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On centralizing automorphisms and Jordan left derivations on σ-prime gamma rings

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Abstract

Let M be a 2-torsion free σ-prime Γ-ring and U be a non-zero σ-square closed Lie ideal of M. If $T : M \to M$ is an automorphism on U such that $T \neq 1$ and $T\sigma = \sigma T$ on U, then we prove that $U \subseteq Z(M)$. We also study the additive maps $d : M \to M$ such that $d(u\alpha u) = 2u\alpha d(u)$, where $u \in U$ and $\alpha \in \Gamma$, and show that $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

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1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [8] as a generalization of a classical ring and afterwards Barnes [2] improved the concepts of Nabusawa's Γ-ring and developed the more general Γ-ring in which all the classical rings are contained in this Γ-ring. Throughout this paper, we consider M as a Γ-ring in the sense of Barnes [2] and we denote the center of M by $Z(M)$. In [3], Ceven proved that every Jordan left derivation on a completely prime Γ-ring is a left derivation. Halder and Paul [5] extended this result in a Lie ideal of a Γ-ring. In Γ-rings, Paul and Uddin [13, 14] studied the Lie and Jordan structures and developed a few number of significant results made by Herstien [6] in Γ-rings. In [15] Paul and Uddin initiated the involution mapping in Γ-rings and studied characterizations of simple Γ-rings by means of involution. In [4], Halder and Paul studied the commutativity properties of σ-prime Γ-rings with a non-zero derivation. Hoque and Paul [7] studied on centralizers of semiprime Γ-rings and proved

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that every Jordan left centralizer on M is a left centralizer on M if M is a 2-torsion free semiprime Γ-ring. They also proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. A number of papers have been developed by Oukhtite and Salhi [10, 11, 12] on σ-prime rings made characterizations of σ-prime rings by means of Lie ideals, derivations and centralizers. By the motivation of the works of Oukhtite and Salhi we initiate to work on σ-prime Γ-rings and generalize the remarkable results of classical ring theories in Γ-ring theories. In the present paper, we work on centralizing automorphisms and Jordan left derivations on σ-prime Γ-rings. We consider M to be a 2-torsion free σ-prime Γ-ring and U to be a non-zero σ-square closed Lie ideal of M. If $T \neq 1$ is an automorphism on U of M which commutes with σ on U, then we show that U is central. We also prove that every Jordan left derivation on U of M is a left derivation on U of M.

2. Preliminaries and Notations

In this section, we give some definitions and preliminary results that we shall use.

2.1. Definition. Let R and $\mathit{\Gamma}$ be two additive abelian groups. If for all $a, b, c \in R$ and $\alpha, \beta \in \mathit{\Gamma}$, the following conditions are satisfied:

(1) $a\alpha b \in R$,
(2) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$,
(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then R is called a $\mathit{\Gamma}$-ring in the sense of Barnes.

Throughout the paper, M will represent a Γ-ring in the sense of Barnes [2] with center $Z(M)$. Then, M is called a 2-torsion free if $2a = 0$ with $a \in M$, then $a = 0$. As usual the commutator $a\alpha b - b\alpha a$ of a and b with respect to α will be denoted by $[a, b]_\alpha$. We make the basic commutator identities

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a[\alpha, \beta]_c b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b[\alpha, \beta]_a c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Let us assume the condition

(2.1) $\quad a\alpha b\beta c = a\beta b\alpha c,$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

According to the condition (2.1), the above two identities reduce to

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, which are used extensively in our paper. An additive mapping satisfying $\sigma : M \to M$ is called an involution on M if $\sigma(a+b) = \sigma(a) + \sigma(b)$, $\sigma(a\alpha b) = \sigma(b)\alpha\sigma(a)$, and $\sigma(\sigma(a)) = a$ are satisfied for all $a, b \in M$ and $\alpha \in \Gamma$. Given an involutorial Γ-ring M with an involution σ, we define $Sa_\sigma(M) = \{m \in M \ : \ \sigma(m) = \pm m\}$, which are known as symmetric and skew symmetric elements of M. Recall that M is σ-prime if

(2.2) $\quad a\Gamma M\Gamma b = a\Gamma M\Gamma\sigma(b) = 0$

implies that $a = 0$ or $b = 0$. It is clear that every prime Γ-ring having an involution is a σ-prime Γ-ring but the converse is in general not true. An additive subgroup U is called a Lie ideal if $[u, m]_\alpha \in U$, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. A Lie ideal U of M is called a σ-Lie ideal, if $\sigma(U) = U$. If U is a σ-Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$, then U is said to be a σ-square closed Lie ideal of M. For $u, v \in U$, $\alpha \in \Gamma$,

$$(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$$

and so $u\alpha v+v\alpha u\in U$. Also, we have $u\alpha v-v\alpha u\in U$. Moreover, from these two relations we obtain $2u\alpha v\in U$. This remark will be freely used in the whole paper. An additive mapping $T:M\to M$ is called centralizing on a subset A of M if $[a,T(a)]_\alpha\subseteq Z(M)$, for every $a\in A$, $\alpha\in\Gamma$. In particular, if T satisfies $[a,T(a)]_\alpha=0$, for all $a\in A$, $\alpha\in\Gamma$, then T is called commuting on A. An additive mapping $d:M\to M$ is said to be a left derivation if $d(a\alpha b)=a\alpha d(b)+b\alpha d(a)$, for all $a,b\in M$, $\alpha\in\Gamma$. And $d:M\to M$ is said to be a Jordan left derivation if $d(a\alpha a)=2a\alpha d(a)$ is satisfied for all $a,b\in M$, $\alpha\in\Gamma$. It is clear that every left derivation is a Jordan left derivation, but the converse need not be true in general.

3. Centralizing automorphisms on σ-square closed Lie ideals

Let M be a 2-torsion free σ-prime Γ-ring and U be a Lie ideal of M such that $U\not\subset Z(M)$. In [5], Halder and Paul proved a lemma (Lemma 2.2) that if $a,b\in M$ such that $a\alpha U\beta b=a\alpha U\beta\sigma(b)=0$, for all $\alpha,\beta\in\Gamma$, then $a=0$ or $b=0$. This lemma is the key of the intensive study of the relationship between several maps (especially derivations and automorphisms) and Lie ideals of σ-prime Γ-rings and by this lemma many results can be extended to σ-prime Γ-rings. In this section, we are primarily interested in centralizing automorphisms on Lie ideals. This lemma will also play an important role in the last section of the present paper. The proof of the following lemma is similar to the proof of Lemma 1.5 in [9]. We give the proof for the sake of completeness.

3.1. Lemma. *Let $U\neq 0$ be a σ-ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (2.1). If $[U,U]_\Gamma=0$, then $U\subseteq Z(M)$.*

Proof. Let $u\in U\cap Sa_\sigma(M)$. From $[U,U]_\Gamma=0$, it follows that $[u,[u,m]_\alpha]_\alpha=0$, for all $x\in M$, $\alpha\in\Gamma$. Let $d_u(x)=[u,x]_\alpha$, for all $x\in M$ and $\alpha\in\Gamma$. Then, d_u is a derivation and by the condition (2.1), $d_u(d_u(x))=[u,[u,x]_\alpha]_\alpha=0$. Hence, $d_u^2(x)=0$, for all $x\in M$. Now, replacing x by $x\beta y$, we have

$$\begin{aligned}0&=d_ud_u(x\beta y)\\&=d_u(d_u(x)\beta y+x\beta d_u(y))\\&=d_u^2(x)\beta y+d_u(x)\beta d_u(y)+d_u(x)\beta d_u(y)+x\beta d_u^2(y)\\&=2d_u(x)\beta d_u(y).\end{aligned}$$

Since M is 2-torsion free, we have

$$d_u(x)\beta d_u(y)=0. \tag{3.1}$$

For every $z\in M$ we replace x by $x\gamma z$ in (3.1), we obtain

$$\begin{aligned}0&=d_u(x\gamma z)\beta d_u(y)\\&=d_u(x)\gamma z\beta d_u(y)+x\gamma d_u(z)\beta d_u(y)\\&=d_u(x)\gamma z\beta d_u(y),\end{aligned}$$

for all $x,y,z\in M$, $\beta,\gamma\in\Gamma$. That is $d_u(x)\gamma M\beta d_u(y)=0$, for all $x,y,z\in M$, $\beta,\gamma\in\Gamma$. As $d_u\sigma=\pm\sigma d_u$, then $d_u(x)\Gamma M\Gamma d_u(y)=0=\sigma(d_u(x))\Gamma M\Gamma d_u(y)$. Since M is σ-prime, $d_u=0$ and hence $[u,x]_\alpha=0$, i.e., $u\in Z(M)$. Therefore, $U\cap Sa_\sigma(M)\subseteq Z(M)$. Let $u\in U$, as $u+\sigma(u)$ and $u-\sigma(u)$ are in $U\cap Sa_\sigma(M)$. Therefore, $u+\sigma(u)$ and $u-\sigma(u)$ are in $Z(M)$, so that $2u\in Z(M)$. Consequently, u in $Z(M)$ proving that $U\subseteq Z(M)$. □

3.2. Lemma. *([3], Lemma 2.2). If $U\not\subset Z(M)$ is a Lie ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (2.1) and $a,b\in M$ such that $a\alpha U\beta b=a\alpha U\beta\sigma(b)=0$, for all $\alpha,\beta\in\Gamma$, then $a=0$ or $b=0$.*

3.3. Lemma. *Let U be a σ-square closed Lie ideal of a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) having a non-trivial automorphism T centralizing on U and commuting with σ on U. If u in $U \cap Sa_\sigma(M)$ is such that $T(u) \neq u$, then $u \in Z(M)$.*

Proof. If $U \subseteq Z(M)$, then $u \in Z(M)$. So, let $U \not\subset Z(M)$. Then, by Lemma 3.1, $[U,U]_\Gamma \neq 0$. Since $[u,T(u)]_\alpha \in Z(M)$, after linearization of it, we obtain $[u,T(v)]_\alpha + [v,T(u)]_\alpha \in Z(M)$, for all $u,v \in U$, $\alpha \in \Gamma$. In particular, we have $[u,T(u\beta u)]_\alpha + [u\beta u,T(u)]_\alpha \in Z(M)$. Hence,

$$\begin{aligned}
&[u,T(u)\beta T(u)]_\alpha + [u\beta u,T(u)]_\alpha \\
&= T(u)\beta[u,T(u)]_\alpha + [u,T(u)]_\alpha\beta T(u) + u\beta[u,T(u)]_\alpha + [u,T(u)]_\alpha\beta u \\
&= T(u)\beta[u,T(u)]_\alpha + T(u)\beta[u,T(u)]_\alpha + u\beta[u,T(u)]_\alpha + u\beta[u,T(u)]_\alpha \\
&= 2(u+T(u))\beta[u,T(u)]_\alpha \in Z(M).
\end{aligned}$$

Since M is 2-torsion free, we obtain $(u+T(u))\beta[u,T(u)]_\alpha \in Z(M)$. Hence,

$$0 = [u,(u+T(u))\beta[u,T(u)]_\alpha]_\alpha = [u,T(u)]_\alpha\beta[u,T(u)]_\alpha,$$

since $[u,T(u)]_\alpha \in Z(M)$. Thus, $[u,T(u)]_\alpha = 0$, for all $u \in U$ and $\alpha \in \Gamma$. Again, linearizing this equality, we obtain

(3.2) $\qquad [u,T(v)]_\alpha = [T(u),v]_\alpha.$

Let $u \in U \cap Sa_\sigma(M)$ with $T(u) \neq u$. By replacing v by $2u\beta v$ in (3.2) we obtain $0 = (u - T(u))\beta[T(u),v]_\alpha$, for all $v \in U$. By putting $2w\gamma v$ instead of v, we obtain $(u-T(u))\beta w\gamma[T(u),v]_\alpha = 0$, for all $w \in U$ and $\gamma \in \Gamma$. This shows that $(u - T(u))\Gamma U\Gamma[T(u),v]_\alpha = 0$. Therefore,

$$(u-T(u))\Gamma U\Gamma[T(u),v]_\alpha = (u-T(u))\Gamma U\Gamma\sigma([T(u),v]_\alpha) = 0,$$

for all $v \in U$. Since $T(u) \neq u$, by Lemma 3.2, $[T(u),v]_\alpha = 0$, for all $v \in U$. Hence, for all $m \in M$, $[T(u),[v,m]_\beta]_\alpha = 0$, and so $[T(u),m\beta v]_\alpha = [T(u),v\beta m]_\alpha$. Thus, $[T(u),m]_\alpha\beta v = v\beta[T(u),m]_\alpha$, for all $m \in M$ and $\beta \in \Gamma$. Replacing m by $m\gamma u$, where $u \in U$, we find that $[T(u),m]_\alpha\gamma u\beta v = v\beta[T(u),m]_\alpha\gamma u = [T(u),m]_\alpha\beta v\gamma u$. Now, by using (2.1), we obtain $[T(u),m]_\alpha\gamma[u,v]_\beta = 0$. This implies that $[T(u),m]_\alpha\delta y\gamma[u,v]_\beta = 0$, for all $y \in M$ and $\delta \in \Gamma$. Hence, $[T(u),m]_\alpha\delta M\gamma[U,U]_\beta = 0$. Since $\sigma(U) = U$,

$$[T(u),m]_\alpha\delta M\gamma[U,U]_\Gamma = 0 = [T(u),m]_\alpha\delta M\gamma\sigma([U,U]_\Gamma).$$

Since $[U,U]_\Gamma \neq 0$, the σ-primeness of M yields $[T(u),m]_\alpha = 0$. This gives that $T(u) \in Z(M)$. As T is an automorphism, it then follows that $u \in Z(M)$. □

Now, we have in position to prove the main result.

3.4. Theorem. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1). Let $T : M \to M$ be an automorphism centralizing on a σ-square closed Lie ideal U of M such that $T \neq 1$ and $T\sigma = \sigma T$ on U. Then, $U \subseteq Z(M)$.*

Proof. If $[U,U]_\Gamma = 0$, then $U \subseteq Z(M)$ by Lemma 3.1. So, let us assume that $[U,U]_\Gamma \neq 0$. If T is the identity on U, then for all $m \in M$, $u \in U$, $\alpha \in \Gamma$,

(3.3) $\qquad T([m,u]_\alpha) = [m,u]_\alpha = [T(m),u]_\alpha.$

Replacing m by $m\beta v$ in (3.3), for $v \in U$ and $\beta \in \Gamma$, we obtain,

$$\begin{aligned}
&[m\beta v,u]_\alpha = [T(m\beta v),u]_\alpha \\
&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = [T(m)\beta T(v),u]_\alpha \\
&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = T(m)\beta[T(v),u]_\alpha + [T(m),u]_\alpha\beta T(v) \\
&\Rightarrow m\beta[v,u]_\alpha + [m,u]_\alpha\beta v = T(m)\beta[v,u]_\alpha + [m,u]_\alpha\beta v
\end{aligned}$$

Thus,

(3.4) $\quad m\beta[v,u]_\alpha = T(m)\beta[v,u]_\alpha.$

For any $y \in M$ and $\gamma \in \Gamma$, we write $m\gamma y$ instead of m in (3.4), we obtain $m\gamma y\beta[v,u]_\alpha = T(m)\gamma T(y)\beta[v,u]_\alpha$ which implies that $m\gamma y\beta[v,u]_\alpha = T(m)\gamma y\beta[v,u]_\alpha$. Thus, $(T(m) - m)\gamma y\beta[v,u]_\alpha = 0$ for all $u, v \in U$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since $\sigma(U) = U$, we obtain

$$(T(m) - m)\gamma M\beta[v,u]_\alpha = 0 = (T(m) - m)\gamma M\beta\sigma([v,u]_\alpha).$$

By the fact that $[U,U]_\Gamma \neq 0$, it yields that $T(m) - m = 0$, for all $m \in M$, which is impossible. So, T is non-trivial on U. By the 2-torsion freeness of M, T is also non-trivial on $U \cap Sa_\sigma(M)$. Therefore, there is an element u in $U \cap Sa_\sigma(M)$ such that $u \neq T(u)$ and $u \in Z(M)$ by Lemma 3.3. Let $v \neq 0$ be in $U \cap Sa_\sigma(M)$ and not be in $Z(M)$. Again in view of Lemma 3.3, we conclude that $T(v) = v$. But we have $T(u\alpha v) = T(u)\alpha v = u\alpha v$, so that $(T(u) - u)\alpha v = 0$. Since $v \in Sa_\sigma(M) \cap U$, it yields that $(T(u) - u)\beta m\alpha v = (T(u) - u)\beta\alpha\sigma(v) = 0$, for all $m \in M$, $\beta \in \Gamma$. Since M is σ-prime and $T(u) \neq u$, we obtain that $v = 0$, a contradiction. Therefore, for all v in $U \cap Sa_\sigma(M)$, v must be in $Z(M)$. Now, let $u \in U$, the fact that $u - \sigma(u)$ and $u + \sigma(u)$ are in $U \cap Sa_\sigma(M)$ gives that both $u - \sigma(u)$ and $u + \sigma(u)$ are in $Z(M)$ and therefore $2u \in Z(M)$. Consequently, $u \in Z(M)$ which proves $U \subseteq Z(M)$. $\square$

4. Jordan left derivations on σ-square closed Lie ideals

In this section M will always denote a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) and U a σ-square closed Lie ideal of M. For proving the main result, we first state a few known results which will be used in subsequent discussion.

4.1. Lemma. *([5], Lemma 3) Let M be a 2-torsion free Γ-ring and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$, $\alpha \in \Gamma$ and M satisfies the condition (2.1), then*

(1) $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$,
(2) $d(u\alpha v\beta u) = u\alpha u\beta d(v) + 3u\alpha v\beta d(u) - v\alpha u\beta d(u)$,
(3) $d(u\alpha v\beta w + w\alpha v\beta u) = (u\alpha w + w\alpha u)\beta d(v) + 3u\alpha v\beta d(w) + 3w\alpha v\beta d(u) - v\alpha u\beta d(w) - v\alpha w\beta d(u)$,
(4) $[u,v]_\alpha\gamma u\beta d(u) = u\gamma[u,v]_\alpha\beta d(u)$,
(5) $[u,v]_\alpha(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0$,

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

4.2. Lemma. *([5], Lemma 4). Let M be a 2-torsion free Γ-ring satisfying the condition (2.1) and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$, then*

(1) $[u,v]_\alpha\beta d([u,v]_\alpha) = 0$,
(2) $(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\beta u)\beta d(v) = 0$,

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Now, similar to the proof of Theorem in [1] and Theorem 1.6 in [9], the main result of this section is given as follows:

4.3. Theorem. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1) and let U be a σ-square closed Lie ideal of M. If $d : M \to M$ is an additive mapping which satisfies $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.*

Proof. Let $[U,U]_\Gamma = 0$. Then, we get $U \subseteq Z$ by Lemma 3.1. Using Lemma 4.1, we obtain that $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$, for all $u, v \in U$. Since $u \in Z(M)$ and M is 2-torsion free, we arrive at $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$. So, we suppose that $[U,U]_\Gamma \neq 0$. We have

$$(4.1) \qquad (u\alpha u\gamma v - 2u\alpha v\gamma u + v\gamma u\alpha u)\beta d(u) = 0.$$

Writing $[u,w]_\delta$ in place of u in (4.1), where $w \in U$ and $\delta \in \Gamma$. We obtain

$$([u,w]_\delta\alpha[u,w]_\delta\gamma v - 2[u,w]_\delta\alpha v\gamma[u,w]_\delta + v\gamma[u,w]_\delta\alpha[u,w]_\delta)\beta d([u,w]_\delta) = 0,$$

which implies that

$$\begin{aligned}&[u,w]_\delta\alpha[u,w]_\delta\gamma v\beta d([u,w]_\delta) - 2[u,w]_\delta\alpha v\gamma[u,w]_\delta\beta d([u,w]_\delta)\\ &\qquad + v\gamma[u,w]_\delta\alpha[u,w]_\delta\beta d([u,w]_\delta) = 0.\end{aligned}$$

In view of Lemma 4.2(1), we have $[u,w]_\delta\alpha[u,w]_\delta\gamma v\beta d([u,w]_\delta) = 0$. This implies that $[u,w]_\delta\alpha[u,w]_\delta\gamma U\beta d([u,w]_\delta) = 0$, for all $u, w \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Let $a, b \in Sa\sigma(M) \cap U$. We have $[a,b]_\delta\alpha[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0 = \sigma([a,b]_\delta\alpha[a,b]_\delta)\gamma U\beta d([a,b]_\delta)$ and by virtue of Lemma 3.2 either $[a,b]_\delta\alpha[a,b]_\delta = 0$ or $d([a,b]_\delta) = 0$. If $d([a,b]_\delta) = 0$, then by using Lemma 4.2(1) and the 2-torsion freeness of M, we have seen that $d(a\delta b) = a\delta d(b) + b\delta d(a)$, for all $\delta \in \Gamma$. Now, assume that $[a,b]_\delta\alpha[a,b]_\delta = 0$. From Lemma 4.2(2), it follows that $(u\alpha u\gamma v - 2u\alpha v\gamma u + v\gamma u\alpha u)\beta d(u) = 0$, for all $u, v \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Linearizing this relation in u, we obtain

$$((u+w)\alpha(u+w)\gamma v - 2(u+w)\alpha v\gamma(u+w) + v\alpha(u+w)\gamma(u+w))\beta d(v) = 0,$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$. After calculation we obtain

$$(u\alpha w\gamma v + w\alpha u\gamma v - 2u\alpha v\gamma w - 2w\alpha v\gamma u + v\alpha u\gamma w + v\alpha w\gamma u)\beta d(v) = 0,$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing v by $[a,b]_\delta$ and using Lemma 4.2(1), we have

$$(4.2) \qquad (-2u\alpha[a,b]_\delta\gamma w - 2w\alpha[a,b]_\delta\gamma u + [a,b]_\delta\alpha u\gamma w + [a,b]_\delta\alpha w\gamma u)\beta d([a,b]_\delta) = 0.$$

Putting $u\mu[a,b]_\delta$ instead of u in (4.2), applying $[a,b]_\delta\alpha[a,b]_\delta = 0$ and Lemma 4.2(1), we obtain $[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma w\beta d([a,b]_\delta) = 0$ by using (2.1), for all $u, w \in M$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. Accordingly, $[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0$, for all $u \in M$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. Since $[a,b]_\delta \in U \cap Sa_\sigma(M)$ and $\sigma(U) = U$, we obtain

$$[a,b]_\delta\alpha u\mu[a,b]_\delta\gamma U\beta d([a,b]_\delta) = 0 = \sigma([a,b]_\delta\alpha u\mu[a,b]_\delta)\gamma U\beta d([a,b]_\delta),$$

for all $u \in U$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$. By Lemma 3.2, we obtain $d([a,b]_\delta) = 0$ or $[a,b]_\delta\alpha u\mu[a,b]_\delta = 0$, for all $u \in U$ and $\alpha, \gamma, \delta, \mu \in \Gamma$. If $[a,b]_\delta\alpha u\mu[a,b]_\delta = 0$, then we have $[a,b]_\delta\alpha u\mu\sigma([a,b]_\delta) = 0$, since $\sigma(u) = u$ for all $u \in U$ and $[a,b]_\delta \in U$ for all $a, b \in U$, $\delta \in \Gamma$. Therefore, by Lemma 3.2, $[a,b]_\delta = 0$, which shows that $[U,U]_\Gamma = 0$, which is contradiction to our assumption that $[U,U]_\Gamma \neq 0$. So, let $d([a,b]_\delta) = 0$. Then, by previous arguments, we have $d(a\delta b) = a\delta d(b) + b\delta d(a)$. Therefore, in the both cases we find that

$$(4.3) \qquad d(a\alpha b) = a\alpha d(b) + b\alpha d(a),$$

for all $a, b \in U \cap Sa_\sigma(M)$ and $\alpha \in \Gamma$. Now, let us assume that $u, v \in U$, set $u_1 = u + \sigma(u)$, $u_2 = u - \sigma(u)$, $v_1 = v + \sigma(v)$, $v_2 = v - \sigma(v)$. Then, we have $2u = u_1 + u_2$ and $2v = v_1 + v_2$. For the fact that $u_1, u_2, v_1, v_2 \in U \cap Sa_\sigma(M)$, and application of (4.3) gives

$$\begin{aligned} d(2u\alpha 2v) &= d((u_1+u_2)\alpha(v_1+v_2))\\ &= d(u_1\alpha v_1 + u_1\alpha v_2 + u_2\alpha v_1 + u_2\alpha v_2)\\ &= u_1\alpha d(v_1) + v_1\alpha d(u_1) + u_1\alpha d(v_2) + v_2\alpha d(u_1) + u_2\alpha d(v_1) + v_1\alpha d(u_2)\\ &\qquad + u_2\alpha d(v_2) + v_2\alpha d(u_2)\\ &= (u_1+u_2)\alpha d(v_1+v_2) + (v_1+v_2)\alpha d(u_1+u_2)\\ &= 2u\alpha d(2v) + 2vd(2u).\end{aligned}$$

This implies that $4d(u\alpha v) = 4(u\alpha d(v) + v\alpha d(u))$. Since M is 2-torsion free, we have $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. $\square$

4.4. Corollary. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (2.1). Then, every Jordan left derivation on M is a left derivation on M.*

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