The existence and location of eigenvalues of the one particle Hamiltonians on lattices

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Abstract

We consider a quantum particle moving in the one dimensional lattice $\mathbb{Z}$ and interacting with a indefinite sign external field $\hat{v}$. We prove that the associated Hamiltonian H can have one or two eigenvalues, situated as below the bottom of the essential spectrum, as well as above the its top. Moreover, we show that the operator H can have two eigenvalues outside of the essential spectrum and one of them is situated below the bottom of the essential spectrum, and other one above its top.

1. Introduction

We consider the Hamiltonian H of a quantum particle moving in the one-dimensional lattice $\mathbb{Z}$ and interacting with a *indefinite sign* external field $\hat{v}$, i.e., the potential has positive and negative values.

In [9] of B.Simon the existence of eigenvalues of a family of continuous Schrödinger operators $H = -\Delta + \lambda V$, $\lambda > 0$ in one and two-dimensional cases have been considered. The result that H has bound state for all $\lambda > 0$ if only if $\int V(x)dx < 0$ is proven there for all $V(x)$ with $\int (1 + |x|^2)|V(x)|dx < +\infty$.

In [3] it is presented that under certain conditions on the potential a one-dimensional Schrödinger operator has a unique bound state in the limit of weak coupling while under other conditions no bound state in this limit. This question is studied for potentials obeying $\int (1 + |x|)|V(x)|dx < +\infty$.

The questions further discussed in R. Blankenbecker M.N. Goldberger and B.Simon [1].

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All these results require the use of the modified determinant. Throughout physics, stable composite objects are usually formed by the way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions [10].

The Bose-Hubbard model, which have been used to describe the repulsive pairs, is the theoretical basis for explanation of the experimental results obtained in [10].

Since the continuous Schrödinger operator has essential spectrum fulfilling semi-axis $[0, +\infty)$ and its eigenvalues appear below the bottom of the essential spectrum, it is a model, which well described the systems of two-particles with the attractive interaction.

Zero-range potentials are the mathematically correct tools for describing contact interactions. The latter reflects the fact that the zero-range potential is effective only in the s-wave [11].

The existence of eigenvalues of a family of Schrödinger operators $H = -\Delta - \mu V$, $\lambda > 0$ with perturbation V of rank one in one and two-dimensional lattices have been considered in [7]. The result that H has a unique bound state for all $\mu > 0$ is proven there and for the unique eigenvalue $e(\mu)$ lying below the bottom of the essential spectrum an asymptotic is found as $\mu \to 0$.

In [2] for the Hamiltonian H of two fermions with attractive interaction on a neighboring sites in the one-dimensional lattice $\mathbb{Z}$ has been considered and an asymptotics of the unique eigenvalue lying below the bottom of its essential spectrum has been proven.

For a family of the generalized Friedrichs models $H_\mu(p)$, $\mu > 0, p \in T^2$ with the perturbation of rank one, associated to a system of two particles moving on the two-dimensional lattice $\mathbb{Z}$ has been considered in [6] and the existence or absence of a positive coupling constant threshold $\mu = \mu_0(p) > 0$ depending on the parameters of the model has been proved.

In[5]a family $H_\mu(p)$, $\mu > 0$, $p \in T$ of the generalized Friedrichs models with the perturbation of rank one, associated to a system of two particles, moving on the one-dimensional lattice $\mathbb{Z}$ is considered. The existence of a unique eigenvalue $E(\mu, p)$, of the operator $H_\mu(p)$ lying below the essential spectrum is proved. For any p from a neighborhood of the origin, the Puiseux series expansion for eigenvalue $E(\mu, p)$ at the point $\mu = \mu(p) \geq 0$ is found.

The main goal of this paper is to investigate the existence and location of eigenvalues of the one-particle Hamiltonian H with the zero-range interaction $\mu \neq 0$ and with interactions $\lambda \neq 0$ on a neighboring sites. We prove that the Hamiltonian H may have one or two eigenvalues, situating as below the bottom of the essential spectrum, as well as above its top. Moreover, the operator H can have two eigenvalues outside of the essential spectrum, where one of them is situated below the bottom of the essential spectrum and other one above its top.

This results are new and in accord with the known results of [9, 3, 1, 7, 6, 5].

2. The coordinate representation of the one particle Hamiltonian

Let $\mathbb{Z}$ be the one dimensional lattice(integer numbers) and $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable functions on $\mathbb{Z}$ and $\ell^{2,e}(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ be the subspace of functions(elements) $\hat{f} \in \ell^2(\mathbb{Z})$ satisfying the condition

$$\hat{f}(x) = \hat{f}(-x), x \in Z$$

The one particle operator $\hat{H}_{\mu\lambda}$ acting on $\ell^{2,e}(\mathbb{Z})$ is of the form

(2.1) $\hat{H}_{\mu\lambda} := \hat{H}_0 + \hat{V}_{\mu\lambda},$

where $\hat{H}_0$ is the Teoplitz type operator

(2.2) $(\hat{H}_0\hat{\varphi})(x) := \sum_{s \in Z} \hat{\varepsilon}(s)\hat{\varphi}(x+s), \; \hat{\varphi} \in \ell^{2,e}(\mathbb{Z}),$

and

(2.3) $(\hat{V}_{\mu\lambda}\hat{\varphi})(x) := \hat{v}_{\mu\lambda}(x)\hat{\varphi}(x), \quad \hat{\varphi} \in \ell^{2,s}(\mathbb{Z}).$

The functions $\hat{\varepsilon}(s)$ and $\hat{v}_{\mu\lambda}(s)$ are defined on $\mathbb{Z}$ as follows

$$\hat{\varepsilon}(s) = \left\{ \begin{array}{cc} 1, & |s| = 0 \\ -\frac{1}{2}, & |s| = 1 \\ 0, & |s| > 1, \end{array} \right.$$

and

$$\hat{v}_{\mu\lambda}(s) = \left\{ \begin{array}{cc} \mu, & |s| = 0 \\ \frac{\lambda}{2}, & |s| = 1 \\ 0, & |s| > 1, \end{array} \right.$$

where $\mu, \lambda \in R$ are real numbers.

We remark that $\hat{H}_{\mu\lambda}$ is a bounded self-adjoint operator on $\ell^{2,e}(\mathbb{Z})$.

3. The momentum representation of the Hamiltonian

Let $\mathbb{T} = (-\pi; \pi]$ be the one dimensional torus and $L^2(\mathbb{T}, d\nu)$ be the Hilbert space of integrable functions on $\mathbb{T}$, where $d\nu$ is the (normalized) Haar measure on $\mathbb{T}$, $d\nu(p) = \frac{dp}{2\pi}$.

Let $L^{2,e}(\mathbb{T}, d\nu) \subset L^2(\mathbb{T}, d\nu)$

be the subspace of elements $f \in L^2(\mathbb{T}, d\nu)$ satisfying the condition

$$f(p) = f(-p), \quad \text{a.e. } p \in T.$$

In the momentum representation the operator $H_{\mu\lambda}$ acts on $L^{2,e}(\mathbb{T}, d\nu)$ and is of the form

$$H_{\mu\lambda} = H_0 + V_{\mu\lambda},$$

where H_0 is the multiplication operator by function $\varepsilon(p) = 1 - \cos p$:

$$(H_0 f)(p) = \varepsilon(p)f(p), \quad f \in L^{2,e}(\mathbb{T}, d\nu),$$

and $V_{\mu\lambda}$ is the integral operator of rank 2

$$(V_{\mu\lambda}f)(p) = \int_{\mathbb{T}} \left(\mu + \lambda \cos p \cos t \right) f(t)dt, \quad f \in L^{2,e}(\mathbb{T}, d\nu).$$

4. Spectral properties of the operators $H_{\mu 0}$ and $H_{0\lambda}$

Since the perturbation operator $V_{\mu 0}$ resp.$V_{0\lambda}$ is of rank 1, according the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\mu 0})$ resp.$\sigma_{ess}(H_{0\lambda})$ of $H_{\mu 0}$ resp. $H_{0\lambda}$ doesn't depend on $\mu \in R$ resp.$\lambda \in R$ and coincides to the spectrum $\sigma(H_0)$ of H_0 (see [8]), i.e.,

$$\sigma_{ess}(H_{\mu 0}) = \sigma_{ess}(H_{0\lambda}) = \sigma(H_0) = [\min_{p \in \mathbb{T}} \varepsilon(p), \max_{p \in \mathbb{T}} \varepsilon(p)] = [0, 2].$$

For any $\mu, \lambda \in R$ we introduce the Fredholm determinant $\Delta(\mu, \lambda; z)$, associating to the one particle Hamiltonian $H_{\mu,\lambda}$, as follows

(4.1) $\Delta(\mu, \lambda; z) = \left[1 - \mu a(z)\right]\left[1 - \lambda c(z)\right] - \mu\lambda b^2(z),$

where

$$(4.2) \qquad a(z) := \int_{\mathbb{T}} \frac{d\nu}{z - \varepsilon(q)},$$

$$(4.3) \qquad b(z) := -\int_{\mathbb{T}} \frac{\cos q \, d\nu}{z - \varepsilon(q)},$$

$$(4.4) \qquad c(z) := \int_{\mathbb{T}} \frac{\cos^2 q \, d\nu}{z - \varepsilon(q)}.$$

are regular functions in $z \in \mathrm{C} \setminus [0, 2]$.

In the following theorem we have collected results on a unique eigenvalue of the operator $H_{\mu 0}$ resp. $H_{0\lambda}$ depending on the sign of $\mu \neq 0$ resp. $\lambda \neq 0$.

4.1. Theorem. *For any $0 \neq \mu \in \ R$ resp. $0 \neq \lambda \in \ R$ the operator $H_{\mu 0}$ resp. $H_{0\lambda}$ has a unique eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ lying outside of the essential spectrum:*

   (i) *If $\mu > 0$ resp. $\lambda > 0$, then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(2, +\infty)$.*
   (ii) *If $\mu < 0$ resp. $\lambda < 0$, then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(-\infty, 0)$.*
   (iii) *If $\mu > 0$ resp. $\lambda < 0$ then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(2, +\infty)$ resp. $(-\infty, 0)$.*
   (iv) *If $\mu < 0$ resp. $\lambda > 0$ then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(-\infty, 0)$ resp. $(2, +\infty)$.*

The proof of Theorem 4.1 is a consequence of the formulated below Lemmas and corollaries, which can be deduced from the simple properties of determinant $\Delta(\mu, 0; z)$ resp.$\Delta(0, \mu; z)$.

4.2. Lemma. *The number $z \in \mathrm{C} \setminus [0, 2]$ is an eigenvalue of the operator $H_{\mu,0}$ resp. $H_{0,\lambda}$ if and only if $\Delta(\mu, 0; z) = 0$ resp. $\Delta(0, \lambda; z) = 0$.*

4.3. Lemma. *Let $\mu, \lambda \in \ R. \, Then$*

$$\lim_{z \to \pm\infty} \Delta(\mu, 0\,; z) = 1,$$
$$\lim_{z \to \pm\infty} \Delta(0, \lambda\,; z) = 1,$$
$$\lim_{z \to \pm\infty} \Delta(\mu, \lambda\,; z) = 1.$$

4.4. Lemma. *The functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ are regular in the region $\mathrm{C} \setminus [0, 2]$, positive and monotone decreasing in the intervals $(-\infty, 0)$ and $(2, +\infty)$ and the following asymptotics are true:*

$$a(z) = C_1 (z - 2)^{-\frac{1}{2}} + O(z - 2)^{\frac{1}{2}}, as \ z \to 2+,$$
$$b(z) = C_1 (z - 2)^{-\frac{1}{2}} + 1 + O(z - 2)^{\frac{1}{2}}, as \ z \to 2+,$$
$$c(z) = C_1 (z - 2)^{-\frac{1}{2}} - 1 + O(z - 2)^{\frac{1}{2}}, as \ z \to 2+,$$

where $C_1 > 0$ and

$$a(z) = -C_0 (-z)^{-\frac{1}{2}} + O(-z)^{\frac{1}{2}}, as \ z \to 0-,$$
$$b(z) = -C_0 (-z)^{-\frac{1}{2}} - 1 + O(-z)^{\frac{1}{2}}, as \ z \to 0-,$$
$$c(z) = -C_0 (-z)^{-\frac{1}{2}} - 1 + O(-z)^{\frac{1}{2}}, as \ z \to 0-,$$

where $C_0 > 0$.

Proof. Since the functions under integral sign are positive the monotonicity of the Lebesgue integral gives that the functions $a(z)$ and $c(z)$ are positive. Now, we show that the function

$$b(z) := -\int_{\mathbb{T}} \frac{\cos q d\nu}{z - \varepsilon(q)}$$

is positive. Representing $b(z)$ as

$$b(z) = -\int_{-\pi}^{0} \frac{\cos q d\nu}{z - \varepsilon(q)} - \int_{0}^{\pi} \frac{\cos q d\nu}{z - \varepsilon(q)}$$

and then changing of variables $q := q + \pi$ we have that

$$b(z) := \int_{0}^{\pi} \frac{2 \cos^2 q d\nu}{(z-1)^2 - \cos^2 q} > 0$$

The asymptotics of functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ can be found in [2]. □

The Lemma 4.4 yields the following Corollary, which gives asymptotics for the functions $\Delta(\mu, 0; z)$ and $\Delta(0, \lambda; z)$.

4.5. Corollary. *The following asymptotics are true:*

(i) *If μ, $\lambda > 0$. Then*

$$\lim_{z \to 2+} \Delta(\mu, 0; z) = -\infty,$$
$$\lim_{z \to 2+} \Delta(0, \lambda; z) = -\infty,$$

(ii) *If μ, $\lambda < 0$. Then*

$$\lim_{z \to 2+} \Delta(\mu, 0; z) = +\infty,$$
$$\lim_{z \to 2+} \Delta(0, \lambda; z) = +\infty,$$

(iii) *If μ, $\lambda > 0$. Then*

$$\lim_{z \to 0-} \Delta(\mu, 0; z) = +\infty,$$
$$\lim_{z \to 0-} \Delta(0, \lambda; z) = +\infty,$$

(iv) *If μ, $\lambda < 0$. Then*

$$\lim_{z \to 0-} \Delta(\mu, 0; z) = -\infty,$$
$$\lim_{z \to 0-} \Delta(0, \lambda; z) = -\infty,$$

5. Spectral properties of the operator $H_{\mu\lambda}$

The perturbation operator $V_{\mu\lambda}$ is of rank 2 and hence by the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\mu\lambda})$ of $H_{\mu\lambda}$ doesn't depend on $\mu, \lambda \in R$ and coincides to the spectrum $\sigma(H_0)$ of H_0 (see [8]), i.e.,

$$\sigma_{ess}(H_{\mu\lambda}) = \sigma(H_0) = [\min_{p \in \mathbb{T}} \varepsilon(p), \max_{p \in \mathbb{T}} \varepsilon(p)] = [0, 2].$$

5.1. Remark. Note that since

$$(V_{\mu\lambda}f, f) = \mu |\int_{\mathbb{T}} f(t)d\nu|^2 + \lambda |\int_{\mathbb{T}} \cos t f(t)d\nu|^2, \quad f \in L^{2,e}(\mathbb{T}, d\nu)),$$

the operator $V_{\mu\lambda}$ is not only positive or only negative and hence the operator $H_{\mu\lambda}$ may have eigenvalues as below the bottom of the essential spectrum, as well as above the its top.

The following lemma describes the relations between the operator $H_{\mu,\lambda}$ and determinant $\Delta(\mu, \lambda; z)$ defined in (4.1).

5.2. Lemma. *The number $z \in \mathrm{C} \setminus [0, 2]$ is an eigenvalue of the operator $H_{\mu,\lambda}$ if and only if $\Delta(\mu, \lambda; z) = 0$.*

Proof. Let the operator $H_{\mu,\lambda}$ has an eigenvalue $z \in \mathrm{C} \setminus [0, 2]$, i.e., the equation

$$(5.1) \qquad (z - H_{\mu,\lambda})\psi(q) = (z - \varepsilon(q))\psi(q) - \mu \int_T \psi(t)d\nu(t) - \lambda \cos p \int_T \cos t \psi(t)d\nu(t) = 0$$

has a non-zero solution $\psi \in L^{2,e}(T, d\nu)$. We introduce the following linear continuous functionals defined on the Hilbert space $\psi \in L^{2,e}(T, d\nu)$

$$(5.2) \qquad c_1 := c_1(\psi) := \int_T \psi(t)d\nu(t)$$

$$(5.3) \qquad c_2 := c_2(\psi) := \int_T \cos(t)\psi(t)$$

Then we easily find that the solution of the equation (5.1) has form

$$(5.4) \qquad \psi(q) = \mu \frac{c_1}{z - \varepsilon(q)} + \lambda \frac{c_2 \cos(q)}{z - \varepsilon(q)}.$$

Putting the expression (5.6) for ψ to (5.2) and (4.7) we get the following homogeneous system of linear equations with respect to the functionals c_1 and c_2

$$(5.5) \qquad \begin{cases} c_1 = \mu c_1 \int_T \frac{d\nu}{z - \varepsilon(q)} + \lambda c_2 \int_T \frac{\cos(q)d\nu}{z - \varepsilon(q)} \\ c_2 = \mu c_1 \int_T \frac{\cos q d\nu}{z - \varepsilon(q)} + \lambda c_2 \int_T \frac{\cos^2 q d\nu}{z - \varepsilon(q)} \end{cases}$$

Hence, we can conclude that this homogenous system of linear equations has nontrivial solutions if and only if the associated determinant $\Delta(\mu, \lambda; z)$ has zero $z \in \mathbf{C} \setminus [0, 2]$.

On the contrary, let a number $z \in \mathbf{C} \setminus [0, 2]$ be a zero of determinant $\Delta(\mu, \lambda; z)$. Then it easily can be checked that z is eigenvalue of $H_{\mu,\lambda}$ and the function

$$(5.6) \qquad \psi(q) = \mu \frac{c_1}{z - \varepsilon(q)} + \lambda \frac{c_2 \cos q}{z - \varepsilon(q)},$$

is the associated eigenfunction, where the vector (c_1, c_2) is a non-zero solution of the system (5.5). $\qquad \square$

The following asymptotics for the determinant $\Delta(\mu, \lambda, z)$ can be received applying the asymptotics of the functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ in Lemma 4.4.

5.3. Lemma.

(5.7) $\quad \Delta(\mu, \lambda, z) = C^{+}_{-\frac{1}{2}}(\mu, \lambda)(z-2)^{-\frac{1}{2}} + C^{+}_0(\mu, \lambda) + O(z-2)^{\frac{1}{2}}, as\ z \to 2+,$

(5.8) $\quad \Delta(\mu, \lambda, z) = C^{-}_{-\frac{1}{2}}(\mu, \lambda)(-z)^{-\frac{1}{2}} + C^{-}_0(\mu, \lambda) + O(-z)^{\frac{1}{2}}, as\ z \to 0-,$

where

(5.9) $\quad C^{+}_{-\frac{1}{2}}(\mu, \lambda) = B_2(\mu\lambda - \mu - \lambda),\ B_2 > 0$

(5.10) $\quad C^{+}_0(\mu, \lambda) = 1 + \lambda - \mu\lambda,$

(5.11) $\quad C^{-}_{-\frac{1}{2}}(\mu, \lambda) = B_0(\mu\lambda + \mu + \lambda),\ B_0 > 0$

(5.12) $\quad C^{-}_0(\mu, \lambda) = 1 - \lambda - \mu\lambda.$

The Lemma 5.3 yields the following results for the determinant $\Delta(\mu, \lambda; z)$.

5.4. Corollary. *For the determinant $\Delta(\mu, \lambda; z)$ the following results are true:*

(i) *Assume $C^{+}_{-\frac{1}{2}}(\mu, \lambda) > 0$ and $C^{-}_{-\frac{1}{2}}(\mu, \lambda) > 0$. Then*

$$\lim_{z \to 2+} \Delta(\mu, \lambda; z) = +\infty.$$
$$\lim_{z \to 0-} \Delta(\mu, \lambda; z) = +\infty.$$

(ii) *Assume $C^{+}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu > 1$ and $C^{-}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu < -1$. Then*

$$\lim_{z \to 2+} \Delta(\mu, \lambda; z) < 0,$$
$$\lim_{z \to 0-} \Delta(\mu, \lambda; z) < 0.$$

(iii) *Assume $C^{+}_{-\frac{1}{2}}(\mu, \lambda) < 0$ and $C^{-}_{-\frac{1}{2}}(\mu, \lambda) < 0. Then$*

$$\lim_{z \to 2+} \Delta(\mu, \lambda; z) = -\infty,$$
$$\lim_{z \to 0-} \Delta(\mu, \lambda; z) = -\infty.$$

(iv) *Assume $C^{+}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu < 1$ and $C^{-}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu > -1. Then$*

$$\lim_{z \to 2+} \Delta(\mu, \lambda; z) > 0,$$
$$\lim_{z \to 0-} \Delta(\mu, \lambda; z) > 0.$$

To formulate the main theorem we introduce the regions $\mathbb{G}^{+}_{02}, \mathbb{G}^{+}_{11}$ and $\mathbb{G}^{+}_{20}$ associated to the function $C^{+}_{-\frac{1}{2}}(\mu, \lambda)$ and also the regions $\mathbb{G}^{-}_{20}, \mathbb{G}^{-}_{11}$ and $\mathbb{G}^{-}_{02}$ associated to the function $C^{-}_{-\frac{1}{2}}(\mu, \lambda)$ as follows

(5.13) $\quad \mathbb{G}_{2,+} = \{(\mu, \lambda) \in\ R^2 : C^{+}_{-\frac{1}{2}}(\mu, \lambda) > 0,\ \mu > 1\},$

(5.14) $\quad \mathbb{G}_{1,+} = \{(\mu, \lambda) \in\ R^2 : C^{+}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu > 1\ or\ C^{+}_{-\frac{1}{2}}(\mu, \lambda) < 0\},$
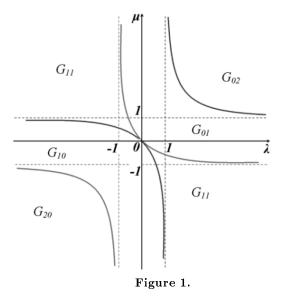
(5.15) $\quad \mathbb{G}_{0,+} = \{(\mu, \lambda) \in\ R^2\} : C^{+}_{-\frac{1}{2}}(\mu, \lambda) = 0,\ \mu < 1\ or\ C^{+}_{-\frac{1}{2}}(\mu, \lambda) > 0$

(5.16)

and

$$(5.17) \quad \mathbb{G}_{2,-} = \{(\mu,\lambda) \in \ R^2 : C^-_{-\frac{1}{2}}(\mu,\lambda) > 0, \ \mu < -1, \}$$

$$(5.18) \quad \mathbb{G}_{1,-} = \{(\mu,\lambda) \in \ R^2 : C^-_{-\frac{1}{2}}(\mu,\lambda) = 0, \ \mu < -1 \ \text{or} \ C^-_{-\frac{1}{2}}(\mu,\lambda) < 0\},$$

$$(5.19) \quad \mathbb{G}_{0,-} = \{(\mu,\lambda) \in \ R^2\} : C^+_{-\frac{1}{2}}(\mu,\lambda) = 0, \ \mu > -1 \ \text{or} \ C^-_{-\frac{1}{2}}(\mu,\lambda) > 0.$$

$$(5.20)$$

The main results are given in the following theorem, where the existence and location of eigenvalues of the one-particle Hamiltonian H with indefinite sign interaction $v_{\mu\lambda}$ are stated.

The Hamiltonian $H_{\mu\lambda}$ can have one or two eigenvalues, situating as below the bottom of the essential spectrum, as well as above its top. Moreover, the operator $H_{\mu\lambda}$ has two eigenvalues outside of the essential spectrum, depending on $\mu \neq 0$ and $\lambda \neq 0$, where one of them is situated below the bottom of the essential spectrum and the other one above its top.

Figure 1.

5.5. Theorem.　　(i) *Assume* $(\mu,\lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{2,+}$. *Then the operator* $H_{\mu\lambda}$ *has no eigenvalue below the essential spectrum and it has two eigenvalues* $\zeta_1(\mu,\lambda)$ *and* $\zeta_2(\mu,\lambda)$ *satisfying the following relations*

$$2 < \zeta_1(\mu,\lambda) < \zeta_{\min}(\mu,\lambda) \leq \zeta_{\max}(\mu,\lambda) < \zeta_2(\mu,\lambda).$$

(ii) *Assume* $(\mu,\lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$. *Then the operator* $H_{\mu\lambda}$ *has no eigenvalue below the essential spectrum and it has one eigenvalue* $\zeta_2(\mu,\lambda)$ *satisfying the following relation*
$\zeta_2(\mu,\lambda) > 2.$

(iii) *Let* $(\mu,\lambda) \in \mathbb{G}_{1,-} \cap \mathbb{G}_{1,+}$. *Then the operator* $H_{\mu\lambda}$ *has two eigenvalues* $\zeta_1(\mu,\lambda)$ *and* $\zeta_2(\mu,\lambda)$ *satisfying the following relations*

$$\zeta_1(\mu,\lambda) < 0 \ \text{and} \ \zeta_2(\mu,\lambda) > 2.$$

(iv) *Assume $(\mu, \lambda) \in \mathbb{G}_{1,-} \cap \mathbb{G}_{0,+}$. Then the operator $H_{\mu\lambda}$ has one eigenvalue $\zeta_1(\mu, \lambda)$ satisfying the relation $\zeta_1(\mu, \lambda) < 0$ it has no eigenvalue above the essential spectrum.*

(v) *Assume $(\mu, \lambda) \in \mathbb{G}_{2,-} \cap \mathbb{G}_{0,+}$. Then the operator $H_{\mu\lambda}$ has two eigenvalues $\zeta_1(\mu, \lambda)$ and $\zeta_2(\mu, \lambda)$ satisfying the following relations*

$$\zeta_1(\mu, \lambda) < \zeta_{\min}(\mu, \lambda) \leq \zeta_{\max}(\mu, \lambda) < \zeta_2(\mu, \lambda) < 0$$

and it has no eigenvalue above the essential spectrum.

5.6. Remark. The sets $G_{02}, G_{01}, G_{11}, G_{10}$ and G_{20} which appears in Theorem 5.5 are shown in the figure 1.

Proof. (i) Assume $(\mu, \lambda) \in (\mu, \lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{2,+}$ and $z < 0$. Then an application the Cauchy–Schwarz inequality for the functions $[\varepsilon(q) - z]^{-\frac{1}{2}}$ and $\cos q \, [\varepsilon(q) - z]^{-\frac{1}{2}}$ yields the inequality

$$\Delta(\mu, \lambda; z) = \left(1 + \mu \int_{\mathbb{T}} \frac{d\nu}{\varepsilon(q) - z}\right) + \left(1 + \lambda \int_{\mathbb{T}} \frac{\cos^2 q \, d\nu}{\varepsilon(q) - z}\right)$$

$$+ \mu\lambda\left[\int_{\mathbb{T}} \frac{d\nu}{\varepsilon(q) - z} \int_{\mathbb{T}} \frac{\cos^2 q \, d\nu}{\varepsilon(q) - z} - \left(\int_{\mathbb{T}} \frac{\cos q \, d\nu}{\varepsilon(q) - z}\right)^2\right] > 0,$$

i.e., $\Delta(\mu, \lambda; z)$ has no zero in the interval $(-\infty, 0)$. Lemma 5.2 gives that the operator $H_{\mu\lambda}$ has no eigenvalue below the bottom of the essential spectrum.

Let $(\mu, \lambda) \in (\mu, \lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{2,+}$ and $z > 2$.

Since $\mu, \lambda > 0$ the function $\Delta(\mu, 0; \cdot)$ resp. $\Delta(0, \lambda; \cdot)$ is monotone increasing in $(1, +\infty)$. Applying Lemma 4.3 we have

$$\lim_{z \to +\infty} \Delta(\mu, 0; z) = 1 \text{ resp. } \lim_{z \to +\infty} \Delta(0, \lambda; z) = 1.$$

Corollary 4.5 gives that

$$\lim_{z \to 1+} \Delta(\mu, 0; z) = -\infty, \text{ resp. } \lim_{z \to 1+} \Delta(0, \lambda; z) = -\infty.$$

The continuous function $\Delta(\mu, 0; \cdot)$ and $\Delta(0, \lambda; \cdot)$ has a zero $\zeta(\mu)$ resp. $\zeta(\lambda)$ in the interval $(1, +\infty)$. The representation (4.1) of the determinant $\Delta(\mu, \lambda; z)$ gives the inequality $\Delta(\mu, \lambda; \zeta(\mu)) < 0$ resp. $\Delta(\mu, \lambda; \zeta(\lambda)) < 0$. Denote by

$\zeta_{\min}(\mu, \lambda) = \min\{\zeta(\mu), \zeta(\lambda)\}$

$\zeta_{\max}(\mu, \lambda) = \max\{\zeta(\mu), \zeta(\lambda)\}$.

The representation (4.1) of determinant $\Delta(\mu, \lambda; z)$ gives the inequality $\Delta(\mu, \lambda; \zeta_{\min}(\mu, \lambda)) < 0$. Corollary 5.3 yields

$$\lim_{z \to 1+} \Delta(\mu, \lambda; z) = +\infty$$

Hence there exist a number $z_1(\mu, \lambda) \in (1, \zeta_{\min}(\mu, \lambda))$ such that

$\Delta(\mu, \lambda; z_1(\mu, \lambda; 0)) = 0$.

Lemma 5.2 gives the existence of the eigenvalue of the operator in the interval $(1, \zeta_{\min}(\mu, \lambda))$.

The monotonicity of function $\Delta(\mu, 0; z)$ resp. $\Delta(\lambda, 0; z)$ gives for $z > \zeta(\mu)$ resp. $z > \zeta(\lambda)$ the relation

$\Delta(\mu, 0; z) > \Delta(\mu, 0; \zeta(\mu)) = 0$, resp. $\Delta(\lambda, 0; z) > \Delta(\lambda; \zeta(\lambda)) = 0$.

Applying Lemma 4.4 we have in the interval $(2, +\infty)$ the inequality

$$\frac{\partial \Delta(\mu, \lambda; z)}{\partial z} = -\mu \Delta(0, \lambda; z) a'(z) - \lambda \Delta(\mu, 0,; z) c'(z) - 4\mu\lambda b(z) b'(z) > 0,$$

i.e., the function $\Delta(\mu, \lambda; \cdot)$ is monotone increasing in the interval $(\zeta_{\max}(\mu, \lambda), +\infty)$. Lemma 4.3, i.e., the relation

$$\lim_{z \to +\infty} \Delta(\mu, \lambda; z) = 1,$$

yields the existence a unique number $z_2(\mu, \lambda) \in (\zeta_{\max}(\mu, \lambda)$ such that

$$\Delta(\mu, \lambda; z_2(\mu, \lambda; 0)) = 0.$$

Lemma 5.2 gives that the operator has two eigenvalues above the top of the essential spectrum. These eigenvalues obeys the relations (5.5).

(ii) Assume $(\mu, \lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$ and $z < 0$.

As in the case (i) we can show that operator $H_{\mu\lambda}$ has no eigenvalue below the essential spectrum.

It is easy to show that the operator $H_{\mu 0}$ has only one eigenvalue at the point $(\mu, 0) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}, \mu > 0$.

Lemma 4.3 and Corollary 5.4 give that

$$\lim_{z \to -\infty} \Delta(\mu, 0; z) = 1$$

and

$$\lim_{z \to 2+} \Delta(\mu, 0; z) < 0.$$

Hence, the continuous function $\Delta(\mu, 0; \cdot)$ in $z \in (2, +\infty)$ has a unique zero $\zeta_1(\mu, 0) \in (2, +\infty)$.

If $(\mu, \lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$ is an other point belonging to the region, then there is a line
$\Gamma[(\mu, 0), (\mu, \lambda)] \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$, which connects the points $(\mu, 0)$ and (μ, λ)(because this is a region).The compactness of $\Gamma[(\mu, 0), (\mu, \lambda)] \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$ yields that at the point (μ, λ) the function $\Delta(\mu, \lambda; z)$ has only one zero. Thus, Lemma 5.2 yields that the operator has only one eigenvalue above the top of the essential spectrum.

(iii) Assume $(\mu, \lambda) \in \mathbb{G}_{1,-} \cap \mathbb{G}_{1,+}$.

In this case applying Lemma 4.3 and Corollary 5.4 we have

$$\lim_{z \to 2+} \Delta(\mu, \lambda; z) = -\infty,$$

$$\lim_{z \to 0-} \Delta(\mu, \lambda; z) = -\infty,$$

and

$$\lim_{z \to \pm\infty} \Delta(\mu, \lambda; z) = 1.$$

Hence, the continuous function $\Delta(\mu, \lambda; \cdot)$ in $z \in (-\infty, 0) \cup (2, +\infty)$ has two zeros $\zeta_1(\mu, \lambda)$ in the interval $(-\infty, 0)$ and $\zeta_2(\mu, \lambda)$ in the interval $(2, +\infty)$.

Thus, Lemma 5.2 yields that the operator has two eigenvalues: one of them lays below the bottom of the essential spectrum and other one lays above the top.

The other cases (iv) and (v) of Theorem 5.5 can be proven by the same way as the cases (i) and (ii).

$\square$

6. Acknowledgment

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