minimal operator P_0 is generated by delay differential expression

$$P(y) = y'(t) + c(t)y(t-\tau)$$

and boundary condition

$$(k+1)y(0) = k exp(-\int_0^1 c(t)dt)y(1), k \in \mathbb{C}$$

and vice versa.

Consequently, all solvable extension P_k of the minimal operator P_0 is generated by delay differential expression

$$l(u) = u'(t) + a(t)u(t) + b(t)u(t-\tau)$$

and boundary condition

$$(k+1)u(0) = kexp(-\int_0^1 b(t)exp(\int_{t-\tau}^{t} a(x)dx)dt)exp(\int_0^1 a(x)dx)u(1), k \in \mathbb{C}$$

and vice versa.
Moreover, spectrum of solvable extension L_k is in form

$$\sigma(L_k) = \{\lambda \in \mathbb{C} : \lambda = \ln|\frac{k+1}{k}| + i\arg(\frac{k+1}{k}) + 2n\pi i, n \in \mathbb{Z}\}, k \in \mathbb{C}.$$

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