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Comparison of some set open and uniform topologies and some properties of the restriction maps

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Abstract

Let X be a Tychonoff space, Y an equiconnected space and $C(X,Y)$ be the set of all continuous functions from X to Y. In this paper, we provide a criterion for the coincidence of set open and uniform topologies on $C(X,Y)$ when these topologies are defined by a family α consisting of Y-compact subsets of X. For a subspace Z of a topological space X, we also study the continuity and the openness of the restriction map $\pi_Z : C(X,Y) \to C(Z,Y)$ when both $C(X,Y)$ and $C(Z,Y)$ are endowed with the set open topology.

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1. Introduction

Let X, Y be topological spaces and $C(X,Y)$ be the set of all continuous functions from X to Y. The set $C(X,Y)$ has a number of classical topologies; among them the topology of uniform convergence and the set open topology. Since their introduction by Arens and Dugundji [1], set open topologies have been studied and the comparison between them and the topology of uniform convergence have been considered by many authors (see, for example, [4, 7, 9, 10]).

In [4], Bouchair and Kelaiaia have established a criterion for the coincidence of the set open topology and the topology of uniform convergence on $C(X,Y)$ defined on a family α of compact subsets of X. They also have studied the comparison between some set open topologies on $C(X,Y)$ for various families α. In this paper we continue the study of the comparison between these topologies in the case when α is a family consisting of Y-compact sets and give a criterion for their coincidence.

One of the most useful tools normally used for studying function spaces is the concept of restriction map. If Z is a subspace of a topological space X, then the restriction map $\pi_Z : C(X,Y) \to C(Z,Y)$ is defined by $\pi_Z(f) = f_{|Z}$ for any $f \in C(X,Y)$. The properties of the restriction map $\pi_Z : C(X,\mathbb{R}) \to C(Z,\mathbb{R})$, when both $C(X,\mathbb{R})$ and $C(Z,\mathbb{R})$ are endowed with the topology of the pointwise convergence, have been studied by Arhangel'skii in

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[2, 3]. In the present paper, we give a criteria for the continuity and for the openness of the restriction map in the case when Y is an equiconnected topological space and $C(X,Y)$ and $C(Z,Y)$ are equipped with the set open topology.

Our paper is organized as follows. In Section 3, we prove that the set open and uniform topologies on $C(X,Y)$ coincide if and only if α is a functional refinement family. Section 4 is devoted to compare the spaces $C_\alpha(X,Y)$ and $C_\beta(X,Y)$ for two given families α and β of Y-compact subsets of X. In Section 5, we consider, for a subspace Z of a topological space X, the restriction map $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ and we give necessary and sufficient condition for the continuity and for the openness of the restriction map in the framework of set open topology. We prove that, if α is a functional refinement family consisting of closed Y-compact subsets of X and β is a family of closed Y-compact subsets of Z, then π_Z is continuous if and only if the quadruplet (β, α, X, Y) satisfies the property (P). We also show that, if α and β are two admissible families of compact subsets of X and Z respectively, then $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ is open onto its image if and only if β approximates $\alpha_{|\overline{Z}}$.

2. Definitions and preliminaries

Throughout this paper, X is a Tychonoff space, Y is an equiconnected topological space, $C(X,Y)$ is the set of all continuous functions from X to Y, and α is always a nonempty family of subsets of X. The set open topology on $C(X,Y)$ has a subbase consisting of all sets of the form $[A,V] = \{f \in C(X) : f(A) \subseteq V\}$, where $A \in \alpha$ and V is an open subset of Y, and the function space $C(X,Y)$ endowed with this topology is denoted by $C_\alpha(X,Y)$. If V is not arbitrary but is restricted to some collection $\mathcal{B}$ of open subsets of Y, then we denote by $C_\alpha^{\mathcal{B}}(X,Y)$ the corresponding function space.

For a metric space (Y,ρ), the topology of uniform convergence on members of α has as base at each point $f \in C(X,Y)$ the family of all sets of the form

$$< f, A, \epsilon >= \{g \in C(X,Y) : \sup_{x\in A} \rho(f(x), g(x)) < \epsilon\},$$

where $A \in \alpha$ and $\epsilon > 0$. The space $C(X,Y)$ having the topology of uniform convergence on α is denoted by $C_{\alpha,u}(X,Y)$.

The symbols $\emptyset$ and $\mathbb{N}$ will stand for the empty set and the positive integers, respectively. We denote by $\mathbb{R}$ the real numbers with the usual topology. The complement and the closure of a subset A in X is denoted by A^c and $\overline{A}$, respectively. If $A \subseteq X$, the restriction of a function $f \in C(X,Y)$ to the set A is denoted by $f_{|A}$. Let Z be a subspace of X, then $\alpha_{|Z}$ denotes the family $\{A \cap Z : A \in \alpha\}$.

Let β be a nonempty family of subsets of X. We say that α refines β if every member of α is contained in some member of β. We say that β approximates α provided that for every $A \in \alpha$ and every open neighborhood U of A in X, there exist $B_1, ..., B_n \in \beta$ such that $A \subseteq B_1 \cup B_2 \cup ... \cup B_n \subseteq U$. A family α is said to be admissible if for every $A \in \alpha$ and every finite sequence $U_1, ..., U_n$ of open subsets of X such that $A \subseteq \bigcup_{i=1}^{n} U_i$, there exists a finite sequence $A_1, ..., A_m$ of members of α which refines $U_1, ..., U_n$ and whose union contains A. For example, the family of all compact sets as well as the family of all finite sets in a topological space is an admissible family.

A topological space Y is said to be equiconnected [6] if there exists a continuous map $\Psi : Y \times Y \times [0,1] \to Y$ such that $\Psi(x,y,0) = x$, $\Psi(x,y,1) = y$, and $\Psi(x,x,t) = x$ for all $x, y \in Y$ and $t \in [0,1]$. The map Ψ is called an equiconnecting function. A subset V of an equiconnected space Y is called a Ψ-convex subset of Y if $\Psi(V,V,[0,1]) \subseteq V$. It is a known fact that any topological vector space or any convex subset of any topological vector space is an equiconnected space, and any equiconnected space is a pathwise connected space.

For topological space X and Y, we write $X = Y$ ($X \leq Y$) to mean that X and Y have the same underlying set and the topology on Y is the same to (finer than or equal to) the topology on X.

Definition 2.1. ([10]). Let $A \subseteq X$ and let Y be an arbitrary topological space. For a fixed natural number n, we will say that A is Y^n-compact if, for any continuous function $f \in C(X, Y^n)$, the set $f(A)$ is compact in Y^n.

We would like to mention that there are Y-compact sets which are not closed. Indeed, it is proved in [10, Example 1] that if X is the set of all ordinals that are less than or equal to ω_1 and $Y = \mathbb{R}$, then the subset of all countable ordinals from X is $\mathbb{R}$-compact but it is not closed in X. It was proved also that there are closed sets that are not Y-compact, see [10, Example 4]. So in our comparison of topologies on $C(X, Y)$ we consider the family α in the class of closed and Y-compact sets. Notice that, in the case when $A = X$ and $Y = \mathbb{R}$, the $\mathbb{R}$-compactness of the set A coincides with the pseudocompactness of the space X.

Definition 2.2. ([11]). A space Y is called *cub*−space (or quadra- space) if for any $x \in Y \times Y$ there are a continuous map f from $Y \times Y$ to Y and a point $y \in Y$ such that $f^{-1}(y) = x$.

For example, any Tychonoff space with G_δ-diagonal containing a nontrivial path or a zero-dimensional space with G_δ-diagonal containing a nontrivial convergent sequence is a cub-space. Also a pathwise connected metric space is a *cub*−space.

Proposition 2.3. *Let X be a topological space, and (Y, ρ) be a pathwise connected metric space. If A is an Y-compact subset of X and n is a natural number, then A is also an Y^n-compact subset of X.*

Proof. For the proof see Proposition 2.2 in [11]. □

The following lemma will be useful in the sequel which is a particular case of the Proposition 2.4 in [11].

Lemma 2.4. *Let X be a topological space and (Y, ρ) be a pathwise connected metric space. Then the intersection of an Y-compact subset of X and the inverse image, by any continuous function from X to Y, of any closed subset of Y is Y-compact.*

Proof. Let $A \subseteq X$ be an Y-compact set, F a closed subset of Y, and $g \in C(X, Y)$. We will show that $A \cap g^{-1}(F)$ is Y-compact. Take an arbitrary element f of $C(X, Y)$ and prove that $f(A \cap g^{-1}(F))$ is compact in Y^2. Define the function $h : X \to Y^2$ by $h(x) = (f(x), g(x))$, for every $x \in X$. It is clear that the function h is continuous. By Proposition 2.3, the set $h(A)$ is compact in Y^2. Consider the set $T = Y \times F$. We claim that $h(A \cap g^{-1}(F)) = h(A) \cap T$. Indeed, if $z \in h(A \cap g^{-1}(F))$, then $z \in h(A)$. On the other hand, there exists a point $x \in A \cap g^{-1}(F)$ such that $z = h(x) = (f(x), g(x))$. Therefore $z \in T$. Conversely, let $y \in h(A) \cap T$. Then, $y = h(x) = (f(x), g(x))$ for some $x \in A$. Since $y \in T$, we have $x \in g^{-1}(F)$. Therefore, $x \in A \cap g^{-1}(F)$ and $y \in h(A \cap g^{-1}(F))$. Thus, $h(A \cap g^{-1}(F)) = h(A) \cap T$ which is compact as the intersection of the compact set $h(A)$ with the closed set T. To finish the proof of the lemma it suffices to see that $f(A \cap g^{-1}(F))$ is the projection of the set $h(A \cap g^{-1}(F))$ on T. □

We give the following definition.

Definition 2.5. A family α of subsets of X is called a functional refinement if for every $A \in \alpha$, every finite sequence $U_1, ..., U_n$ of open subsets of Y, and every $f \in C(X, Y)$ such that $A \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$, there exists a finite sequence $A_1, ..., A_m$ of members of α which refines $f^{-1}(U_1), ..., f^{-1}(U_n)$ and whose union contains A.

It is clear that every admissible family is a functional refinement family.

Proposition 2.6. *Let X be a topological space, and (Y,ρ) be a metric space. Then, the family of all Y-compact subsets of X is a functional refinement family.*

Proof. Let $A \subseteq X$ be an Y-compact set, $\{U_1, ..., U_n\}$ a finite sequence of open subsets of Y, and $f \in C(X,Y)$ such that $A \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$. For every $y \in f(A)$, there exist $i_y \in \{1, ..., n\}$ and $\varepsilon_y > 0$ such that $\overline{B(y,\varepsilon_y)} \subseteq U_{i_y}$, where $B(y,\varepsilon_y)$ is the open ball with center y and radius ε_y. Then the family $\{B(y,\varepsilon_y) : y \in f(A)\}$ is an open covering of $f(A)$. From this covering, let us choose a finite subcover $\left\{B(y_j,\varepsilon_{y_j})\right\}_{j=1}^{m}$ and for each $j = 1, ..., m$, let us choose some i_{y_j} such that $\overline{B(y_j,\varepsilon_{y_j})} \subseteq U_{i_{y_j}}$. By Lemma 2.4, the set $A_j = f^{-1}\left(\overline{B(y_j,\varepsilon_{y_j})}\right) \cap A$ is Y-compact, for each j. Therefore, the family $A_1, ..., A_m$ of Y-compact subsets of X covers A and refines $f^{-1}(U_1), ..., f^{-1}(U_n)$. Hence, the family of all Y-compact subsets of X is a functional refinement. □

3. Coincidence of set open and uniform topologies

In this section, we study necessary and sufficient condition for the coincidence of the set open topology and the uniform topology on $C(X,Y)$ in the case when the family α consists of closed Y-compact sets. We first give subbase for the space $C_\alpha(X,Y)$ that help us to study the comparison of these topologies.

Theorem 3.1. *Let α be a functional refinement family consisting of Y-compact subsets of X and $\mathcal{B}$ be an arbitrary base for Y. Then, the family*

$$\{[A,V] : A \in \alpha, V \in \mathcal{B}\}$$

is a subbase for the space $C_\alpha(X,Y)$.

Proof. Let $f \in C(X,Y)$ and take a subbasic open neighborhood $[K,U]$ of f in $C_\alpha(X,Y)$, where $K \in \alpha$ and U open in Y . The open set U will be written as the union of some subfamily $\{V_i : \ i \in I\}$ of $\mathcal{B}$ which covers $f(K)$. Since $f(K)$ is compact, there exists $n \in \mathbb{N}^*$ such that $f(K) \subseteq \bigcup_{i=1}^{n} V_i$. Since α is a functional refinement, there exists a sequence $K_i, ..., K_m$ of members of α which refines $\{f^{-1}(V_i) : \ i = 1, ..., n\}$ and whose union contains K. For each $j \in \{1, ..., m\}$, let us choose $i_j \in \{1, ..., n\}$ such that $K_i \subseteq f^{-1}(V_{i_j})$. It is easy to see that $f \in \bigcap_{j=1}^{m} \left[K_j, V_{i_j}\right] \subseteq [K,U]$. □

The following result was obtained in [11, Theorem 3.3].

Theorem 3.2. *For every Hausdorff space X and any uniform cub-space $(Y,\mathcal{U})$ the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}|\alpha$ of uniform convergence on the saturation family α coincides with the set open topology on $C(X,Y)$, where Y has the topology induced by $\mathcal{U}$.*

Because every equiconnected metric space is a cub-space, the following result follows immediately from the above theorem.

Theorem 3.3. *Let X be a topological space, and (Y,ρ) be an equiconnected metric space. If α is a functional refinement family consisting of Y-compact subsets of X, then $C_\alpha(X,Y) = C_{\alpha,u}(X,Y)$.*

Corollary 3.4. *Let X be a topological space, and (Y,ρ) be an equiconnected metric space. If α is a family consisting of Y-compact subsets of X which contains all Y-compact subsets of its elements, then $C_\alpha(X,Y) = C_{\alpha,u}(X,Y)$.*

We will now give a necessary and sufficient condition for which $C_{\alpha}^{\mathcal{B}}(X,Y) = C_{\alpha,u}(X,Y)$. To this end, we will introduce, for a family α of subsets of X, the following family

$$\alpha_1 = \{A'/A' \text{ is } Y\text{-compact subset of } X \text{ and } \exists A \in \alpha : A' \subseteq A\}.$$

Proposition 2.6 leads us to the following corollary.

Corollary 3.5. *If α is a family of Y-compact subsets of X, then the family α_1 is always a functional refinement family.*

We give a definition.

Definition 3.6. Let X and Y be two topological spaces. Let α and β be two families of subsets of X. We will say that the quadruplet (α, β, X, Y) satisfies the property (P) provided that for every $A \in \alpha$, every open subset U in Y, and every $f \in C(X,Y)$ such that $A \subseteq f^{-1}(U)$, there exist $B_1, ..., B_n \in \beta$ with $A \subseteq \bigcup_{i=1}^{n} B_i \subseteq f^{-1}(U)$.

From the above definition, we observe that if a family β approximates α then (α, β, X, Y) satisfies the property (P).

Lemma 3.7. *Let α be a family of Y-compact subsets of X. Then α is a functional refinement if and only if the quadruplet (α_1, α, X, Y) satisfies the property (P).*

Proof. Suppose that α is a functional refinement family. We will show that (α_1, α, X, Y) satisfies the property (P). Let $A' \in \alpha_1$, U be an open subset of Y and $f \in C(X,Y)$ such that $A' \subseteq f^{-1}(U)$. Let $A \in \alpha$, with $A' \subseteq A$. If $A \subseteq f^{-1}(U)$ the proof is finished. If $A \nsubseteq f^{-1}(U)$; then the family $\{f^{-1}(f(A')^c), f^{-1}(U)\}$ is an open cover of A. Since α is a functional refinement family, there exists a finite sequence $A_1, ..., A_n$ of elements of α which refines $\{f^{-1}(f(A')^c), f^{-1}(U)\}$ and whose union contains A. Put $\mathcal{I} = \{i : A_i \subseteq f^{-1}(U)\}$. It is clear that $A' \subseteq \bigcup_{i\in\mathcal{I}} A_i \subseteq f^{-1}(U)$.

Conversely, suppose that (α_1, α, X, Y) satisfies the property (P). Let $A \in \alpha$, $\{U_1, ..., U_n\}$ a finite family of open subsets of Y and let $f \in C(X,Y)$ such that $A \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$. From Proposition 2.6, there exists a finite sequence $A_1, ..., A_m$ of Y-compact subsets of X which refines $\{f^{-1}(U_1), ..., f^{-1}(U_n)\}$ and whose union contains A. We set $A_j' = A_j \cap A$ for each $1 \leq j \leq m$. Then the subfamily $\{A_1', ..., A_m'\}$ of α_1 covers A and refines $f^{-1}(U_1), ..., f^{-1}(U_n)$. For each $j = 1, ..., m$, let us choose some i_j such that $A_j' \subseteq f^{-1}(U_{i_j})$. By our hypothesis there is, for every $j = 1, ..., m$, a finite family $\mathcal{A}_j = \{A_1^j, ..., A_{m_j}^j\}$ of members of α, such that $A_j' \subseteq \bigcup_{k=1}^{m_j} A_k^j \subseteq f^{-1}(U_{i_j})$. Put $\mathcal{A} = \bigcup_{i=1}^{m} \mathcal{A}_i$. This is a finite family of elements of α which covers A and refines $f^{-1}(U_1), ..., f^{-1}(U_n)$. Therefore α is a functional refinement family. □

Corollary 3.8. *For any family α of Y-compact subsets of X, we have $C_{\alpha}(X,Y) \leq C_{\alpha_1}(X,Y) = C_{\alpha_1,u}(X,Y) = C_{\alpha,u}(X,Y)$.*

Let $\overline{\alpha} = \{\overline{A} : A \in \alpha\}$. We have the following result.

Proposition 3.9. *Let X be a topological space, and (Y, ρ) be a metric space. For any family α of Y-compact subsets of X, we have $C_{\alpha,u}(X,Y) = C_{\overline{\alpha},u}(X,Y)$.*

Proof. Let us prove that $C_{\overline{\alpha},u}(X,Y) = C_{\alpha,u}(X,Y)$. Let $A \in \alpha$, $\varepsilon > 0$ and $f \in C(X,Y)$. As $A \subseteq \overline{A}$, we have $\left\langle f, \overline{A}, \varepsilon \right\rangle \subseteq \langle f, A, \varepsilon \rangle$, then $C_{\alpha,u}(X,Y) \leq C_{\overline{\alpha},u}(X,Y)$. Let $\left\langle f, \overline{A}, \varepsilon \right\rangle$ be an open subbasic set of $C_{\overline{\alpha},u}(X,Y)$. Let us show that $\langle f, A, \frac{\varepsilon}{3} \rangle \subseteq \left\langle f, \overline{A}, \varepsilon \right\rangle$. Let $g \in \langle f, A, \frac{\varepsilon}{3} \rangle$ and x be an arbitrary point of the set $\overline{A}$. Since g and f are continuous, then for every $\varepsilon > 0$, there exists a point $y_\varepsilon \in A$ such that $\rho(f(x), f(y_\varepsilon)) < \frac{\varepsilon}{3}$ and $\delta(g(x), g(y_\varepsilon)) < \frac{\varepsilon}{3}$

(this is possible, since $f^{-1}(B(f(x),\frac{\varepsilon}{3}))\cap g^{-1}(B(g(x),\frac{\varepsilon}{3})$ is a neighborhood of the point x and $x\in\overline{A}$). Hence $\rho(f(x),g(x))\leq\rho(f(x),f(y))+\rho(f(y),g(y))+\rho(g(x),g(y))<\varepsilon$; therefore $C_{\overline{\alpha},u}(X,Y)\leq C_{\alpha,u}(X,Y)$. □

Theorem 3.10. *Let X be a topological space, and (Y,ρ) be an equiconnected metric space with a bounded metric ρ and having a base $\mathcal{B}$ consisting of $\Psi-$convex sets. Let α be a family of closed Y-compact subsets of X. Then $C_{\alpha}^{\mathcal{B}}(X,Y)=C_{\alpha,u}(X,Y)$ if and only if α is a functional refinement family.*

Proof. If α is a functional refinement family, then by Theorems 3.1 and 3.3 we have $C_{\alpha}^{\mathcal{B}}(X,Y)=C_{\alpha}(X,Y)=C_{\alpha,u}(X,Y)$. Conversely, suppose that $C_{\alpha}^{\mathcal{B}}(X,Y)=C_{\alpha,u}(X,Y)$ and let us show that α is a functional refinement family. From Lemma 3.7 it suffices to prove that (α_1,α,X,Y) satisfies the property (P). Let $A'\in\alpha_1$, $f\in C(X,Y)$ and let U be an open subset in Y such that $A^{'}\subseteq f^{-1}(U)$. Since $f(A^{'})$ is compact, there exists a continuous function $g:Y\to[0,1]$ such that $g(f(A'))=\{1\}$ and $g(U^c)=\{0\}$. Take a nontrivial path p in Y with $p(0)\neq p(1)$, and put $h=p\circ g\circ f$. Since the topologies of the spaces $C_{\alpha_1,u}(X,Y)$ and $C_{\alpha,u}(X,Y)$ coincide, the set $\langle h,A',\epsilon\rangle$, where ϵ is strictly inferior to the distance between $p(0)$ and $p(1)$ in Y, is an open neighborhood of h in $C_{\alpha,u}(X,Y)$. Moreover, since $C_{\alpha}^{\mathcal{B}}(X,Y)=C_{\alpha,u}(X,Y)$ there exist $A_1,...,A_n\in\alpha$ and $V_1,...,V_n\in\mathcal{B}$ such that

$$h\in\bigcap_{i=1}^{n}[A_i,V_i]\subseteq\langle h,A',\epsilon\rangle\,.$$

Equiconnectedness of Y leads us, by [12, Corollary 1], to the fact that $A'\subseteq\bigcup_{i=1}^{n}A_i$. We set $\mathcal{I}=\{i:A_i\subseteq f^{-1}(U)\}$. By the same argument as in [4, Theorem 3], it follows that $A'\subseteq\bigcup_{i\in\mathcal{I}}A_i$. This means that (α_1,α,X,Y) satisfies the property (P), and so the family α is a functional refinement. □

Corollary 3.11. *Let X be a topological space, and (Y,ρ) be an equiconnected metric space with a bounded metric ρ and having a base $\mathcal{B}$ consisting of $\Psi-$convex sets. Let α be a family of closed Y-compact subsets of X. Then $C_{\alpha}^{\mathcal{B}}(X,Y)=C_{\alpha_1}(X,Y)$ if and only if α is a functional refinement family.*

4. Comparison of $C_\alpha(X,Y)$ and $C_\beta(X,Y)$

In this section, we are going to compare the topologies of $C_\alpha(X,Y)$ and $C_\beta(X,Y)$ when α and β are two families of Y-compact subsets of X.

Theorem 4.1. *Let α and β be two families of subsets of X. If (α,β,X,Y) satisfies the property (P), then $C_\alpha(X,Y)\leq C_\beta(X,Y)$.*

Proof. The proof is the same of [4, Theorem 5]. □

Theorem 4.2. *Let α and β be two families of closed Y-compact subsets of X, and Y be an equiconnected topological space having a base $\mathcal{B}$ consisting of Ψ-convex sets. If $C_\alpha(X,Y)\leq C_{\beta}^{\mathcal{B}}(X,Y)$, then the quadruplet (α,β,X,Y) satisfies the property (P).*

Proof. Let $A\in\alpha$, V an open subset in Y and $f\in C(X,Y)$ such that $A\subseteq f^{-1}(V)$. Since $f(A)$ is compact in Y, there exists a continuous function $g:Y\to[0,1]$ such that $g(f(A))=\{0\}$ and $g(V^c)=\{1\}$; Let $p:[0,1]\to Y$ be a path in Y with $p(0)\neq p(1)$, and let $h=p\circ g\circ f$. Let $W\in B$ which contains the point $p(0)$ and does not contain $p(1)$. Then $[A,W]$ is an open neighborhood of h in $C_{\alpha}^{\mathcal{B}}(X,Y)$. Since the topology of $C_{\beta}^{\mathcal{B}}(X,Y)$ is finer than the topology of $C_\alpha(X,Y)$, there exist $B_1,...,B_n\in\beta$ and $V_1,...,V_n\in\mathcal{B}$ such that

$$h\in\bigcap_{i=1}^{n}[B_i,V_i]\subseteq[A,W]\,.$$

We have then $A \subseteq \bigcup_{i=1}^{n} B_i$. Put $I = \{i : B_i \subseteq f^{-1}(V)\}$. As in the proof of Theorem 3.10, we obtain that $A \subseteq \bigcup_{i \in \mathfrak{I}} B_i$ and hence (α, β, X, Y) satisfies the property (P). □

Corollary 4.3. *Let α and β be two families consisting of closed Y-compact subsets of X, and Y be an equiconnected topological space having a base $\mathcal{B}$ consisting of Ψ-convex sets. If β is a functional refinement family, then $C_\alpha(X,Y) \leq C_\beta(X,Y)$ if and only if (α, β, X, Y) satisfies the property (P).*

5. Restriction map

In this section, we use the results obtained above to study and generalize some results due to Arhangel'skii about the properties of the so-called restriction map on function spaces. Let Z be a subspace of a topological space X. The restriction map $\pi_Z : C(X,Y) \to C(Z,Y)$ is defined by $\pi_Z(f) = f_{|Z}$ for any $f \in C(X,Y)$. We begin by examining the continuity of π_Z. The following result is stated in [3].

Proposition 5.1. [3, Proposition 1] *Let X, Y be topological spaces, and Z be a subspace of X. Let α be a network in X and β be a network in Z. If $\beta \subset \alpha$, then the restriction map $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ is continuous.*

Proposition 5.1 can be strengthened as follows.

Proposition 5.2. *Let X, Y be topological spaces, and Z be a subspace of X. Let α be a family of subsets of X and β be a family of subsets of Z. If (β, α, X, Y) satisfies the property (P), then $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ is continuous.*

Proof. Let $f \in C(X,Y)$ and $[B,V]$ be an open neighborhood of $f_{|Z}$ in $C_\beta(Z,Y)$, where $B \in \beta$ and V open in Y. Then $B \subseteq f^{-1}(V)$. Since (β, α, X, Y) satisfies the property (P), there exist $A_1, ..., A_n \in \alpha$ such that $B \subseteq \bigcup_{i=1}^{n} A_i \subseteq f^{-1}(V)$. Thus $\bigcap_{i=1}^{n} [A_i, V]$ is an open neighborhood of f in $C_\alpha(X,Y)$. It is easy to see that $\pi_Z(\bigcap_{i=1}^{n} [A_i, V_i]) \subseteq [B,V]$. Therefore π_Z is continuous. □

Proposition 5.3. *Let X, Y be topological spaces, and Z be a subspace of X. Let α be a family of subsets of X and β be a family of subsets of Z. If Z is dense in X and (β, α, X, Y) satisfies the property (P), then $\pi_Z : C_\alpha(X,Y) \to \pi_Z(C_\alpha(X,Y))$ is a bijective continuous map, i.e. a condensation.*

Proof. Let f and g be distinct elements in $C(X,Y)$. The continuity of the functions f and g and the fact that $\overline{Z} = X$ imply that $f_{|Z} \neq g_{|Z}$. Hence $\pi_Z(f) \neq \pi_Z(g)$. This means that π_Z is one-to-one. By Proposition 5.2, π_Z is continuous. □

Corollary 5.4. *Let X, Y be topological spaces, and Z be a subspace of X. Let α be a family of subsets of X and β be a family of subsets of Z. If α approximates β, then $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ is continuous.*

Let $B \subseteq Z \subseteq X$ and $V \subseteq Y$. Recall that $[B,V] = \{f \in C(X,Y) : f(B) \subseteq V\}$; let us denote by $[B,V]_Z = \{f \in C(Z,Y) : f(B) \subseteq V\}$. For the converse of Theorem 5.2, we have the following.

Proposition 5.5. *Let X be topological space, Y an equiconnected topological space having a base $\mathcal{B}$ consisting of Ψ-convex sets, and Z be a subspace of X. Let α be a functional refinement family consisting of closed Y-compact subsets of X and β be a family of closed Y-compact subsets of Z. If $\pi_Z : C_\alpha(X,Y) \to C_\beta(Z,Y)$ is continuous, then (β, α, X, Y) satisfies the property (P).*

Proof. By Corollary 4.3, it suffices to show that $C_\beta(X,Y) \leq C_\alpha(X,Y)$. Let $f \in C_\beta(X,Y)$ and $[B,V]$ be an open neighborhood of it in $C_\beta(X,Y)$, where $B \in \beta$ and V open in Y. Then $\pi_Z(f) = f_{|Z} \in [B,V]_Z$. The continuity of π_Z leads to the existence of $A_1,...,A_n \in \alpha$ and open subsets $V_1,...,V_n$ of Y such that

$$f \in \bigcap_{i=1}^{n} [A_i, V_i] \subseteq [B,V].$$

Hence $C_\beta(X,Y) \leq C_\alpha(X,Y)$, and so (β,α,X,Y) satisfies the property (P). □

Now, to find out when π_Z is open we first recall the following result obtained by Arhangel'skii [2, Proposition 3] for the topology of poinwise convergence in the case $Y = \mathbb{R}$.

Theorem 5.6. *If Z is a closed subset od X, then π_Z maps the space $C_p(X)$ openly onto the subspace $\pi_Z(C_p(X))$ of $C_p(Z)$.*

In order to study the openness of the restriction map when $C(X,Y)$ and $C(Z,Y)$ are equipped with set open topologies, we will need the following lemmas.

Lemma 5.7. *Let X be a topological space, and Y be an equiconnected space. Let K be compact subset of X, F be closed subset of X, and let $f : X \to Y$ be a continuous function such that $f(K \cap F) \subseteq V$, where V is an open Ψ-convex subset of Y. Then there exists a continuous function $f_1 : X \to Y$ such that $f_1(K) \subseteq V$ and $f_{1|F} = f_{|F}$.*

Proof. We observe that V is pathwise connected. Let $p : [0,1] \to V$ be a path in V. Put $K_1 = K \cap f^{-1}(V^c)$ which is compact. Let $g : X \to [0,1]$ be a continuous function such that $g(F) = \{1\}$ and $g(K_1) = \{0\}$. Define the function $f_1 : X \to Y$ by $f_1(z) = \Psi(p \circ g(z), f(z), g(z))$ for each $z \in X$. It is clear that f_1 is continuous, and one can easily verify that $f_{1|F} = f_{|F}$ and $f_1(K) \subseteq V$. □

Lemma 5.8. *Let X be a topological space, Y an equiconnected space with equiconnecting function Ψ, Z a subspace of X, α a family of compact subsets of X with $A \cap Z = A \cap \overline{Z}$ for each $A \in \alpha$ and let $g \in C(Z,Y)$ be a function continuously extendable over X. Let $A_1,...,A_n \in \alpha$, and $V_1,...,V_n$ are Ψ-convex open subsets of Y such that $g(A_i \cap \overline{Z}) \subseteq V_i$ for each $i = 1,...,n$. Then there exists a continuous extension $g' : X \to Y$ of g such that $g'(A_i) \subseteq V_i$ for each $i = 1,...,n$.*

Proof. We proceed by recurrence. Let $g \in C(Z,Y)$, $A \in \alpha$ and V be an open Ψ-convex subset of Y with $g(A \cap \overline{Z}) \subseteq V$. By applying Lemma 5.7 with $F = \overline{Z}$ and $K = A$, we obtain a continuous extension g' of g over X with $g'(A) \subseteq V$.

Suppose that the property is true up to n. We show that it remains true for $n+1$; let $A_1,...,A_{n+1} \in \alpha$, and $V_1,...,V_{n+1}$ are Ψ-convex open subsets of Y such that $g(A_i \cap \overline{Z}) \subseteq V_i$ for each $i = 1,...,n+1$, and let us show the existence of a continuous extension $g' \in C(X,Y)$ of g such that $g'(A_i) \subseteq V_i$ for each $i = 1,...,n$. By our assumption, we have $g((A_i \cap A_{n+1}) \cap \overline{Z}) \subseteq V_i \cap V_{n+1}$, for each $i = 1,...,n$. Then the family $\{A_1 \cap A_{n+1},...,A_n \cap A_{n+1}\}$ verifies $(A_i \cap A_{n+1}) \cap Z = (A_i \cap A_{n+1}) \cap \overline{Z}$ for each $i = 1,...,n$. Therefore, by the recurrence hypothesis, we find a function $g_1^{'} \in C(X,Y)$ extending g and such that $g_1^{'}(A_i \cap A_{n+1}) \subseteq V_i \cap V_{n+1}$, for each $i = 1,...,n$. We set $X_1 = X \cup (\bigcup_{i=1}^{n}(A_i \cap A_{n+1}))$. Clearly we have $A_i \cap X_1 = A_i \cap \overline{X}_1$ and $g_1^{'}(A_i \cap \overline{X}_1) \subseteq V_i$ for each $i = 1,...,n$. Applying Lemma 5.7 once again for $F = \overline{X}_1$ and $K = A_{n+1}$, we get an extension $g_2^{'} \in C(X,Y)$ of $g^{'}_{1|\overline{X}_1}$ such that $g_2^{'}(A_{n+1}) \subseteq V_{n+1}$. We observe that $g_2^{'}(A_i \cap A_{n+1}) \subseteq V_i$, for each i. Again we put $X_2 = X_1 \cup A_{n+1}$. Then, we have, for each $i = 1,...,n$, $A_i \cap X_2 = A_i \cap \overline{X}_2$ and $g_2^{'}(A_i \cap \overline{X}_2) \subseteq V_i$. By the recurrence hypothesis, there exists $g_3^{'} \in C(X,Y)$ which extending $g^{'}_{2|\overline{X}_2}$ such that $g_3^{'}(A_i) \subseteq V_i$, for each $i = 1,...,n$, and we have $g_3^{'}(A_{n+1}) = g_2^{'}(A_{n+1}) \subseteq V_{n+1}$. Whence $g' = g_3^{'}$ is our required function. □

Lemma 5.9. *Let X be a topological space, Z a subspace of X, α and β are two families of compact subsets of X and Z, respectively. If β approximates $\alpha_{|\overline{Z}}$, then $A \cap Z = A \cap \overline{Z}$ for each $A \in \alpha$.*

Proof. Suppose that there exists $A \in \alpha$ such that $A \cap (\overline{Z} \setminus Z) \neq \emptyset$. Let $x \in A \cap (\overline{Z} \setminus Z)$. Then there is no member of β for which x belongs. Therefore $A \cap \overline{Z}$ does not contained in any finite union of members of β; this contradicts the fact that β approximates $\alpha_{|\overline{Z}}$. □

Theorem 5.10. *Let X be a topological space, Z a subspace of X, and Y is an equiconnected space with a base $\mathcal{B}$ consisting of Ψ-convex sets. Let α be an admissible family of compact subsets of X and β be a family of compact subsets of Z. If β approximates $\alpha_{|\overline{Z}}$, then π_Z is an open map from $C_\alpha(X,Y)$ onto the subspace $\pi_Z(C_\alpha(X,Y))$ of $C_\beta(Z,Y)$.*

Proof. Let $\cap_{i=1}^n [A_i, V_i]$, where $A_1, ..., A_n \in \alpha$ and $V_1, ..., V_n \in \mathcal{B}$, be a basic open subset of $C_\alpha(X,Y)$ and $f \in \pi_Z(\cap_{i=1}^n [A_i, V_i])$. Let $f' \in C(X,Y)$ be an extension of f over X such that $f' \in \cap_{i=1}^n [A_i, V_i]$. Since β approximates $\alpha_{|\overline{Z}}$, there exists, for each $i = 1, ..., n$, a finite subfamily β_i of β such that

$$A_i \cap \overline{Z} \subseteq \bigcup \{B : B \in \beta_i\} \subseteq f^{-1}(V_i).$$

Then

$$f \in \left(\bigcap_{i=1}^{n} \bigcap_{B \in \beta_i} [B, V_i] \right) \bigcap \pi_Z(C(X,Y)) = W.$$

We have $W \subseteq \pi_Z(\cap_{i=1}^n [A_i, V_i])$. Indeed, let $g \in W$. Because $g(\bigcup \{B : B \in \beta_i\}) \subseteq V_i$, then $g(A_i \cap \overline{Z}) \subseteq V_i$ for every $i = 1, ..., n$. Also, from Lemma 5.9, we have $A \cap Z = A \cap \overline{Z}$ for each $A \in \alpha$. Then, by Lemma 5.8, there exists a function $g' \in C(X,Y)$ which agrees with g on Z and belongs to $\cap_{i=1}^n [A_i, V_i]$. We have then $g \in \pi_Z(\cap_{i=1}^n [A_i, V_i])$. Therefore $W \subseteq \pi_Z(\cap_{i=1}^n [A_i, V_i])$, which means that $\pi_Z : C_\alpha(X,Y) \to \pi_Z(C_\alpha(X,Y))$ is open. □

Lemma 5.11. *Let X be a topological space, Z a subspace of X, and α, β are two families of compact subsets of X and Z, respectively. Let Y be an equiconnected T_1-space, with equiconnecting function Ψ. If π_Z is an open map from $C_\alpha(X,Y)$ onto the subspace $\pi_Z(C_\alpha(X,Y))$ of $C_\beta(Z,Y)$, then $A \cap Z = A \cap \overline{Z}$ for each $A \in \alpha$.*

Proof. Suppose that there exists $A \in \alpha$ such that $A \cap (\overline{Z} \setminus Z) \neq \emptyset$. Let $x \in A \cap (\overline{Z} \setminus Z)$. Let $p : [0,1] \to Y$ be a path in Y with $p(0) \neq p(1)$ and put $V = Y \setminus \{p(0)\}$. Let $f \in \pi_Z([A,V])$, then we have $f = f'_{|Z}$ for some $f' \in [A,V]$. Since $\pi_Z : C_\alpha(X,Y) \to \pi_Z(C_\alpha(X,Y))$ is open, there exist $B_1, ..., B_n \in \beta$ and $V_1, ..., V_n$ open subsets in Y such that

$$f \in (\bigcap_{i=1}^{n} [B_i, V_i]) \cap \pi_Z(C(X,Y)) \subseteq \pi_Z([A,V]).$$

Since $x \in A \cap (\overline{Z} \setminus Z)$, we have $x \notin \cup_{i=1}^n B_i$. Complete regularity of X gives us a continuous function $h : X \to [0,1]$ such that $h(x) = 0$ and $h(\cup_{i=1}^n B_i) = \{1\}$. Consider the function $h_1 : X \to Y$ defined by $h_1(z) = \Psi(p \circ h(z), f'(z), h(z))$ for each $z \in X$. It is clear that h_1 is continuous, $h_1(\cup_{i=1}^n B_i) \subseteq V$ and $h_1(x) = p(0) \notin V$. So $h_{1|Z} \in (\bigcap_{i=1}^{n} [B_i, V_i]) \cap \pi_Z(C(X,Y))$ and h_1 does not belong to $[A,V]$. Assume that $h_{1|Z}$ admits another extension $h_2 \in C(X,Y)$. By continuity we have $h_{1|\bar{Z}} = h_{2|\bar{Z}}$. Thus $h_1(x) = h_2(x) \notin V$, and so $h_2 \notin [A,V]$. This leads that no continuous extension of $h_{1|Z}$ over X belongs to $[A,V]$, which contradicts the fact that $(\bigcap_{i=1}^{n} [B_i, V_i]) \cap \pi_Z(C(X,Y)) \subseteq \pi_Z([A,V])$. Hence, we have $A \cap Z = A \cap \overline{Z}$ for every $A \in \alpha$. □

Theorem 5.12. *Let X be a topological space, Z a subspace of X, and Y is an equiconnected T_1-space with a base $\mathcal{B}$ consisting of Ψ-convex sets.Let α be a family of compact subsets of X and β an admissible family of compact subsets of Z. If π_Z is an open map from $C_\alpha(X,Y)$ onto the subspace $\pi_Z(C_\alpha(X,Y))$ of $C_\beta(Z,Y)$, then β approximates $\alpha_{|\overline{Z}}$.*

Proof. Notice first that, from Lemma 5.11, we have $A\cap\overline{Z}=A\cap Z$ for every $A\in\alpha$. Furthermore, $C_\beta(Z,Y)=C_\beta^{\mathcal{B}}(Z,Y)$ because β is a functional refinement family. Let $A\in\alpha$, G an open subset of Z with $A\cap\overline{Z}=A\cap Z\subseteq G$. let G_1 be an open subset in X with $G_1\cap Z=G$. Now the subset $G_2=G_1\cup\left(X\backslash\overline{Z}\right)$, which is open in X, contains A and verifies $G_2\cap Z=G_1\cap Z=G$. Let $f:X\to[0,1]$ be a continuous function such that $f(A)=\{1\}$ and $f(X\backslash G_2)=\{0\}$. Let $V=Y\setminus\{p(0)\}$, where p is a path in Y, and put $g=p\circ f$. Then $g^{-1}(V)\subseteq G_2$. Thus $g_{|Z}^{-1}(V)\subseteq G$. Consider in $C_\alpha(X,Y)$ the subbasic open subset $[A,V]$. We have then $g\in[A,V]$. Thus $\pi_Z(g)=g_{|Z}$ belongs to $\pi_Z([A,V])$ which is open in $\pi_Z(C(X,Y))$, by our assumption. Therefore, there exist $B_1,...,B_n\in\beta$ and open sets $V_1,...,V_n\in\mathcal{B}$ such that

$$g_{|Z}\in(\cap_{i=1}^n[B_i,V_i])\cap\pi_Z(C(X,Y))\subseteq\pi_Z([A,V]).$$

By the same reasoning as in the proof of Lemma 5.11, we obtain that $A\cap Z=A\cap\overline{Z}\subseteq\cup_{i=1}^nB_i$. To continue our proof we will introduce the following notation. By B^1 we denote a subset $B\subseteq X$, and by B^0 its complementary in X, i.e., B^c. Let $\mathcal{I}=\{1,...,n\}$ and define the following set

$$\triangle=\left\{(\delta_1,\delta_2,...,\delta_n)\in\{0,1\}^n\setminus\{(0,...,0)\}:\left(\bigcap_{i\in\mathcal{I}}B_i^{\delta_i}\right)\cap\left(A\cap\overline{Z}\right)\neq\phi\right\}.$$

We have then

$$A\cap\overline{Z}=A\cap Z\subseteq\bigcup\left\{\bigcap_{i=1}^nB_i^{\delta_i}\ /\ (\delta_1,...,\delta_n)\in\triangle\right\}.$$

Fixing an element $(\delta_1,...,\delta_n)$ in $\triangle$ and let us show that $\bigcap_{\delta_i=1}V_i\subseteq V$. Assume the contrary. Let $y_0\in\bigcap_{\delta_i=1}V_i\setminus V$. Let $x_0\in\left(\bigcap_{i\in\mathcal{I}}B_i^{\delta_i}\right)\cap(A\cap\overline{Z})$, then $x_0\notin(\bigcup_{\delta_i=0}B_i)$ and $g(x_0)\in\bigcap_{\delta_i=1}V_i$. By continuity of g and the fact that $x_0\notin(\cup_{\delta_i=0}B_i)$, we can take an open neighborhood U of x_0 such that $g(U)\subseteq\bigcap_{\delta_i=1}V_i$ and $U\cap(\bigcup_{\delta_i=0}B_i)=\emptyset$. Consider a continuous function $\varphi:X\to[0,1]$ such that $\varphi(x_0)=0$ and $\varphi(U^c)=\{1\}$, and $0\le\varphi(x)\le1$ for all $x\in X$. Then the function $h:X\to Y$ defined by $h(x)=\Psi(y_0,g(x),\varphi(x))$, for all $x\in X$, is continuous and $h_{|Z}$ does not belong to $\pi_Z([A,V])$ because $h(x_0)=y_0$. But $h_{|Z}\in(\cap_{i=1}^n[B_i,V_i])\cap\pi_Z(C(X,Y))$. In fact, if $x\in B_i\cap U$ then $h(x)=\Psi(y_0,g(x),\varphi(x))\in V_i$, because V_i is Ψ-convex subset, for each $1\le i\le n$. If $x\in B_i\setminus U$, then $h(x)=g(x)\in V_i$ for each i with $1\le i\le n$. This gives us a contradiction, so we have $\bigcap_{\delta_i=1}V_i\subseteq V$. Moreover, since $\bigcap_{i\in\mathcal{I}}B_i^{\delta_i}\subseteq\bigcap_{\delta_i=1}B_i$ and $A\subseteq\bigcup\left\{\bigcap_{i\in\mathcal{I}}B_i^{\delta_i}\ /\ (\delta_i)\in\triangle\right\}$ we obtain that

$$A\cap Z\subseteq\bigcup\left\{\bigcap_{\delta_i=1}B_i\ /\ (\delta_i)\in\triangle\right\}\subseteq f_{|Z}^{-1}(V).$$

Moreover, the admissibility of β gives us, by [4, Lemma 1] , that for each $(\delta_i)\in\triangle$, there exist $\beta_{(\delta_i)}$ finite subfamily of β such that

$$\bigcap_{\delta_i=1}B_i\subseteq\bigcup\left\{B:B\in\beta_{(\delta_i)}\right\}\subseteq f_{|Z}^{-1}(V).$$

Hence

$$A\cap\overline{Z}=A\ \cap Z\subseteq\bigcup_{(\delta_i)\in\triangle}(\bigcup_{B\in\beta_{(\delta_i)}}B)\subseteq f_{|Z}^{-1}(V)\subseteq G.$$

Thus, we have β approximates $\alpha_{|\overline{Z}}$. □

Corollary 5.13. *Let X be a topological space, Z a subspace of X, and Y is an equiconnected T_1-space with a base $\mathcal{B}$ consisting of Ψ-convex sets.Let α be an admissible family of compact subsets of X and β an admissible family of compact subsets of Z. Then π_Z is an open map from $C_\alpha(X,Y)$ onto the subspace $\pi_Z(C_\alpha(X,Y))$ of $C_\beta(Z,Y)$ if and only if β approximates $\alpha_{|\overline{Z}}$.*

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