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On Gini mean difference bounds via generalised Iyengar results

Pietro Cerone *

Abstract

A variety of mathematical inequalities have been utilised to obtain approximation and bounds of the Gini mean difference.The Gini mean difference or the related index is a widely used measure of inequality in numerous areas such as in health, finance and population attributes arenas.The paper extends the Iyengar inequality to a Riemann-Stieltjes setting and obtains new results relating to the Gini mean difference.

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1. Introduction

Let $f:\mathbb{R}\to[0,\infty)$ be a *probability density function* (pdf), meaning that f is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(t)\,dt = 1$, and define

$$F(x) := \int_{-\infty}^{x} f(t)\,dt,\ x\in\mathbb{R} \quad \text{and} \quad E(f) := \int_{-\infty}^{\infty} x f(x)\,dx, \tag{1.1}$$

to be its *cumulative distribution function* and the *expectation* or *mean* provided that the integrals exist and are finite.

The *mean difference*

$$R_G(f) := \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\,dF(x)\,dF(y) \tag{1.2}$$

was proposed by Gini in 1912 [14], after whom it is usually named, but it was discussed by Helmert and other German writers in the 1870's (cf. H.A. David [12]). The mean difference has a certain theoretical attraction, being dependent on the spread of the variate values among themselves rather than on the deviations from some central value

*Department of Mathematics and Statistics,La Trobe University,Bundoora VIC 3086, Australia.
Email: p.cerone@latrobe.edu.au

([21, p. 48]). Further, as noted by Kendall and Stuart ([21, p. 48]), its defining integral (1.2) may converge when the *variance* $\sigma^2(f)$,

$$(1.3) \qquad \sigma^2(f) := \int_{-\infty}^{\infty} (x - E(f))^2 \, dF(x),$$

does not. It can, however, be more difficult to compute than (1.3).

Another useful concept is the *mean deviation* $M_D(f)$, defined by [21, p. 48]

$$(1.4) \qquad M_D(f) := \int_{-\infty}^{\infty} |x - E(f)| \, dF(x) = 2 \int_{\mu}^{\infty} (x - E(f)) \, dF(x).$$

As G.M. Giorgi noted in [15], some of the many reasons for the success and the relevance of the Gini mean difference or *Gini index* $I_G(f)$,

$$(1.5) \qquad I_G(f) = \frac{R_G(f)}{E(f)},$$

are their simplicity, certain interesting properties and useful decomposition possibilities, and these attributes have been analysed in an earlier work by Giorgi [16]. For a bibliographic portrait of the Gini index, see [15] where numerous references are given.

The Gini index given by (1.5) is a measure of relative inequality since it is a ratio of the Gini mean difference, a measure of dispersion, to the average value $\mu = E(f)$. Other measures are the coefficient of variation $V = \frac{\sigma}{\mu}$ and half the relative mean deviation $\frac{M_D(f)}{2\mu}$ where $M_D(f)$ is as defined in (1.4).

From (1.1), $F(x)$ is assumed to strictly increase on its support and its mean $\mu = E(f)$ exist. These assumptions imply that $F^{-1}(p)$ is well defined and is the population's p^{th} quantile. The theoretical Lorenz curve (Gastwirth [13]) corresponding to a given $F(x)$ is defined by

$$(1.6) \qquad L(p) = \frac{1}{\mu} \int_0^p F^{-1}(x) \, dx, \qquad 0 \le p \le 1.$$

Now $F^{-1}(x)$ is non decreasing and so from (1.6) $L(p)$ is convex and $L'(p) = 1$ at $p = F(\mu)$.

The area between the Lorenz curve and the line p, is known as the area of concentration.

The most common measure of inequality is the Gini index defined by (1.5) which may be shown to be equivalent to twice the area of concentration ([13])

$$(1.7) \qquad C = \int_0^1 c(p) \, dp, \quad c(p) = p - L(p).$$

$c(p)$ vanishes at $p = 0$ or 1 and is concave since $L(p)$ is convex. Further, there is a *point of maximum discrepancy* p^* between the Lorenz curve and the line of equality which satisfies

$$(1.8) \qquad c(p^*) \ge c(p) \qquad \text{for all } \; p \in [0,1].$$

The point $p^* = F(\mu)$ and $c(p^*) = \frac{M_D(f)}{2\mu}$ where $M_D(f)$ is given by (1.4).

The study of income inequality has gained considerable importance and the the Lorenz curve and the associated Gini mean or Gini index are certainly the most popular meausres of income inequality. These have also however found application in many other problems within the health, finance and population arenas.

In a sequence of four papers, Cerone and Dragomir ([6] – [10]) developed approximation and bounds from identities involving the Gini mean difference $R_G(f)$. Some of these results involved using the well known Sonin and Korkine identities. Cerone [3] procured some approximations and bounds utilising the well known Steffensen and Karamata inequalities. Further, the characteristics of the Lorenz curve, $L(p)$ and its connection to the Gini index via (1.7) to obtain upper and lower bounds for both $L(p)$ and $I_G(f)$ was analysed by the author in [4]. This was accomplished by utilising the well known Young's integral inequality and some less well known reverse inequalities.

The main aim of the current paper is to develop generalisations and extensions of the Iyengar inequality to allow the approximation and bounds of Riemann-Stieltjes integrals and weighted integrals in a less restrictive framework.These developments are then used to procure novel results for the approximation and bounds of the Gini mean difference.

2. Some identities Associated with the Gini mean difference

Some identities for the Gini mean difference, $R_G(f)$ through which results for the Gini index $I_G(f)$ may be procured via the relationship (1.5) will be stated here. These have been used in [6] – [10] to obtain approximations and bounds. The reader is referred to the book [21], Exercise 2.9, p. 94 or [6].

The following results hold (see for instance [21, p. 54] or [6];[7],using the well known Sonin identity; and [8] using the Korkine identity respectively.

2.1. Theorem. *With the above notation, the identities*

$$R_G(f)=\int_{-\infty}^{\infty}(1-F(y))F(y)\,dy=2\int_{-\infty}^{\infty}xf(x)F(x)\,dx-E(f); \tag{2.1}$$

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty}(x-E(f))(F(x)-\gamma)f(x)\,dx \\ &= 2\int_{-\infty}^{\infty}(x-\delta)\left(F(x)-\frac{1}{2}\right)f(x)\,dx \end{aligned} \tag{2.2}$$

for any $\gamma,\delta\in\mathbb{R}$; and

$$R_G(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-y)(F(x)-F(y))f(x)f(y)\,dxdy, \tag{2.3}$$

hold.

The following lemma was proven in [4] bounding the Gini index via the Lorenz curve and the area of concentration C. The identity is also proven in [21, p. 49] in a different way.

2.2. Lemma. *The following identity holds*

$$R_G(f)=\mu I_G(f)=2\mu C, \tag{2.4}$$

where the quantities are defined by (1.2), (1.5), (1.6) – (1.7).

3. Iyengar Inequality for Riemann-Stieltjes Integrals

In 1938 Iyegar using geometric arguments developed the following result in the paper [18] .

3.1. Theorem. *Let $h : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ and for some $M > 0$ we have $\left|h^{'}(x)\right| \leq M$ then,*

$$\left|\int_a^b h(x)\,dx - \frac{h(a)+h(b)}{2}(b-a)\right| \leq \frac{M}{4}(b-a)^2 - \frac{(h(b)-h(a))^2}{4M}. \tag{3.1}$$

Remark

It should be noted that for $m \leq h^{'}(x) \leq M$ then $\left|h^{'}(x) - \frac{m+M}{2}\right| \leq \frac{M-m}{2}$ so that the Iyengar's result may be extended by applying it to $k(x) = h(x) - \frac{m+M}{2}x$ with bound $M_k = \frac{M-m}{2}$.

The following result extends the Iyengar inequality to involve Riemann-Stieltjes integrals while also relaxing the differentiability condition.

3.2. Theorem. *Let $h, g : [a,b] \to \mathbb{R}$ be such that g is non decreasing function and for all $x \in [a,b]$ and $M > 0$ the following conditions hold,*

$$|h(x) - h(a)| \leq M \cdot (x-a) \text{ and } |h(x) - h(b)| \leq M \cdot (b-x)\ . \tag{3.2}$$

Then for any $t \in [a,b]$

$$\left|\int_a^b h(x)\,dg(x) - \{[g(t)-g(a)]\,h(a) + [g(b)-g(t)]\,h(b)\}\right| \tag{3.3}$$

$$\leq M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right]. \tag{3.4}$$

Proof. We have from (3.2)

$$\begin{aligned} h(a) - M(x-a) &\leq h(x) \leq h(a) + M(x-a) \text{ and} \\ h(b) - M(b-x) &\leq h(x) \leq h(b) + M(b-x) \end{aligned}$$

so that since $g(x)$ is non decreasing on $[a,b]$ it follows that

$$h(a)\int_a^t dg(x) - M\int_a^t (x-a)dg(x) \leq \int_a^t h(x)dg(x) \leq h(a)\int_a^t dg(x) + M\int_a^t (x-a)dg(x)$$

and

$$h(b)\int_t^b dg(x) - M\int_t^b (b-x)dg(x) \leq \int_t^b h(x)dg(x) \leq h(b)\int_t^b dg(x) + M\int_t^b (b-x)dg(x).$$

Combining the last two results produces

$$\begin{aligned} & -M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right] \\ \leq & \int_a^b h(x)dg(x) - \left\{h(a)\int_a^t dg(x) + h(b)\int_t^b dg(x)\right\} \\ \leq & M\left[\int_a^t (x-a)dg(x) + \int_t^b (b-x)dg(x)\right]. \end{aligned} \tag{3.5}$$

Simplifying and using the properties of the modulus produces (3.3). □

3.3. Corollary. *Let the conditions of Theorem 3.2 persist then the coarser but simpler bound is given by,*

$$\begin{aligned} & \left|\int_a^b h(x)\,dg(x) - \{[g(t)-g(a)]\,h(a) + [g(b)-g(t)]\,h(b)\}\right| \\ \leq & M\left[\frac{b-a}{2} + \left|t - \frac{a+b}{2}\right|\right](g(b)-g(a)), \end{aligned} \tag{3.6}$$

with the smallest bound occuring at $t = \frac{a+b}{2}$.

Proof. Let the bound from (3.3) be denoted by

$$B(t) := \int_a^t (x-a)dg(x) \; + \; \int_t^b (b-x)dg(x)$$

so that

$$\begin{aligned} |B(t)| &= \left| \int_a^b K(x,t)dg(x) \right| \le \int_a^b |K(x,t)|\, dg(x) \\ &\le \sup_{x\in[a,b]} |K(x,t)| \int_a^b dg(x) = \max\{t-a, b-t\}\,(g(b)-g(a)). \end{aligned}$$

Now, using the fact that $\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|$ produces (3.6) and the fact that the best of these occurs at $t = \frac{a+b}{2}$ is obvious. □

3.4. Theorem. *Let* $h, g : [a,b] \to \mathbb{R}$ *be such that* g *is non decreasing for all* $x \in [a,b]$ *and for* $M > 0$ *the following conditions hold,*

$$|h(x) - h(a)| \le M \cdot (x-a) \text{ and } |h(x) - h(b)| \le M \cdot (b-x) \; . \tag{3.7}$$

Then for $t \in [a,b]$ *the tightest bound is given by*

$$-MD(t^*) \le \int_a^b h(x)\, dg(x) \; - [h(b)g(b) - h(a)g(a)] \le MD(t_*), \tag{3.8}$$

or

$$-2M\delta(t_m)g(t_m) \le \int_a^b h(x)\, dg(x) \; - [h(b)g(b) - h(a)g(a)] \le 2M\Delta(t_m)g(t_m), \tag{3.9}$$

where for $\alpha = \frac{a+b}{2}$ *and* $\beta = \frac{h(b)-h(a)}{2}$; $t^* = \alpha - \frac{\beta}{M}$ *and* $t_* = \alpha + \frac{\beta}{M}$ *or* $D(t_m) = 0$ *with*

$$\begin{aligned} D(t) &= \int_t^b g(x)dx - \int_a^t g(x)dx \quad \text{and} \\ \delta(t) &= \frac{h(b)-h(a)}{2M} - \left(t - \frac{a+b}{2}\right), \\ \Delta(t) &= -\left[\frac{h(b)-h(a)}{2M} + \left(t - \frac{a+b}{2}\right)\right]. \end{aligned}$$

Here, $t^* \in [a, \frac{a+b}{2}]$ *and* $t_* \in [\frac{a+b}{2}, b]$.

Proof. From (3.5) we have on integration by parts of the Riemann-Stieltjes integrals $\int_t^b (b-x)dg(x)$ and $\int_a^t (x-a)dg(x)$,

$$L(t) \le \int_a^b h(x)\, dg(x) \; - [h(b)g(b) - h(a)g(a)] \le R(t) \tag{3.10}$$

where

$$\begin{aligned} L(t,-M) &= -2M\left[\frac{h(b)-h(a)}{2M} - \left(t - \frac{a+b}{2}\right)\right] g(t) \\ &\quad -M\left[\int_t^b g(x)dx - \int_a^t g(x)dx\right] \end{aligned} \tag{3.11}$$

and $R(t) = L(t,M)$.

We notice that (3.11) may be simplified by choosing $t = t^* = \alpha - \frac{\beta}{M}$ or $t = t_m$ where $D(t_m) = 0$ to produce the two lower bounds in (3.8) and (3.9). A similar reasoning provides the two upper bounds where $t = t_* = \alpha + \frac{\beta}{M}$.

The best bounds may be procured from the supremum of the lower bounds and the infimum of the upper bounds for $t \in [a,b]$ .Further, using the conditions in (3.7) it may be demonstrated that $t^* \in [a, \frac{a+b}{2}]$ and $t_* \in [\frac{a+b}{2}, b]$. □

The following theorem develops a weighted Iyengar inequality.

3.5. Theorem. *Let $h, w : [a,b] \to \mathbb{R}$ be such that $w(x) > 0$ for $x \in (a,b)$ and for $M > 0$ the following conditions hold,*

$$|h(x) - h(a)| \leq M \cdot (x-a) \text{ and } |h(x) - h(b)| \leq M \cdot (b-x) \ . \tag{3.12}$$

Then for $t \in (a,b)$ the tightest bound is given by

$$\left| \int_a^b w(x)h(x)\,dx \; - \{h(b)W(b) + M[I(t_*) - I(t^*)]\} \right| \leq M \left\{ \int_a^b (b-x)w(x)dx \; - [I(t^*) + I(t_*)] \right\}, \tag{3.13}$$

where for $\alpha = \frac{a+b}{2}$ and $\beta = \frac{h(b)-h(a)}{2}$; $t^* = \alpha - \frac{\beta}{M}$ and $t_* = \alpha + \frac{\beta}{M}$ with

$$I(t) = \int_a^t (t-x)w(x)dx \ . \tag{3.14}$$

If $w(a) = 0$ then the bounds at $t = a$ need to be compared with $L(t^*)$ and $R(t_*)$ and similarly for $w(b) = 0$.

Proof. Let $g(x) = \int_a^x w(u)du$ in (3.5) then

$$H(t) - M \cdot K(t) \leq \int_a^b w(x)h(x)\,dx \; \leq H(t) + M \cdot K(t) \tag{3.15}$$

where

$$W(t) = \int_a^t w(x)dx \ , \tag{3.16}$$

$$H(t) = h(a)W(t) + h(b)[W(b) - W(t)] \text{ and,} \tag{3.17}$$

$$K(t) = \int_a^t (x-a)w(x)dx \; + \int_t^b (b-x)w(x)dx.$$

If we now let $L(t, -M)$ represent the lower bound (3.15) $L(t)$, namely

$$L(t) = H(t) - M \cdot K(t) \tag{3.18}$$

and $R(t) = L(t, M)$ represent the upper bound,

$$R(t) = H(t) + M \cdot K(t). \tag{3.19}$$

so that (3.15) may be written in the form

$$\left| \int_a^b w(x)h(x)\,dx - \frac{R(t)+L(t)}{2} \right| \leq \frac{R(t)-L(t)}{2}. \tag{3.20}$$

Then we have that

$$L'(t) = \{[h(a) - h(b)] - M \cdot [2t - (a+b)]\}\, w(t)$$

and so the largest lower bound occurs at $t^* = \frac{a+b}{2} - \frac{h(b)-h(a)}{2M}$ since $w(t) > 0$ for $t \in (a,b)$. In a similar fashion we have that the smallest upper bound occurs at $t_* = \frac{a+b}{2} + \frac{h(b)-h(a)}{2M}$.

Thus we have from (3.15) that

$$L(t^*) \leq \int_a^b w(x)h(x)\,dx \; \leq R(t_*) \tag{3.21}$$

and so

$$\left|\int_a^b w(x)h(x)\,dx - \frac{R(t_*)+L(t^*)}{2}\right| \le \frac{R(t_*)-L(t^*)}{2}. \tag{3.22}$$

where after some simplification,

$$L(t^*) = h(b)W(b) - M\int_a^b (b-x)w(x)dx + 2M\int_a^{t^*}(t^*-x)w(x)dx \tag{3.23}$$

and,

$$R(t_*) = h(b)W(b) + M\int_a^b (b-x)w(x)dx - 2M\int_a^{t_*}(t_*-x)w(x)dx. \tag{3.24}$$

The result (3.13) is procured following some straight forward algebra from (3.22). □

Remark

It should be noted that taking $w(x) = 1$ in Theorem 3.5 recaptures the Iyengar result of Theorem 3.1 under less restrictive conditions (3.12) rather than $\left|h'(x)\right| < M$. It should be further emphasised that for $m \le \frac{h(x)-h(a)}{x-a} \le M$ and $m \le \frac{h(b)-h(x)}{b-x} \le M$ the above results may be extended by taking $k(x) = h(x) - \frac{M+m}{2}x$ to produce the conditions of the above results for $|k(x)-k(a)| \le \frac{M-m}{2}\cdot(x-a)$ and $|k(x)-k(b)| \le \frac{M-m}{2}\cdot(b-x)$.

4. Application of Extended Iyengar Results to Gini Mean Difference

We are now in a positition to obtain bounds utilising the Iyengar type inequalities developed above to obtain approximation and bounds for the Gini mean difference. We shall make use of the following identities, where f is the pdf and F its corresponding distribution,

$$R_G(f) = \int_a^b (1-F(x))F(x)\,dx = 2\int_a^b xf(x)F(x)\,dx - E(f). \tag{4.1}$$

4.1. Theorem. *Let $f(x)$ be a pdf on $[a,b]$, $f(x) \le M$ and $F(x) = \int_a^x f(u)du$ then the Gini Mean Difference $R_G(f)$ satisfies*

$$\begin{aligned} &|R_G(f) + E(f) - 2\{bf(b)E(f) + M[I(t^*) - I(t_*)]\}| \\ \le\ & 2M\left\{\int_a^b (b-x)xf(x)dx - [I(t_*) + I(t^*)]\right\} \end{aligned} \tag{4.2}$$

where $t^* = \frac{a+b}{2} - \frac{bf(b)-af(a)}{2M}$ and $t_* = \frac{a+b}{2} + \frac{bf(b)-af(a)}{2M}$ with,

$$I(t) = \int_a^t (t-x)xf(x)dx\ .$$

For $f(a) = 0$ we have

$$|R_G(f) - [2bf(b)-1]E(f)| \le 2M\int_a^b (b-x)xf(x)dx. \tag{4.3}$$

For $f(b) = 0$ we have

$$|R_G(f) - [2af(a)-1]E(f)| \le 2M\int_a^b (x-a)xf(x)dx. \tag{4.4}$$

Finally, for $f(a) = f(b) = 0$ we have

$$|R_G(f) + E(f)| \le 2M\left\{\frac{b-a}{2}E(f) - \left|\mathcal{M}_2 - \frac{b+a}{2}E(f)\right|\right\} \tag{4.5}$$

where $\mathcal{M}_2 = \int_a^b x^2 f(x)dx$, *the second moment of* $f(x)$.

Proof. In Theorem 3.5 let $w(x) = xf(x)$ and $h(x) = F(x)$ so that $\left|h'(x)\right| = f(x) \leq M$.

Now from (4.1) we have

$$\frac{R_G(f) + E(f)}{2} = \int_a^b xf(x)F(x)\,dx, \tag{4.6}$$

and so we have three possible cases to consider.

The first is that $w(x) = xf(x) > 0$ for $x \in [a,b]$ which we have from (3.20) that

$$\begin{aligned} &\left|\int_a^b xf(x)F(x)\,dx \; - \{bf(b)E(f) + M\left[I(t^*) - I(t_*)\right]\}\right| \\ \leq\; & M\left\{\int_a^b (b-x)xf(x)dx \; - \left[I(t_*) + I(t^*)\right]\right\} \end{aligned}$$

where $t^* = \frac{a+b}{2} - \frac{bf(b)-af(a)}{2M}$ and $t_* = \frac{a+b}{2} + \frac{bf(b)-af(a)}{2M}$ with,

$$I(t) = \int_a^t (t-x)xf(x)dx\;.$$

Using the identity (4.6) produces the result as stated in (4.2).

Now for $w(a) = af(a) = 0$ we have from (3.15)- (3.20),

$$\begin{aligned} L(a) &= bf(b)E(f) - M\int_a^b (b-x)xf(x)dx \text{ and,} \\ R(a) &= bf(b)E(f) + M\int_a^b (b-x)xf(x)dx \end{aligned}$$

and so $\frac{R(a)\,+\,L(a)}{2} = bf(b)E(f)$ and $\frac{R(a)\,-\,L(a)}{2} = M\int_a^b (b-x)xf(x)dx$ which results in (4.3) on using (3.20) .

For $w(b) = bf(b) = 0$ we have from (3.15)- (3.20),

$$\begin{aligned} L(b) &= af(a)E(f) - M\int_a^b (x-a)xf(x)dx \text{ and,} \\ R(b) &= af(a)E(f) + M\int_a^b (x-a)xf(x)dx \end{aligned}$$

and so $\frac{R(b)\,+\,L(b)}{2} = af(a)E(f)$ and $\frac{R(b)\,-\,L(b)}{2} = M\int_a^b (x-a)xf(x)dx$ from which we obtain (4.4) on using (3.20) .

Finally, for $w(a) = w(b) = 0$ so that $f(a) = f(b) = 0$ then from (4.3) and (4.4) on choosing the minimum of the bounds produces the stated result. □

4.2. Theorem. *Let* $f(x)$ *be a pdf on* $[a,b]$, $f(x) \leq M$ *and* $F(x) = \int_a^x f(u)du$ *then the Gini Mean Difference* $R_G(f)$ *satisfies*

$$\begin{aligned} &\left|R_G(f) - \{E(f) + M\left[J(t_*) - J(t^*)\right]\}\right| \qquad (4.7) \\ \leq\; & M\left\{\frac{1}{2}\left[(b-a)^2 - (t_*-a)^2 - (t^*-a)^2\right]\right. \\ & \left. -J(b) + \left[\,J(t_*) + J(t^*)\right]\right\} \qquad (4.8) \end{aligned}$$

where $t^* = \frac{a+b}{2} - \frac{1}{2M}$ *and* $t_* = \frac{a+b}{2} + \frac{1}{2M}$ *with*

$$J(t) = \int_a^t (t-x)F(x)dx\;.$$

Further, for $F(b) = 1$ we have

$$(4.9) \qquad |R_G(f)| \leq \frac{M}{2}\int_a^b (x-a)^2 f(x)dx = \frac{M}{2}\left\{\mathcal{M}_2 - a\left[2E(f) - a\right]\right\}.$$

where $\mathcal{M}_2 = \int_a^b x^2 f(x)dx$.

Proof. In Theorem 3.5 let $w(x) = 1 - F(x)$ and $h(x) = F(x)$ so that $\left|h'(x)\right| = f(x) \leq M$.Now from (4.1) we have

$$(4.10) \qquad R_G(f) = \int_a^b (1 - F(x))\, F(x)\, dx;$$

and so we have two possible cases to consider namely, that $w(x) = 1 - F(x) > 0$ for $x \in [a, b)$ and $w(b) = 0$.

Now for $t \in [a, b)$ we have from (3.22) that

$$\begin{aligned} & |R_G(f) - \{E(f) + M\left[I(t^*) - I(t_*)\right]\}| \\ \leq \quad & M\left\{\int_a^b (b-x)(1-F(x))dx \; - \left[\, I(t_*) + I(t^*)\right]\right\} \end{aligned}$$

where $t^* = \frac{a+b}{2} - \frac{1}{2M}$ and $t_* = \frac{a+b}{2} + \frac{1}{2M}$ with,

$$I(t) = \int_a^t (t-x)(1-F(x))dx \; .$$

After some algebraic simplification the results as depicted in (4.7) are establisted.

Now, for $w(b) = 1 - F(b) = 0$ we have from (3.18) - (3.20),

$$L(b) = -M\int_a^b (x-a)(1-F(x))dx \text{ and } R(b) = M\int_a^b (x-a)(1-F(x))dx$$

and so $\frac{R(b)+L(b)}{2} = 0$ and $\frac{R(b)-L(b)}{2} = M\int_a^b (x-a)(1-F(x))dx$ from which we obtain (4.9) on using (3.20) and some simplification. □

An investigation of bounds for the Gini mean difference from the Iyengar inequality (3.1) and the identity depicted in Lemma 2.2 reproduces a the result

$$0 \leq R_G(f) \leq \frac{1}{b-a}\left(b - E(f)\right)\left(E(f) - a\right),$$

obtained by Gastwirth [13, p. 308] by a different approach.

Conclusion

The paper has extended results relating to the Ingear inequality to less restrictive conditions and involving Reimann-Stieltjes integrals. This in turn has let to a weighted version in form of Theorem 3.5 which recaptures the Iyengar result when the weight function is 1. The generalised Iyengar results are then used in the final section to obtain approximation and bounds for the Gini Mean Difference.The novel bounds for realistic pdfs such as those contained in (4.5) and (4.9) involve the first and second moments.

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