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Hermite-Hadamard type inequalities for harmonically (α, m)-convex functions

İmdat İşcan*†

Abstract

The author introduces the concept of harmonically (α, m)-convex functions and establishes some Hermite-Hadamard type inequalities of these classes of functions.

Keywords: Harmonically (α, m)-convex function, Hermite-Hadamard type inequalities, Hypergeometric function.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The class of (α, m)-convex functions was first introduced In [8], and it is defined as follows:

1.1. Definition. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

*Department of Mathematics, Faculty of Arts and Sciences,
Giresun University, 28100, Giresun, Turkey., Email: imdat.iscan@giresun.edu.tr
†Corresponding Author.

It can be easily that for $(\alpha, m) \in \{(0,0), (\alpha, 0), (1,0), (1,m), (1,1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex, α-convex.

Denote by $K_m^\alpha(b)$ the set of all (α, m)-convex functions on $[0,b]$ for which

$f(0) \leq 0$. For recent results and generalizations concerning (α, m)-convex functions (see [2, 4, 5, 6, 8, 9, 10, 11, 12]).

In [7], the author gave definition of harmonically convex functions and established some Hermite-Hadamard type inequalities for harmonically convex functions as follows:

1.2. Definition. Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

1.3. Theorem. *Let $f : I \subset \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

The above inequalities are sharp.

1.4. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \geq 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ \leq\ & \frac{ab\,(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_2 &= \frac{-1}{b\,(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_3 &= \frac{1}{a\,(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right) \\ &= \lambda_1 - \lambda_2. \end{aligned}$$

1.5. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ \leq\ & \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1\left|f'(a)\right|^q + \mu_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}, \end{aligned} \tag{1.5}$$

where

$$\mu_1 = \frac{\left[a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q)-a\right]\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2 = \frac{\left[b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q)+b\right]\right]}{2(b-a)^2(1-q)(1-2q)}.$$

In [7], the author gave the following identity for differentiable functions.

1.6. Lemma. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$= \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

The main purpose of this paper is to introduce the concept of harmonically (α, m)-convex functions and establish some new Hermite-Hadamard type inequalities for these classes of functions.

2. Main Results

2.1. Definition. The function $f : (0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be harmonically (α, m)-convex, where $\alpha \in [0,1]$ and $m \in (0,1]$, if

$$f\left(\frac{mxy}{mty+(1-t)x}\right) = f\left(\left(\frac{t}{x}+\frac{1-t}{my}\right)^{-1}\right) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{2.1}$$

for all $x, y \in (0, b^*]$ and $t \in [0,1]$. If the inequality in (2.1) is reversed, then f is said to be harmonically (α, m)-concave.

2.2. Remark. When $m = \alpha = 1$, the harmonically (α, m)-convex (concave) function defined in Definition 2.1 becomes a harmonically convex (concave) function defined in [7]. Thus, every harmonically convex (concave) function is also harmonically $(1,1)$-convex (concave) function.

The following proposition is obvious.

2.3. Proposition. *Let $f : (0, b^*] \to \mathbb{R}$ be a function.*

a) *if f is (α, m)-convex and nondecreasing function then f is harmonically (α, m)-convex.*
b) *if f is harmonically (α, m)-convex and nonincreasing function then f is (α, m)-convex.*

Proof. For all $t \in [0,1]$, $m \in (0,1]$ and $x, y \in (0, b^*]$ we have

$$t(1-t)(x-my)^2 \ge 0,$$

then the following inequality holds

$$\frac{mxy}{mty+(1-t)x} \le tx + m(1-t)y. \tag{2.2}$$

By the inequality (2.2), the proof is completed. □

2.4. Remark. According to Proposition 2.3, every nondecreasing s-convex function in the first sense (or $(s,1)$-convex function) is also harmonically $(s,1)$-convex function.

2.5. Example. Let $s \in (0,1]$, then the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^s$ is a nondecreasing s-convex function in the first sense [3]. According to the above Remark, f is also harmonically $(s,1)$-convex function.

The following result of the Hermite-Hadamard type holds.

2.6. Theorem. *Let $f : (0,\infty) \to \mathbb{R}$ be a harmonically (α, m)-convex function with $\alpha \in [0,1]$ and $m \in (0,1]$. If $0 < a < b < \infty$ and $f \in L[a,b]$, then one has the inequality*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left\{\frac{f(a)+\alpha m f(\frac{b}{m})}{\alpha+1}, \frac{f(b)+\alpha m f(\frac{a}{m})}{\alpha+1}\right\}. \tag{2.3}$$

Proof. Since $f : (0,\infty) \to \mathbb{R}$ is a harmonically (α, m)-convex function, we have, for all $x, y \in I$

$$f\left(\frac{xy}{ty+(1-t)x}\right) = f\left(\frac{m\frac{y}{m}x}{tm\frac{y}{m}+(1-t)x}\right) \le t^\alpha f(x) + m(1-t^\alpha)f(\frac{y}{m})$$

which gives:

$$f\left(\frac{ab}{tb+(1-t)a}\right) \le t^\alpha f(a) + m(1-t^\alpha)f(\frac{b}{m})$$

and

$$f\left(\frac{ab}{ta+(1-t)b}\right) \le t^\alpha f(b) + m(1-t^\alpha)f(\frac{a}{m})$$

for all $t \in [0,1]$. Integrating on $[0,1]$ we obtain

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt \le \frac{f(a)+\alpha m f(\frac{b}{m})}{\alpha+1}$$

and

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \le \frac{f(b)+\alpha m f(\frac{a}{m})}{\alpha+1}.$$

However,

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt = \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

and the inequality (2.3) is obtained. □

2.7. Remark. If we take $\alpha = m = 1$ in Theorem 2.6, then inequality (2.3) becomes the right-hand side of inequality (1.3).

2.8. Corollary. *If we take $m = 1$ in Theorem 2.6, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left\{\frac{f(a)+\alpha f(b)}{\alpha+1}, \frac{f(b)+\alpha f(a)}{\alpha+1}\right\} \tag{2.4}$$

2.9. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α, m)-convex on $[a, b/m]$ for $q \ge 1$, with $\alpha \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab\,(b-a)}{2^{2-1/q}}\left[\lambda(\alpha,q;a,b)\left|f'(a)\right|^{q}+m\mu(\alpha,q;a,b)\left|f'(b/m)\right|^{q}\right]^{\frac{1}{q}},$$

where

$$\begin{aligned}\lambda(\alpha,q;a,b) &= \frac{\beta(1,\alpha+2)}{b^{2q}}\cdot {}_2F_1\left(2q,1;\alpha+3;1-\frac{a}{b}\right)\\ &\quad -\frac{\beta(2,\alpha+1)}{b^{2q}}\cdot {}_2F_1\left(2q,2;\alpha+3;1-\frac{a}{b}\right)\\ &\quad +\frac{2^{2q-\alpha}\beta(2,\alpha+1)}{(a+b)^{2q}}\cdot {}_2F_1\left(2q,2;\alpha+3;1-\frac{2a}{a+b}\right),\\ \mu(\alpha,q;a,b) &= \lambda(0,q;a,b)-\lambda(\alpha,q;a,b),\end{aligned}$$

β is Euler Beta function defined by

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x,y>0,$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\ c>b>0,\ |z|<1$$

(see [1]*).*

Proof. From Lemma 1.6 and using the power mean inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\ \leq\ & \frac{ab\,(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt\\ \leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, by harmonically (α,m)-convexity of $|f'|^q$ on $[a,b/m]$,we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\ \leq\ & \frac{ab\,(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|\left[t^{\alpha}\left|f'(a)\right|^q+m(1-t^{\alpha})\left|f'(b/m)\right|^q\right]}{(tb+(1-t)a)^{2q}}dt\right)^{\frac{1}{q}}\\ \leq\ & \frac{ab\,(b-a)}{2^{2-1/q}}\left[\lambda(\alpha,q;a,b)\left|f'(a)\right|^q+m\left(\lambda(0,q;a,b)-\lambda(\alpha,q;a,b)\right)\left|f'(b/m)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

It is easily check that

$$\int_0^1 \frac{|1-2t|\,t^{\alpha}}{(tb+(1-t)a)^{2q}}dt = 2\int_0^{1/2} \frac{(1-2t)\,t^{\alpha}}{(tb+(1-t)a)^{2q}}dt - \int_0^1 \frac{(1-2t)\,t^{\alpha}}{(tb+(1-t)a)^{2q}}dt$$

$$= \frac{\beta(1,\alpha+2)}{b^{2q}}.{}_2F_1\left(2q,1;\alpha+3;1-\frac{a}{b}\right) - \frac{\beta(2,\alpha+1)}{b^{2q}}.{}_2F_1\left(2q,2;\alpha+3;1-\frac{a}{b}\right)$$
$$+\frac{2^{2q-\alpha}\beta(2,\alpha+1)}{(a+b)^{2q}}.{}_2F_1\left(2q,2;\alpha+3;1-\frac{2a}{a+b}\right) = \lambda(\alpha,q;a,b),$$

$$\int_0^1 \frac{|1-2t|\,(1-t^{\alpha})}{(tb+(1-t)a)^{2q}}dt = \lambda(0,q;a,b) - \lambda(\alpha,q;a,b).$$

This completes the proof. □

If we take $\alpha = m = 1$ in Theorem 2.9 then we get the following a new corrollary for harmonically convex functions:

2.10. Corollary. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^{\circ}$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \geq 1$ then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab\,(b-a)}{2^{2-1/q}}\left[\lambda(1,q;a,b)\left|f'(a)\right|^q + \mu(1,q;a,b)\left|f'(b/m)\right|^q\right]^{\frac{1}{q}}.$$

2.11. Corollary. *If we take $m = 1$ in Theorem 2.9 then we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \tag{2.5}$$

$$\leq \frac{ab\,(b-a)}{2^{2-1/q}}\left[\lambda(\alpha,q;a,b)\left|f'(a)\right|^q + \mu(\alpha,q;a,b)\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

2.12. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^{\circ}$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α,m)-convex on $[a,b/m]$ for $q \geq 1$, with $\alpha \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab\,(b-a)}{2} \tag{2.6}$$

$$\times\lambda^{1-\frac{1}{q}}(0,q;a,b)\left[\lambda(\alpha,1;a,b)\left|f'(a)\right|^q + m\mu(\alpha,1;a,b)\left|f'(b/m)\right|^q\right]^{\frac{1}{q}},$$

where λ and μ is defined as in Theorem 2.9.

Proof. From Lemma 1.6, power mean inequality and the harmonically (α, m)-convexity of $|f'|^q$ on $[a, b/m]$,we have,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{ab\,(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
\leq\ & \frac{ab\,(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{|1-2t|\left[t^\alpha\left|f'(a)\right|^q+m(1-t^\alpha)\left|f'(b/m)\right|^q\right]}{(tb+(1-t)a)^2}dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{ab\,(b-a)}{2}\lambda^{1-\frac{1}{q}}(0,q;a,b)\left[\lambda(\alpha,1;a,b)\left|f'(a)\right|^q+m\mu(\alpha,1;a,b)\left|f'(b/m)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

□

2.13. Remark. If we take $\alpha = m = 1$ in Theorem 2.12 then inequality (2.6) becomes inequality (1.4) of Theorem 1.4.

2.14. Corollary. *If we take $m = 1$ in Theorem 2.12 then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab\,(b-a)}{2} \tag{2.7}$$

$$\times\lambda^{1-\frac{1}{q}}(0,q;a,b)\left[\lambda(\alpha,1;a,b)\left|f'(a)\right|^q+\mu(\alpha,1;a,b)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

2.15. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0,1]$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α,m)-convex on $[a,b/m]$ for $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, with $\alpha \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{2.8}$$

$$\times\left(\nu(\alpha,q;a,b)\left|f'(a)\right|^q+m(\nu(0,q;a,b)-\nu(\alpha,q;a,b))\left|f'(b/m)\right|^q\right)^{\frac{1}{q}}$$

where

$$\nu(\alpha,q;a,b)=\frac{\beta\,(1,\alpha+1)}{b^{2q}}.{}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right).$$

Proof. From Lemma 1.6, Hölder's inequality and the harmonically (α, m)-convexity of $|f'|^q$ on $[a, b/m]$,we have,

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
\leq\ & \frac{ab(b-a)}{2} \left(\int_0^1 |1-2t|^p \, dt \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}} \left| f'\left(\frac{ab}{tb+(1-t)a} \right) \right|^q dt \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \frac{t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q}{(tb+(1-t)a)^{2q}} dt \right)^{\frac{1}{q}} \\
\leq\ & \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \times \left(\nu(\alpha, q; a, b) \left| f'(a) \right|^q + m(\nu(0, q; a, b) - \nu(\alpha, q; a, b)) \left| f'(b/m) \right|^q \right)^{\frac{1}{q}},
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
& \int_0^1 \frac{t^\alpha}{(tb+(1-t)a)^{2q}} dt \\
=\ & \frac{\beta(1, \alpha+1)}{b^{2q}} \cdot {}_2F_1\left(2q, 1; \alpha+2; 1-\frac{a}{b}\right) \\
=\ & \nu(\alpha, q; a, b)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \frac{1-t^\alpha}{(tb+(1-t)a)^{2q}} dt \\
=\ & \nu(0, q; a, b) - \nu(\alpha, q; a, b).
\end{aligned}$$

This completes the proof. □

2.16. Remark. If we take $\alpha = m = 1$ in Theorem 2.15 then inequality (2.8) becomes inequality (1.5) of Theorem 1.5.

2.17. Corollary. *If we take $m = 1$ in Theorem 2.15 then we get*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \tag{2.9}$$

$$\times \left(\nu(\alpha, q; a, b) \left| f'(a) \right|^q + (\nu(0, q; a, b) - \nu(\alpha, q; a, b)) \left| f'(b) \right|^q \right)^{\frac{1}{q}}.$$

3. Some applications for special means

Let us recall the following special means of two nonnegative number a, b with $b > a$:

(1) The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b,\ \ \alpha \in [0,1].$$

(2) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

(3) The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

(4) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}.$$

(5) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\quad p \in \mathbb{R}\backslash\{-1,0\}.$$

3.1. Proposition. *Let $0 < a < b$. Then we have the following inequality*

$$G^2 L_{\alpha-2}^{\alpha-2} \leq \min\left\{A_{1/(\alpha+1)}(a^\alpha, b^\alpha), A_{1/(\alpha+1)}(b^\alpha, a^\alpha)\right\}.$$

Proof. The assertion follows from the inequality (2.4) in Corollary 2.8, for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^\alpha,\ 0<\alpha<1.$ □

3.2. Proposition. *Let $0 < a < b$, $q \geq 1$ and $0 < \alpha < 1$. Then we have the following inequality*

$$\left|A\left(a^{\frac{\alpha}{q}+1}, b^{\frac{\alpha}{q}+1}\right) - G^2 L_{\frac{\alpha}{q}-1}^{\frac{\alpha}{q}-1}\right|$$
$$\leq \frac{ab(b-a)(\alpha+q)}{q2^{2-1/q}}\left[\lambda(\alpha,q;a,b)a^\alpha + \mu(\alpha,q;a,b)b^\alpha\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (2.5) in Corollary 2.11, for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{\frac{\alpha}{q}+1}/\left(\frac{\alpha}{q}+1\right).$ □

3.3. Proposition. *Let $0 < a < b$, $q \geq 1$ and $0 < \alpha < 1$. Then we have the following inequality*

$$\left|A\left(a^{\frac{\alpha}{q}+1}, b^{\frac{\alpha}{q}+1}\right) - G^2 L_{\frac{\alpha}{q}-1}^{\frac{\alpha}{q}-1}\right|$$
$$\leq \frac{ab(b-a)(\alpha+q)}{2q}\lambda^{1-\frac{1}{q}}(0,q;a,b)\left[\lambda(\alpha,1;a,b)a^\alpha + \mu(\alpha,1;a,b)b^\alpha\right]^{\frac{1}{q}},$$

Proof. The assertion follows from the inequality (2.7) in Corollary 2.14, for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{\frac{\alpha}{q}+1}/\left(\frac{\alpha}{q}+1\right).$ □

3.4. Proposition. *Let $0 < a < b$, $q > 1$, $1/p + 1/q = 1$ and $0 < \alpha < 1$. Then we have the following inequality*

$$\left|A\left(a^{\frac{\alpha}{q}+1}, b^{\frac{\alpha}{q}+1}\right) - G^2 L_{\frac{\alpha}{q}-1}^{\frac{\alpha}{q}-1}\right|$$
$$\leq \frac{ab(b-a)(\alpha+q)}{2q}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\nu(\alpha,q;a,b)a^\alpha + (\nu(0,q;a,b)-\nu(\alpha,q;a,b))b^\alpha\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (2.9) in Corollary 2.17, for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{\frac{\alpha}{q}+1}/\left(\frac{\alpha}{q}+1\right)$. □

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