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Sharp Wilker and Huygens type inequalities for trigonometric and hyperbolic functions

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Abstract

In the article, some sharp Huygens and Wilker type inequalities involving trigonometric and hyperbolic functions are established.

Keywords: Huygens inequality, Wilker inequality, Trigonometric function, Hyperbolic function.

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1. Introduction

The trigonometric and hyperbolic inequalities have been in recent years in the focus of many researchers. For many results and a long list of references we quote the papers [6, 10, 24], where many further references may be found. The following inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \quad 0 < x < \frac{\pi}{2} \tag{1.1}$$

is due to Wilker [13]. It has attracted attention of several researchers(see, e. g.,[4],[7], [8], [9],[14],[15],[21]). A hyperbolic counterpart of Wilker's inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{1.2}$$

($x \neq 0$) has been established by L. Zhu[16].

In [12], it was proved that

$$2 + \frac{8}{45}x^3 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \tag{1.3}$$

for $0 < x < \frac{\pi}{2}$. The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ in the inequality (1.3) are the best possible.

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The famous Huygens inequality[11] for the sine and tangent functions states that for $x \in \left(0, \frac{\pi}{2}\right)$

$$2\sin x + \tan x > 3x. \tag{1.4}$$

The hyperbolic counterpart of (1.4) was established in [6] as follows: For $x > 0$

$$2\sinh x + \tanh x > 3x. \tag{1.5}$$

The inequalities (1.4) and (1.5) were respectively refined in [6, Theorem 2.6] as

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3, \tag{1.6}$$

and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0. \tag{1.7}$$

In the most recent paper [5], the inequalities (1.6) ,(1.7) and (1.1) were respectively further refined as

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2\frac{\tan(x/2)}{x/2} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3. \tag{1.8}$$

and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > \frac{\sinh x}{x} + 2\frac{\tanh(x/2)}{x/2} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3. \tag{1.9}$$

and

$$\begin{aligned} &\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > \frac{\sin x}{x} + \left(\frac{\tan(x/2)}{x/2}\right)^2 \\ &> \frac{x}{\sin x} + \left(\frac{x/2}{\tan(x/2)}\right)^2 > 2. \end{aligned} \tag{1.10}$$

The hyperbolic counterparts of the last two inequalities in (1.10) were also given in [5] as follows:

$$\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 > \frac{x}{\sinh x} + \left[\frac{x/2}{\tanh(x/2)}\right]^2 > 2. \tag{1.11}$$

Inspired by (1.3), Jiang et al. [19] first proved

$$3 + \frac{1}{60}x^3 \sin x < 2\frac{x}{\sin x} + \frac{x}{\tan x} < 3 + \frac{8\pi - 24}{\pi^3}x^3 \sin x. \tag{1.12}$$

and

$$2 + \frac{17}{720}x^3 \sin x < \frac{x}{\sin x} + \left(\frac{\frac{x}{2}}{\tan\frac{x}{2}}\right)^2 < 2 + \frac{\pi^2 + 8\pi - 32}{2\pi^3}x^3 \sin x. \tag{1.13}$$

holds for $0 < |x| < \frac{\pi}{2}$. Furthermore the constants $\frac{1}{60}$, $\frac{8\pi-24}{\pi^3}$ in (1.12) and the constants $\frac{17}{720}$, $\frac{\pi^2+8\pi-32}{2\pi^3}$ in (1.13) are the best possible.

Recently, Chen and Sándor [20] proved that

$$3 + \frac{3}{20}x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x.$$

for $0 < |x| < \frac{\pi}{2}$. The constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

This paper is a continuation of our work [25] and is organized as follows. In Section 2, we give some lemmas and preliminary results. In Section 3, we prove some new sharp Wilker- and Huygens-type inequalities for trigonometric and hyperbolic functions.

2. some Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

2.1. Lemma. *The Bernoulli numbers B_{2n} for $n \in \mathbb{N}$ have the property*

$$(-1)^{n-1}B_{2n} = |B_{2n}|, \tag{2.1}$$

where the Bernoulli numbers B_i for $i \geq 0$ are defined by

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!}x^i = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i}\frac{x^{2i}}{(2i)!}, \quad |x| < 2\pi. \tag{2.2}$$

Proof. In [2, p. 16 and p. 56], it is listed that for $q \geq 1$

$$\zeta(2q) = (-1)^{q-1}\frac{(2\pi)^{2q}}{(2q)!}\frac{B_{2q}}{2}, \tag{2.3}$$

where ζ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In [22, p.18, theorem 3.4], the following formula was given

$$\sum_{n=1}^{\infty} \frac{1}{n^{2q}} = \frac{2^{2q-1}\pi^{2q}|B_{2q}|}{(2q)!}. \tag{2.4}$$

From (2.3) and (2.4), the formula (2.1) follows. □

2.2. Lemma. [17, 18] *Let B_{2n} be the even-indexed Bernoulli numbers. Then*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}}, n = 1, 2, 3, \cdots.$$

2.3. Lemma. *For $0 < |x| < \pi$, we have*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n}. \tag{2.5}$$

Proof. This is an easy consequence of combining the equality

$$\frac{1}{\sin x} = \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2\left(2^{2n-1}-1\right)B_{2n}}{(2n)!}x^{2n-1}, \tag{2.6}$$

see [1, p. 75, 4.3.68], with Lemma 2.1. □

2.4. Lemma ([1, p. 75, 4.3.70])**.** *For $0 < |x| < \pi$,*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}. \tag{2.7}$$

The following Lemma 2.5 and Lemma 2.6 can be found in [25].

2.5. Lemma. *For $0 < |x| < \pi$,*

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{2.8}$$

2.6. Lemma. *For $0 < |x| < \pi$,*

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2(2n-1)\big(2^{2n-1}-1\big)|B_{2n}|}{(2n)!} x^{2(n-1)}. \tag{2.9}$$

2.7. Lemma. *For $0 < |x| < \pi$,*

$$\begin{aligned}\frac{1}{\sin^3 x} =& \frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2)x^{2n-3} \\ &+ \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1},\end{aligned} \tag{2.10}$$

and

$$\frac{\cos x}{\sin^3 x} = \frac{1}{x^3} - \sum_{n=2}^{\infty} \frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!} x^{2n-3}. \tag{2.11}$$

Proof. Combining

$$\frac{1}{\sin^3 x} = \frac{1}{2\sin x} - \frac{1}{2}\left(\frac{\cos x}{\sin^2 x}\right)'$$

with Lemma 2.6, the identity (2.6), and Lemma 2.1 gives (2.10).

The equality (2.11) follows from combination of

$$\frac{\cos x}{\sin^3 x} = -\frac{1}{2}\left(\frac{1}{\sin^2 x}\right)'$$

with Lemma 2.5. □

2.8. Lemma. [23, 3, 15] *Let a_n and $b_n (n = 0, 1, 2, \cdots)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \cdots$, and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \cdots$, then the function $\frac{A(t)}{B(t)}$ is strictly increasing (or decreasing) on $(0, R)$.*

3. Main results

Now we are in a position to state and prove our main results.

3.1. Theorem. *For $0 < |x| < \frac{\pi}{2}$, we have*

$$2 + \frac{23}{720}x^3 \sin x < \frac{\sin x}{x} + \left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2 < 2 + \frac{128 - 16\pi^2 + 16\pi}{\pi^5}x^3 \sin x. \tag{3.1}$$

The constants $\frac{23}{720}$ and $\frac{128-16\pi^2+16\pi}{\pi^5}$ in (3.1) are the best possible.

Proof. Let

$$\begin{aligned}f(x) =& \frac{\frac{\sin x}{x} + \left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2 - 2}{x^3 \sin x} \\ =& \frac{x\sin^3 x - 8\cos x - 4\sin^2 x - 2x^2\sin^2 x + 8}{x^5 \sin^3 x} \\ =& \frac{1}{x^5}\left(x + \frac{8}{\sin^3 x} - \frac{8\cos x}{\sin^3 x} - \frac{4}{\sin x} - \frac{2x^2}{\sin x}\right)\end{aligned}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. By virtue of (2.10), (2.11), and (2.6), we have

$$f(x) = \frac{1}{x^5}\Bigg[x + \frac{8}{x^3} + \sum_{n=2}^{\infty}\frac{4(2n-1)(2n-2)(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-3}$$
$$+ \frac{4}{x} + \sum_{n=1}^{\infty}\frac{4(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-1}$$
$$- \frac{8}{x^3} + \sum_{n=2}^{\infty}\frac{8\cdot 2^{2n}(2n-1)(n-1)}{(2n)!}|B_{2n}|x^{2n-3}$$
$$- \frac{4}{x} - \sum_{n=1}^{\infty}\frac{4(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-1}$$
$$- 2x - \sum_{n=1}^{\infty}\frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\Bigg]$$
$$= \frac{1}{x^5}\left[-x + \sum_{n=2}^{\infty}\frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty}\frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \frac{1}{x^5}\left[\sum_{n=3}^{\infty}\frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty}\frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \frac{1}{x^5}\left[\sum_{n=1}^{\infty}\frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}|x^{2n+1} - \sum_{n=1}^{\infty}\frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \sum_{n=2}^{\infty}\left[\frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|\right]x^{2n-4}.$$

Let $a_n = \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|$ for $n \geq 2$.
By a simple computation, we have $a_2 = \frac{23}{720}$.
Furthermore, when $n \geq 3$, From Lemma 2.2 one can get

$$\begin{aligned} a_n =& \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}| \\ >& \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}\cdot\frac{2(2n+4)!}{(2\pi)^{2n+4}}\frac{1}{1-2^{-2n-4}} \\ & - \frac{2(2^{2n}-2)}{(2n)!}\cdot\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}} \\ =& \frac{4}{(\pi)^{2n}}\left[\frac{8(2n+3)(n+1)}{\pi^4} - 1\right] > 0. \end{aligned}$$

So the function $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Moreover, it is easy to obtain

$$\lim_{x\to 0^+} f(x) = a_2 = \frac{23}{720} \quad \text{and} \quad \lim_{x\to(\pi/2)^-} f(x) = \frac{128 - 16\pi^2 + 16\pi}{\pi^5}.$$

The proof of Theorem 3.1 is complete. □

3.2. Remark. Since $f(x)$ is an even function we conclude that Theorem 3.1 holds for all x which satisfy $0 < |x| < \frac{\pi}{2}$.

3.3. Theorem. *For $x \neq 0$, we have*

$$3 + \frac{1}{40}x^3\tanh x < \frac{\sinh x}{x} + 2\left(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\right) < 3 + \frac{1}{40}x^3\sinh x. \tag{3.2}$$

The constant $\frac{1}{40}$ is the best possible.

Proof. Without loss of generality, we assume that $x > 0$.

We firstly prove the first inequality of (3.2).

Consider the function $F(x)$ defined by

$$F(x) = \frac{\frac{\sinh x}{x} + 2\frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3}{x^3 \tanh x} = \frac{\cosh 3x - 17\cosh x + 8\cosh 2x - 6x\sinh 2x + 8}{2x^4(\cosh 2x - 1)}.$$

and let

$$f(x) = \cosh 3x - 17\cosh x + 8\cosh 2x - 6x\sinh 2x + 8 \quad \text{and} \quad g(x) = 2x^4(\cosh 2x - 1).$$

From the power series expansions

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \tag{3.3}$$

it follows that

$$\begin{aligned} f(x) &= \cosh 3x - 17\cosh x + 8\cosh 2x - 6x\sinh 2x + 8 \\ &= \sum_{n=0}^{\infty} \frac{3^{2n}x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{17x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{2^{2n+3}x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6\cdot 2^{2n+1}x^{2n+2}}{(2n+1)!} + 8 \\ &= \sum_{n=0}^{\infty} \frac{\left(3^{2n} + 2^{2n+3} - 17\right)x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6\cdot 2^{2n+1}x^{2n+2}}{(2n+1)!} + 8 \\ &= \sum_{n=1}^{\infty} \frac{\left(3^{2n} + 2^{2n+3} - 17\right)x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{6n2^{2n}x^{2n}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{3^{2n} + 2^{2n+3} - 17 - 6n2^{2n}}{(2n)!}x^{2n} \\ &\triangleq \sum_{n=3}^{\infty} a_n x^{2n} \end{aligned}$$

and

$$\begin{aligned} g(x) &= 2x^4(\cosh 2x - 1) \\ &= \sum_{n=1}^{\infty} \frac{2^{2n+1}x^{2n+4}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-3)(2n-1)2^{2n-3}x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} b_n x^{2n}. \end{aligned}$$

It is easy to see that the quotient

$$c_n = \frac{a_n}{b_n} = \frac{3^{2n} + 2^{2n+3} - 17 - 6n2^{2n}}{4n(n-1)(2n-3)(2n-1)2^{2n-3}}$$

satisfies $c_3 = \frac{1}{40}, c_4 = \frac{51}{1120}, c_5 = \frac{507}{8960}$ and

$$c_{n+1} - c_n = \frac{f_1 + f_2 + f_3}{2n(2n+3)\left(4n^2-1\right)\left(n^2-1\right)}, (n \geq 6),$$

where

$$f_1 = \left(\frac{9}{4}\right)^n \left(10n^2 - 57n + 23\right) = \left(\frac{9}{4}\right)^n \left(10n(n-6) + 3(n-6) + 41\right) > 0,$$

$$f_2 = \frac{1}{4^n}(102n^2 + 298n + 17) > 0,$$

$$f_3 = 144n^2 - 184n - 8 = 144n(n-6) + 680(n-6) + 4072 > 0.$$

for $n \geq 6$. This means that the sequence c_n is increasing. By Lemma 2.8, the function $F(x)$ is increasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} F(x) = c_3 = \frac{1}{40}$. Therefore, the first inequality in (3.2) holds.

Finally, we prove the second inequality of (3.2).

Define a function $G(x)$ by

$$G(x) = \frac{\frac{\sinh x}{x} + 2\frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3}{x^3 \sinh x} = \frac{\cosh 2x + 8\cosh x - 6x \sinh x - 9}{x^4(\cosh 2x - 1)}.$$

and let

$$f(x) = \cosh 2x + 8\cosh x - 6x\sinh x - 9 \quad \text{and} \quad g(x) = x^4(\cosh 2x - 1).$$

By using (3.3), it follows that

$$\begin{aligned} f(x) &= \cosh 2x + 8\cosh x - 6x\sinh x - 9 \\ &= \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{8x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6x^{2n+2}}{(2n+1)!} - 9 \\ &= \sum_{n=1}^{\infty} \frac{\left(2^{2n} + 8\right)x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6x^{2n+2}}{(2n+1)!} \\ &= \sum_{n=1}^{\infty} \frac{\left(2^{2n} + 8\right)x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{12nx^{2n}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{\left(2^{2n} + 8 - 12n\right)x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} a_n x^{2n} \end{aligned}$$

and

$$\begin{aligned} g(x) &= x^4(\cosh 2x - 1) \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}x^{2n+4}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-1)(2n-3)2^{2n-4}x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} b_n x^{2n}. \end{aligned}$$

Let

$$c_n = \frac{a_n}{b_n} = \frac{2^{2n} - 12n + 8}{4n(n-1)(2n-1)(2n-3)2^{2n-4}}$$

satisfies $c_3 = \frac{1}{40}$. Furthermore, when $n \geq 3$, by a simple computation, we have

$$c_{n+1} - c_n = -4\frac{(8n-2)4^n - (18n^3 + 33n^2 - 16n - 11)}{n(2n-3)\big(4n^2-1\big)(n^2-1)4^n},$$

for $n \geq 3$.

Since

$$\begin{aligned}
&(8n-2)4^n - (18n^3 + 33n^2 - 16n - 11)\\
&> (8n-2)4n^2 - (18n^3 + 33n^2 - 16n - 11)\\
&= 14n^3 - 41n^2 + 16n + 11\\
&= 14n(n-3)^2 + 43n(n-3) + 19(n-3) + 68 > 0.
\end{aligned}$$

This means that the sequence c_n is decreasing. By Lemma 2.8, the function $G(x)$ is decreasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} G(x) = c_3 = \frac{1}{40}$.

This completes the proof of Theorem 3.3 .

□

3.4. Remark. Since $F(x)$ and $G(x)$ both are even functions, we conclude that Theorem 3.3 holds for all $x \neq 0$.

3.5. Theorem. *For $x \neq 0$,*

$$2 + \frac{23}{720}x^3 \tanh x < \frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 < 2 + \frac{23}{720}x^3 \sinh x. \tag{3.4}$$

The both constants $\frac{23}{720}$ in (3.4) are the best possible.

Proof. The left-hand side of inequality in (3.4) has been proved in [19], so we only need to prove the right-hand side of the inequality in (3.4).

Without loss of generality, we assume that $x > 0$.

Consider the function $H(x)$ defined by

$$\begin{aligned}
H(x) &= \frac{\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 - 2}{x^3 \sinh x}\\
&= \frac{x \sinh x \cosh x + x \sinh x + 4\cosh x - 2x^2 \cosh x - 2x^2 - 4}{x^5 \sinh x(1 + \cosh x)}
\end{aligned}$$

and let

$$f(x) = x \sinh x \cosh x + x \sinh x + 4\cosh x - 2x^2 \cosh x - 2x^2 - 4$$

and

$$g(x) = x^5 \sinh x(1 + \cosh x).$$

By the power series expansions in (3.3), we obtain

$$
\begin{aligned}
f(x) &= x\sinh x\cosh x + x\sinh x + 4\cosh x - 2x^2\cosh x - 2x^2 - 4\\
&= \sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}x^{2n+2} + \sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n)!} + \sum_{n=0}^{\infty}\frac{4x^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{2x^{2n+2}}{(2n)!} - 2x^2 - 4\\
&= \sum_{n=0}^{\infty}\frac{2^{2n}+1-2(2n+1)}{(2n+1)!}x^{2n+2} + \sum_{n=2}^{\infty}\frac{4}{(2n)!}x^{2n}\\
&= \sum_{n=1}^{\infty}\frac{2^{2n-2}+1-2(2n-1)}{(2n-1)!}x^{2n} + \sum_{n=2}^{\infty}\frac{4}{(2n)!}x^{2n}\\
&= \sum_{n=3}^{\infty}\frac{2n\Big(2^{2n-2}-4n+3\Big)+4}{(2n)!}x^{2n}\\
&\triangleq \sum_{n=3}^{\infty}a_n x^{2n}
\end{aligned}
$$

and

$$
\begin{aligned}
g(x) &= x^5\left[\frac{1}{2}\sinh(2x) + \sinh x\right]\\
&= \sum_{n=0}^{\infty}\frac{1+2^{2n}}{(2n+1)!}x^{2n+6} = \sum_{n=3}^{\infty}\frac{1+2^{2n-6}}{(2n-5)!}x^{2n}\\
&= \sum_{n=3}^{\infty}\frac{\big(1+2^{2n-6}\big)(2n-4)(2n-3)(2n-2)(2n-1)2n}{(2n)!}x^{2n}\\
&\triangleq \sum_{n=3}^{\infty}b_n x^{2n}.
\end{aligned}
$$

Let

$$
c_n = \frac{a_n}{b_n} = \frac{2n\Big(2^{2n-2}-4n+3\Big)+4}{\big(1+2^{2n-6}\big)(2n-4)(2n-3)(2n-2)(2n-1)2n}
$$

satisfies

$$
c_3 = \frac{23}{720} = 0.031\ldots, \qquad c_4 = \frac{17}{336} = 0.01226\ldots.
$$

Furthermore, when $n \geq 4$, by a simple computation, we have

$$
c_{n+1} - c_n = -4\frac{f_1(n)+f_2(n)+f_3(n)}{n(16+4^n)(64+4^n)(n-2)(2n-3)\big(4n^2-1\big)(n^2-1)},
$$

where

$$
\begin{aligned}
f_1(n) &= 16^n\left(8n^2+2n-6\right)\\
f_2(n) &= 4^n\left(-24n^4-138n^3+391n^2+153n-382\right)\\
f_3(n) &= -1536n^3-256n^2+2944n-256
\end{aligned}
$$

Since $n \geq 4$, one can easily check that $4^n \geq 16n^2$, this implies that

$$
\begin{aligned}
f_1(n)+f_2(n) &> 4^n 16n^2(8n^2+2n-6) + 4^n\left(-24n^4-138n^3+391n^2+153n-382\right)\\
&= 4^n(104n^4-106n^3+295n^2+153n-382)
\end{aligned}
$$

By a simple computation, one has

$$\begin{aligned}&104n^4 - 106n^3 + 295n^2 + 153n - 382\\&= 104n(n-4)^3 + 1142n(n-4)^2 + 4439n(n-4) + 6293(n-4) + 24790 > 0.\end{aligned}$$

On the other hand, when $n \geq 4$, one has $4^n > 16$, Hence

$$\begin{aligned}&f_1(n) + f_2(n) + f_3(n)\\&> 4^n(104n^4 - 106n^3 + 295n^2 + 153n - 382) - 1536n^3 - 256n^2 + 2944n - 256\\&> 16(104n^4 - 106n^3 + 295n^2 + 153n - 382) - 1536n^3 - 256n^2 + 2944n - 256\\&= 1664n^4 - 3232n^3 + 4464n^2 + 5392n - 6368\\&= 1664n(n-4)^3 + 16736n(n-4)^2 + 58480n(n-4) + 78032(n-4) + 305760 > 0.\end{aligned}$$

This means that the sequence c_n is decreasing. By Lemma 2.8, the function $H(x)$ is decreasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} H(x) = c_3 = \frac{23}{720}$. □

3.6. Remark. Since $H(x)$ is an even function, we conclude that Theorem 3.5 holds for all $x \neq 0$.

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