Hacettepe Journal of Mathematics and Statistics
Volume 45 (1) (2016), 49 – 55

Differential identities on Jordan ideals of rings with involution

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Abstract

In this paper we investigate generalized derivations satisfying certain differential identities on Jordan ideals of rings with involution and discuss related results. Moreover, we provide examples to show that the assumed restriction cannot be relaxed.

Keywords: $*$-prime rings, Jordan ideals, generalized derivations, commutativity.

2000 AMS Classification: 16W10, 16W25, 16U80.

Received : 21.02.2013 *Accepted* : 19.01.2015 *Doi* : 10.15672/HJMS.20164512481

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Recall that R is 2-torsion free if $2x = 0$ yields $x = 0$. The ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive map $* : R \longrightarrow R$ is an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ forces $a = 0$ or $b = 0$. It is straightforward to check that a $*$-prime ring is necessarily semiprime, that is $xRx = 0$ forces $x = 0$. Furthermore, every prime ring having an involution $*$ is $*$-prime, but the converse need not be true in general. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.
In all that follows $Sa_*(R) = \{x \in R : x^* = \pm x\}$ will denote the set of symmetric and skew-symmetric elements of R. We will write for all $x, y \in R, [x, y] = xy - yx$ and $x \circ y = xy + yx$ for the commutator and anticommutator, respectively. An additive subgroup U of R is a Lie ideal if $[x, r] \in U$ for all $x \in U$ and $r \in R$. An additive subgroup

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J of R is a Jordan ideal if $x \circ r \in J$ for all $x \in J$ and $r \in R$. Moreover, if $J^* = J$, then J is called a $*$-Jordan ideal. We shall use without explicit mention the fact that if J is a Jordan ideal of R, then $2[R,R]J \subseteq J$ and $2J[R,R] \subseteq J$ ([7], Lemma 1). Moreover, From [1] we have $4jRj \subset J$, $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$. An additive mapping $d : R \longrightarrow R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Many results in literature indicate how the global structure of a ring R is often tightly connected to the behavior of derivations defined on R. More recently several authors consider similar situation in the case the derivation d is replaced by a generalized derivation. More specifically an additive map $F : R \longrightarrow R$ is a generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [2] and [8]).

Recently many authors have studied commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings. Moreover, many of obtained results extend other ones proven previously just for the action of the considered mapping on the whole ring. In this paper we continue the line of investigation regarding the study of commutativity for rings with involution satisfying certain differential identities involving generalized derivations acting on Jordan ideals.

2. Differential commutator identities

In 2002 Rehman [9] established that if a 2-torsion free prime ring admits a generalized derivation F associated with a nonzero derivation such that $F\Big([x,y]\Big) = [x,y]$ (or $F\Big([x,y]\Big) = -[x,y]$) for all x, y in a nonzero square closed Lie ideal U, then $U \subset Z(R)$. Quadri et al. [8], without 2-torsion freeness hypothesis, proved that a prime ring must be commutative if it admits a generalized derivation F, associated with a nonzero derivation, such that $F\Big([x,y]\Big) = [x,y]$ (or $F\Big([x,y]\Big) = -[x,y]$) for all x, y in a nonzero ideal I. Motivated by the above results, in this section we explore the commutativity of a $*$-prime ring R in which the generalized derivation F satisfies similar identities on a $*$-Jordan ideal. We shall conclude this section with an application of our results which extend results of [8] and [9] to Jordan ideals with the additional assumption that the ring R be 2-torsion free.

We begin with the following known results which will be used extensively to prove our theorems.

1. Lemma. ([3], Lemma 2) Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $aJb = a^*Jb = 0$ (or $aJb = aJb^* = 0$), then $a = 0$ or $b = 0$.

2. Lemma. ([5], Lemma 3) Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal. If d is a derivation such that $d(x^2) = 0$ for all $x \in J$, then $d = 0$.

3. Lemma. Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If R admits a nonzero derivation d such that $[[r,s],y]Jd(y^2) = 0$ for all $r, s \in R$ and $y \in J$, then $J \cap Z(R) \neq \{0\}$.

Proof. Assume that $J \cap Z(R) = \{0\}$. We have

(2.1) $\qquad [[r,s],y]Jd(y^2) = 0$ for all $y \in J$, $r, s \in R$.

Let $y \in J \cap Sa_*(R)$; then (2.1) implies that

$$[[r,s],y]^* J d(y^2) = 0$$

and combining this equation with (2.1), then Lemma 1 yields that either $d(y^2) = 0$ or $[[r,s],y] = 0$. Suppose

(2.2) $\quad [[r,s],y] = 0$ for all $r,s \in R$.

Substituting sy for s in (2.2) we get

$$0 = [[r,sy],y] = s[[r,y],y] + [s,y][r,y] + [[r,s],y]y$$

and employing (2.2) we find that

(2.3) $\quad [s,y][r,y] = 0$ for all $r,s \in R$.

Replacing r by rs in (2.3) we get $[s,y]r[s,y] = 0$ and thus

(2.4) $\quad [s,y]R[s,y] = 0$ for all $s \in R$.

In view of semi-primeness of R, equation (2.4) assures that $y \in Z(R)$ and thus $y = 0$. Accordingly

(2.5) $\quad d(y^2) = 0$ for all $y \in J \cap Sa_*(R)$.

Let $y \in J$, as $y^* - y,\ y^* + y \in J \cap Sa_*(R)$, then (2.5) forces $d(y^2) = -d((y^*)^2)$. Substituting y^* for y in (2.1) we obtain

$$[[r,s],y^*]Jd(y^2) = 0 \text{ for all } r,s \in R.$$

In particular,

$$[[r^*,s^*],y^*]Jd(y^2) = 0 \text{ for all } r,s \in R$$

which implies that

(2.6) $\quad [[r,s],y]^* J d(y^2) = 0$ for all $y \in J,\ r,s \in R$.

Combining (2.1) and (2.6), we conclude that $d(y^2) = 0$ or $[[r,s],y] = 0$ which, as above, leads to $d(y^2) = 0$. Consequently,

$$d(y^2) = 0 \text{ for all } y \in J$$

and Lemma 2 assures that $d = 0$ which contradicts our hypothesis. □

1. Theorem. Let R be a 2-torsion free $*$-prime ring and J be a nonzero $*$-Jordan ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F\Big([x,y]\Big)\Big) = [x,y]$ for all $x,y \in J$, then R is commutative.

Proof. Assume that

(2.7) $\quad F\Big([x,y]\Big) = [x,y]$ for all $x,y \in J$.

Replacing x by $4xy^2$ in (2.7) we get $F\Big([x,y]y^2\Big) = [x,y]y^2$ and thus

(2.8) $\quad [x,y]d(y^2) = 0$ for all $x,y \in J$.

Substituting $2[r,s]x$ for x in (2.8), where $r,s \in R$, we find that $[[r,s],y]xd(y^2) = 0$ and therefore

(2.9) $\quad [[r,s],y]Jd(y^2) = 0$ for all $y \in J$ and $r,s \in R$.

In view of (2.9), application of Lemma 3 assures that $J \cap Z(R) \neq \{0\}$. Replacing x by $4x^2u$ in (2.7), where $0 \neq u \in J \cap Z(R)$, we get

$$(2.10) \quad F\Big([2x^2, y]u\Big) = [2x^2, y]u \quad \text{for all } \ x, y \in J.$$

Using (2.7), equation (2.10) yields $[x^2, y]d(u) = 0$ and thus

$$(2.11) \quad [x^2, y]Jd(u) = 0 \quad \text{for all } \ x, y \in J.$$

Since J is a $*$-ideal, (2.11) forces

$$(2.12) \quad [x^2, y]^* Jd(u) = 0 \quad \text{for all } \ x, y \in J.$$

Combining (2.11) and (2.12), Lemma 1 yields $d(u) = 0$ or $[x^2, y] = 0$ for all $x, y \in J$.
If $[x^2, y] = 0$ for all $x, y \in J$, then R is commutative by proof of Theorem 3 in [4].
If $d(u) = 0$, then replacing x by $4ru^2$ in (2.7) we obtain $F\Big([r, y]u^2\Big) = [r, y]u^2$ and thus

$$\Big(F([r, y]) - [r, y]\Big)u^2 = 0.$$

Accordingly

$$(2.13) \quad \Big(F\Big([r, y]\Big) - [r, y]\Big)Ju^2 = 0 \quad \text{for all } \ y \in J, r \in R.$$

As $0 \neq u^* \in J \cap Z(R)$, then a similar reasoning as above leads to

$$(2.14) \quad \Big(F\Big([r, y]\Big) - [r, y]\Big)J(u^2)^* = 0 \quad \text{for all } \ y \in J, r \in R.$$

In view of Lemma 1, (2.13) together with (2.14) forces

$$(2.15) \quad F\Big([r, y]\Big) = [r, y] \quad \text{for all } \ y \in J \ \text{ and } \ r \in R.$$

Substituting ry for r in (2.15) we get

$$(2.16) \quad [r, y]d(y) = 0 \quad \text{for all } \ y \in J \ \text{ and } \ r \in R.$$

Replacing r by rs in (2.16), where $s \in R$, we obtain $[r, y]sd(y) = 0$ so that

$$(2.17) \quad [r, y]Rd(y) = 0 \quad \text{for all } \ y \in J \ \text{ and } \ r \in R.$$

Once again using the proof of Theorem 3 in [4], from equation (2.17) it follows that R is commutative. □

As an application of Theorem 1, the following theorem extends ([9], Theorem 3.3) and ([8], Theorem 2.1) to Jordan ideals.

2. Theorem. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F\Big([x, y]\Big) = [x, y]$ for all $x, y \in J$, then R is commutative.

Proof. Assume that F is a generalized derivation associated to a nonzero derivation d such that $F\Big([x, y]\Big) = [x, y]$, for all $x, y \in J$. Let $\mathcal{D}$ be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathcal{D}(x, y) = (d(x), 0)$ and $\mathcal{F}(x, y) = (F(x), y)$. Clearly, $\mathcal{D}$ is a nonzero derivation of $\mathcal{R}$ and $\mathcal{F}$ is a generalized derivation associated with $\mathcal{D}$. Moreover, if we set $\mathcal{J} = J \times J$, then $\mathcal{J}$ is a $*_{ex}$-Jordan ideal of $\mathcal{R}$ and $\mathcal{F}\Big([x, y]\Big) = [x, y]$ for all $x, y \in \mathcal{J}$. Since $\mathcal{R}$

is a $*_{ex}$-prime ring, in view of Theorem 1 we deduce that $\mathcal{R}$ is commutative and a fortiori R is commutative. $\square$

A slight modification in the proof of Theorem 1 yields the following result.

3. Theorem. Let R be a 2-torsion free $*$-prime ring and J be a nonzero $*$-Jordan ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F\Big([x,y]\Big) = -[x,y]$ for all $x, y \in J$, then R is commutative.

Reasoning as in the proof of Theorem 2, where $\mathcal{F}(x,y) = (F(x), -y)$, and using Theorem 3 we extend ([9], Theorem 3.4) and ([8], Theorem 2.2) to Jordan ideals as follows.

4. Theorem. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F\Big([x,y]\Big) = -[x,y]$ for all $x, y \in J$, then R is commutative.

3. Differential anticommutator identities

It is natural to ask what can we say about the commutativity of R if the commutator in the preceding section is replaced by anticommutator. In this section, we have investigated this problem and proved that the commutativity cannot be characterized by the same conditions on anticommutator.

5. Theorem. Let R be a 2-torsion free $*$-prime ring and J be a nonzero $*$-Jordan ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in J$, then $d = 0$ and F is the identity map.

Proof. Assume that

(3.1) $\quad F(x \circ y) = x \circ y \quad \text{for all } x, y \in J.$

Replacing y by $4yx^2$ in (3.1) we find that

(3.2) $\quad (x \circ y)d(x^2) = 0 \quad \text{for all } x, y \in J.$

Substituting $2[r,s]y$ for y in (3.2), where $r, s \in R$, we obtain $(x \circ (2[r,s]y))d(x^2) = 0$ and thus $[x,[r,s]]yd(x^2) = 0$. Hence

(3.3) $\quad [x,[r,s]]Jd(x^2) = 0 \quad \text{for all } x \in J \text{ and } r, s \in R.$

In view of Lemma 3, equation (3.3) assures that $d = 0$ or $J \cap Z(R) \neq 0$.
If there exists $0 \neq u \in J \cap Z(R)$, then replacing y by $4u^2y$ in (3.1), we get

(3.4) $\quad F(2u^2(x \circ y)) = 2u^2(x \circ y) \quad \text{for all } x, y \in J.$

Since by assumption of the theorem $F(2u^2) = F(u \circ u) = 2u^2$, then (3.4) leads to

(3.5) $\quad u^2d(x \circ y) = 0 \quad \text{for all } x, y \in J.$

Using the fact that $u \in Z(R)$, from (3.5) it follows that

(3.6) $\quad u^2Jd(x \circ y) = 0 \quad \text{for all } x, y \in J.$

As $0 \neq u^* \in J \cap Z(R)$, a similar reasoning as above yields

(3.7) $\quad (u^2)^*Jd(x \circ y) = 0 \quad \text{for all } x, y \in J.$

We claim that $u^2 \neq 0$. For contradiction assume that $u^2 = 0$, then $uRu = 0$. Since R is semiprime $u = 0$. This is a contradiction. Thus $u^2 \neq 0$.
Now if we combine (3.6) and (3.7) and apply Lemma 1, we conclude that

$$d(x \circ y) = 0 \quad \text{for all } x, y \in J$$

and a fortiori

$$d(x^2) = 0 \quad \text{for all } \ x \in J. \tag{3.8}$$

In light of Lemma 2, equation (3.8) forces $d = 0$ hence F is a left multiplier.
From $F(x \circ y) = x \circ y$ it then follows $(F(x) - x)y = -(F(y) - y)x$ and replacing y by $y \circ z$ where $z \in J$ we get

$$(F(x) - x)(y \circ z) = 0 \quad \text{for all } \ x, y, z \in J. \tag{3.9}$$

Replacing z by $2z[r, s]$ in (3.9) where $r, s \in R$ we obtain

$$(F(x) - x)J[y, [r, s]] = (F(x) - x)J\Big([y, [r, s]]\Big)^* = 0 \quad \text{for all } \ x, y \in J, \quad r, s \in R. \tag{3.10}$$

Thus, according to Lemma 1, either $F(x) = x$ for all $x \in J$ or $[y, [r, s]] = 0$ for all $y \in J$ and $r, s \in R$.
Assume that $[y, [r, s]] = 0$ for all $y \in J$ and $r, s \in R$, hence as in (2.2) this implies that

$$[s, y]R[s, y] = 0 \quad \text{for all } \ y \in J, \quad s \in R.$$

and the semi-primeness of R forces $[s, y] = 0$ so that $J \subseteq Z(R)$. Therefore [[6], Lemma 3] assures that R is a commutative ring in which case, as F is a left multiplier, equation (3.1) implies that $F(x)y = xy$.
In conclusion, in either case (3.1) becomes

$$(F(x) - x)y = 0 \quad \text{for all } \ x, y \in J \tag{3.11}$$

in such a way that $F(x) = x$ for all $x \in J$. Let $r \in R$ and $x \in J$, from $F(x \circ r) = x \circ r$ it follows that

$$\begin{aligned} xr + rx &= F(xr + rx) \\ &= F(x)r + F(r)x \\ &= xr + F(r)x \end{aligned}$$

so that

$$(F(r) - r)x = 0 \quad \text{for all } \ r \in R, \ x \in J$$

and therefore $F(r) = r$ for all $r \in R$. Hence F is the identity map. □

Using similar arguments as used in the proof of Theorem 2, application of Theorem 5 yields the following result which extends ([9], Theorem 3.7) and ([8], Theorem 2.3) to Jordan ideals in the case of a 2-torsion free ring.

6. Theorem. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If R admits a generalized derivation associated with a derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in J$, then $d = 0$ and F is the identity map.

Reasoning as in proof of Theorem 5, we can prove the following.

7. Theorem. Let R be a 2-torsion free $*$-prime ring and J be a nonzero $*$-Jordan ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(x \circ y) = -\, x \circ y$ for all $x, y \in J$, then $d = 0$ and $(-F)$ is the identity map.

Similarly, application of Theorem 7 yields the following result which improves ([9], Theorem 3.8) and ([8], Theorem 2.4).

8. Theorem. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(x \circ y) = -\, x \circ y$ for all $x, y \in J$, then $d = 0$ and $(-F)$ is the identity map.

1. Example. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c\in\mathbb{R}\right\}$ and consider $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$. It is straightforward to verify that F is a generalized derivation associated with the non zero derivation d defined by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Moreover, if we set $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{*}=\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$, then R is a non $*$-prime ring. Furthermore, it is easy to verify that $J=\left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a\in\mathbb{R}\right\}$ is a nonzero $*$-Jordan ideal of R such that

$$F(A\circ B)=A\circ B,\ F(A\circ B)=-\,A\circ B,\ F[A,B]=[A,B],\ F[A,B]=-[A,B]$$

for al $A,B\in R$. Hence in theorems 1, 3, 5, 7 the $*$-primeness hypothesis is crucial.

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