Hacettepe Journal of Mathematics and Statistics
Volume 45 (1) (2016), 57 – 67

Some properties of AFG and CTF rings

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Abstract

R is said to be a right AFG ring if the right annihilator of every nonempty subset of R is a finitely generated right ideal. R is called a right CTF ring if every cyclic torsionless right R-module embeds in a free module. In this paper, we first give new characterizations of AFG rings and study some closure properties of AFG rings. Then we explore the intimate relationships between AFG rings and CTF rings.

Keywords: AFG ring; CTF ring; pseudo-coherent ring; FP-injective module; singly projective module.

2000 AMS Classification: 16D40, 16P70.

Received : 26.12.2013 *Accepted* : 27.11.2014 *Doi* : 10.15672/HJMS.20164512483

1. Introduction

In [19], we introduced the concept of AFG rings, which is a generalization of Noetherian rings. R is said to be a *right AFG ring* in case the right annihilator of every nonempty subset of R is a finitely generated right ideal, equivalently, every cyclic torsionless right R-module is finitely presented, where a right R-module M is called *torsionless* if M embeds in a direct product of copies of R_R. The concept of AFG rings is very useful in ring theory. For more details about AFG rings, we refer the reader to [19, 20, 21].

In this paper, we gave some new characterizations of AFG rings and further study some properties of AFG rings, such as closure properties under finite direct products, quotients and localizations. On the other hand, we explore the intimate connections between AFG rings and CTF rings, where a ring R is called *right CTF* [27] if every cyclic torsionless right R-module embeds in a free module.

The layout of the paper is as follows:

Section 2 is devoted to AFG rings. We first prove that R is a right AFG ring if and only if the dual module $\mathrm{Hom}_R(M, R)$ of any cyclic torsionless left R-module M is finitely generated if and only if every cyclic torsionless left R-module has a projective preenvelope. It is also shown that R is a right AFG ring if R is a left singly injective left

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CF ring. Next we discuss the closure properties of AFG rings. We prove that: (1) R and S are right AFG rings if and only if $R \times S$ is a right AFG ring. (2) If R is a right AFG ring and I is an ideal which is a right annihilator in R, then R/I is a right AFG ring. (3) If R is a commutative AFG ring and S a multiplicative subset of R without zero-divisors, then $S^{-1}R$ is also an AFG ring. Finally we give some examples to clarify the relationships among AFG rings, AC rings, Π-coherent rings and pseudo-coherent rings.

In Section 3, we deal with some properties of CTF rings. For example, it is shown that R is a right CTF ring if the dual module of every cyclic torsionless right R-module is H-finitely generated, and the converse holds if R is a left f-injective ring. Furthermore, we explore the close connections between AFG rings and CTF rings. We prove that: (1) If R is a left AFG ring, then R is a right CTF ring. (2) If R is a right CTF right pseudo-coherent ring, then R is a right AFG ring. (3) R is a left AFG ring if and only if R is a right CTF ring and $lr(S)$ is a finitely generated left ideal for any finite subset S of R. (4) R is a two-sided AFG two-sided singly injective ring if and only if R is a two-sided CTF two-sided FP-injective ring.

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R (resp. ${}_RM$) denotes a right (resp. left) R-module. For an R-module M, the dual module $\mathrm{Hom}_R(M,R)$ is denoted by M^* and the character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. $E(M)$ denotes the injective envelope of M. M^I (resp. $M^{(I)}$) stands for the direct product (resp. direct sum) of copies of M indexed by a set I. For a subset X of R, the right (resp. left) annihilator of X in R is denoted by $r(X)$ (resp. $l(X)$). We refer to [1, 9, 15, 16, 24, 26] for all undefined notions in this article.

2. AFG rings

In [19], the author gave some characterizations of AFG rings. For example, R is a right AFG ring if and only if the dual module M^* of any cyclic left R-module M is finitely generated if and only if every cyclic left R-module has a projective preenvelope. The following theorem gives an improvement of the above result.

Recall that that a homomorphism $f : M \to P$ is called a *projective preenvelope* of a left R-module M [9] if P is projective, and for any homomorphism g from M to any projective left R-module P', there exists $h : P \to P'$ such that $g = hf$.

We also recall a right R-module M is *FP-injective* (or *absolutely pure*) [25, 17] if $\mathrm{Ext}^1_R(N, M) = 0$ for any finitely presented right R-module N. M is called *$\mathcal{A}$-injective* [18] if $\mathrm{Ext}^1_R(R/I, M) = 0$ for any right annihilator I in R.

2.1. Theorem. *The following are equivalent for a ring R:*

(1) *R is a right AFG ring.*
(2) *The dual module M^* of any cyclic torsionless left R-module M is finitely generated.*
(3) *For any cyclic torsionless left R-module A and $x \in A$, the additive subgroup $H_{A,x} = \{f(x) : f \in \mathrm{Hom}_R(A, R)\}$ of R is a finitely generated right ideal.*
(4) *Every cyclic torsionless left R-module has a projective preenvelope.*
(5) *Every FP-injective right R-module is $\mathcal{A}$-injective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are obvious by [19, Theorem 2.3].

(2) $\Rightarrow$ (1) Let I be any right annihilator in R. Then the exact sequence

$$0 \to I \xrightarrow{i} R_R \xrightarrow{f} R/I \to 0$$

of right R-modules yields the exact sequence of left R-modules

$$0 \to (R/I)^* \xrightarrow{f^*} (R_R)^* \xrightarrow{i^*} I^*.$$

Let $B = \operatorname{im}(i^*)$. Then we get the exact sequence

$$0 \to (R/I)^* \xrightarrow{f^*} (R_R)^* \to B \to 0,$$

which gives rise to the exactness of the sequence

$$0 \to B^* \to (R_R)^{**} \to (R/I)^{**}.$$

By [24, Exercise 2.7, p.27], there exists $\phi : I \to B^*$ such that the following diagram with exact rows commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R_R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \sigma_R} & & \Big\downarrow{\scriptstyle \sigma_{R/I}} & & \\ 0 & \longrightarrow & B^* & \longrightarrow & (R_R)^{**} & \longrightarrow & (R/I)^{**}. & & \end{array}$$

Since $\sigma_{R/I}$ is a monomorphism, $I \cong B^*$ by the Five Lemma. Note that I^* is torsionless by [1, Proposition 20.14], so B is a cyclic torsionless left R-module. Thus $I \cong B^*$ is finitely generated by (2), which implies that R is a right AFG ring.

(2) $\Rightarrow$ (3) Let A be any cyclic torsionless left R-module and $x \in A$. Then there exist $f_1, f_2, \cdots, f_n \in A^*$ such that

$$A^* = f_1 R + f_2 R + \cdots + f_n R.$$

So $H_{A,x} = \sum_{k=1}^{n} f_k(x) R$ is a finitely generated right ideal.

(3) $\Rightarrow$ (2) Let $A = Rx$ be a cyclic torsionless left R-module. Define a right R-homomorphism $\beta : A^* \to H_{A,x}$ via $f \mapsto f(x)$. It is clear that β is an isomorphism. Thus A^* is a finitely generated right R-module by (3).

(4) $\Rightarrow$ (2) Let M be a cyclic torsionless left R-module. Then M has a projective preenvelope $f : M \to P$. We may choose P to be finitely generated since M is cyclic. So we get the exact sequence $P^* \to M^* \to 0$. Thus M^* is finitely generated.

(1) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1) Let M be a cyclic torsionless right R-module. Then $\operatorname{Ext}^1_R(M, N) = 0$ for any FP-injective right R-module N by (5). Therefore M is finitely presented by [8], and so R is a right AFG ring. ☐

Now we investigate AFG rings in terms of singly projective, singly injective and singly flat modules.

Recall that a left R-module M is *singly projective* [2] in case for any cyclic submodule N of M, the inclusion map $N \to M$ factors through a free module.

According to [22], a left R-module M (resp. right R-module N) is called *singly injective* (resp. *singly flat*) if $\operatorname{Ext}^1_R(F/C, M) = 0$ (resp. $\operatorname{Tor}^R_1(N, F/C) = 0$) for any cyclic submodule C of any finitely generated free left R-module F. R is called a *left singly injective ring* if ${}_R R$ is a singly injective left R-module.

Recall that R is a *left CF ring* [13] if every cyclic left R-module embeds in a free module.

2.2. Proposition. *The following are true:*

1. *R is a left singly injective ring if and only if every singly projective left R-module is singly injective.*
2. *R is a left CF ring if and only if every singly injective left R-module is singly projective.*

(3) *If R is a left singly injective left CF ring, then R is a right AFG ring.*

Proof. (1) " $\Rightarrow$ " Let M be a singly projective left R-module. For any cyclic submodule C of any finitely generated free left R-module F and any homomorphism $f : C \to M$, there exist a finitely generated free left R-module G, $g : C \to G$ and $h : G \to M$ such that $f = hg$. Note that G is singly injective, and so there exists $\varphi : F \to G$ such that $\varphi\lambda = g$, where $\lambda : C \to F$ is the inclusion. Hence $(h\varphi)\lambda = hg = f$. Thus M is singly injective.

" $\Leftarrow$ " is clear.

(2) " $\Rightarrow$ " Let M be a singly injective left R-module. For any cyclic submodule N of M, there exists a monomorphism $\gamma : N \to R^n, n \in \mathbb{N}$. Thus there is $\theta : R^n \to M$ such that $\iota = \theta\gamma$, where $\iota : N \to M$ is the inclusion. So M is singly projective.

" $\Leftarrow$ " is obvious by [19, Lemma 3.6].

(3) Let $\{M_i\}_{i\in I}$ be a family of singly projective left R-modules. Then each M_i is singly injective by (1) and so M_i^I is singly injective. Thus M_i^I is singly projective by (2). Hence R is a right AFG ring by [19, Theorem 2.3]. □

It is known that any singly projective R-module is singly flat for any ring R by [22, Lemma 2.4] and any singly flat R-module is singly projective for any commutative domain R by [22, Corollary 2.6]. Here we have the following result.

2.3. Proposition. *The following are equivalent for a ring R:*

(1) *R is right AFG and every singly flat left R-module is singly projective.*
(2) *N^+ is singly projective for every singly injective right R-module N.*
(3) *M^{++} is singly projective for every singly flat left R-module M.*

Proof. (1) $\Rightarrow$ (2) Since R is right AFG, N^+ is singly flat by [22, Theorem 2.10] for any singly injective right R-module N. So N^+ is singly projective by (1).

(2) $\Rightarrow$ (3) Let M be a singly flat left R-module. Then M^+ is singly injective by [22, Lemma 2.4]. So M^{++} is singly projective by (2).

(3) $\Rightarrow$ (1) Let $\{M_i\}_{i\in I}$ be a family of singly projective left R-modules, then the pure exact sequence

$$0 \to (M_i^+)^{(I)} \to (M_i^+)^I$$

induces the split exact sequence

$$((M_i^+)^I)^+ \to ((M_i^+)^{(I)})^+ \to 0.$$

Thus $((M_i^+)^{(I)})^+$ is isomorphic to a direct summand of $((M_i^+)^I)^+$. Note that

$$((M_i^+)^{(I)})^+ \cong (M_i^{++})^I, ((M_i^+)^I)^+ \cong (M_i^{(I)})^{++}.$$

Thus $(M_i^{++})^I$ is singly projective since $(M_i^{(I)})^{++}$ is singly projective by (3). Also M_i^I is a pure submodule of $(M_i^{++})^I$ by [6, Lemma 1(2)]. Hence M_i^I is singly projective by [2, Proposition 14], and so R is right AFG by [19, Theorem 2.3].

On the other hand, let M be any singly flat left R-module, then M^{++} is singly projective by (3). Note that M is a pure submodule of M^{++}, and so M is singly projective by [2, Proposition 14]. □

Recall that R is a *left dual ring* if every left ideal is a left annihilator in R, equivalently, every cyclic left R-module is torsionless.

2.4. Theorem. *The following are equivalent for a ring R:*

(1) *R is a right AFG left dual ring.*
(2) *R is a right AFG ring and the injective envelope of every simple left R-module is singly projective.*

(3) *R is a right AFG ring and the injective envelope of every finitely cogenerated left R-module is singly projective.*
(4) *R is a right AFG ring and $(R_R)^+$ is singly projective.*
(5) *Every cyclic left R-module has a projective preenvelope which is a monomorphism.*

Proof. (1) $\Rightarrow$ (5) holds by [19, Theorem 3.7].

(5) $\Rightarrow$ (4) R is a right AFG ring by [19, Theorem 2.3]. Let N be a cyclic submodule of $(R_R)^+$. Since N embeds in $R^n, n \in \mathbb{N}$ and $(R_R)^+$ is injective, the inclusion $N \to (R_R)^+$ factors through R^n. So $(R_R)^+$ is singly projective.

(4) $\Rightarrow$ (2) Let M be a simple left R-module. Then there is a monomorphism $E(M) \to ((R_R)^+)^I$. So $E(M)$ is isomorphic to a direct summand of $((R_R)^+)^I$. Since $((R_R)^+)^I$ is singly projective by [19, Theorem 2.3], $E(M)$ is singly projective.

(2) $\Rightarrow$ (1) Let N be a cyclic left R-module. It is enough to show that for any $0 \neq m \in N$, there exists $f : N \to R$ such that $f(m) \neq 0$. In fact, there is a maximal submodule K of Rm, and so Rm/K is simple. Let $\iota : Rm \to N$ and $i : Rm/K \to E(Rm/K)$ be the inclusions, and $\pi : Rm \to Rm/K$ be the natural map. Then there exists $j : N \to E(Rm/K)$ such that $j\iota = i\pi$. So $j(m) = j\iota(m) = i\pi(m) \neq 0$. On the other hand, since $E(Rm/K)$ is singly projective by (2), there exist $n \in \mathbb{N}$, $g : N \to R^n$ and $h : R^n \to E(Rm/K)$ such that $j = hg$. Therefore $g(m) = (x_1, x_2, \cdots, x_n) \neq 0$. Let $x_i \neq 0$ and $p_i : R^n \to R$ be the ith projection. Then $p_i g(m) \neq 0$. So N is torsionless. Thus R is a left dual ring.

(2) $\Leftrightarrow$ (3) By [15, Theorem 9.4.3], a left R-module N is finitely cogenerated if and only if $E(N) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$, where $S_1, S_2, \cdots, S_n$ are simple left R-modules. So (2) $\Leftrightarrow$ (3) follows. □

Next we discuss the closure properties of AFG rings.

2.5. Theorem. *R and S are right AFG rings if and only if $R \times S$ is a right AFG ring.*

Proof. " $\Rightarrow$ " Let M be a cyclic torsionless right $(R \times S)$-module. Then M has a unique decomposition that $M = A \oplus B$, where $A = M(R, 0)$ is a right R-module and $B = M(0, S)$ is a right S-module via $xr = x(r, 0)$ for $x \in A$, $r \in R$, and $ys = y(0, s)$ for $y \in B$, $s \in S$. It is easy to verify that A is a cyclic torsionless right R-module and B is a cyclic torsionless right S-module. Thus A is a finitely presented right R-module and B is a finitely presented right S-module by hypothesis. So there exist two exact sequences $P_1 \to P_0 \to A \to 0$ of right R-modules and $Q_1 \to Q_0 \to B \to 0$ of right S-modules, where each P_i is a finitely generated projective right R-module, and each Q_i is a finitely generated projective right S-module.

Regarding the above exact sequences as exact sequences of right $(R \times S)$-modules, we have an exact sequence of right $(R \times S)$-modules

$$P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to A \oplus B \to 0.$$

Note that each $P_i \oplus Q_i$ is a finitely generated projective right $(R \times S)$-module. So $M = A \oplus B$ is a finitely presented right $(R \times S)$-module. Thus $R \times S$ is a right AFG ring.

" $\Leftarrow$ " Let M be a cyclic torsionless right R-module. Note that M may be regarded as a cyclic torsionless right $(R \times S)$-module, so M is a finitely presented right $(R \times S)$-module by hypothesis. Thus there exists an exact sequence $P_1 \to P_0 \to M \to 0$ of right $(R \times S)$-modules, where each P_i is a finitely generated projective right $(R \times S)$-module. Let $P_i = A_i \oplus B_i$, where A_i is a right R-module and B_i is a right S-module, $i = 0, 1$. Then we have the exact sequence $A_1 \to A_0 \to M \to 0$ of right R-modules. Note that each A_i is a finitely generated projective right $(R \times S)$-module, and so is a finitely generated projective right R-module, whence M is a finitely presented right R-module. Thus R is a right AFG ring. Similarly S is a right AFG ring. □

2.6. Proposition. *Let R be a right AFG ring and I be an ideal which is a right annihilator in R. Then R/I is also a right AFG ring.*

Proof. Let $M_{R/I}$ be a cyclic torsionless right R/I-module. Then M_R is clearly a cyclic right R-module. Note that R/I is a torsionless right R-module since I is a right annihilator in R. Thus M_R is also a torsionless right R-module. So M_R is a finitely presented right R-module, i.e., there is an exact sequence of right R-modules

$$R^n \to R^m \to M_R \to 0.$$

Then we get the exact sequence of right R/I-modules

$$R^n \otimes_R R/I \to R^m \otimes_R R/I \to M \otimes_R R/I \to 0,$$

which yields the exact sequence of right R/I-modules

$$(R/I)^n \to (R/I)^m \to \ M_{R/I} \to 0.$$

Hence $M_{R/I}$ is a finitely presented right R/I-module. It follows that R/I is a right AFG ring. □

2.7. Theorem. *Let R be a commutative AFG ring. If S is a multiplicative subset of R without zero-divisors, then $S^{-1}R$ is also an AFG ring.*

Proof. Let M be a cyclic $S^{-1}R$-module. Then there exists a cyclic R-submodule N of M such that $S^{-1}N = M$. Since S contains no zero-divisors, we get the exact sequence of R-modules

$$0 \to R \to S^{-1}R \to S^{-1}R/R \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_R(N, R) \to \mathrm{Hom}_R(N, S^{-1}R) \to \mathrm{Hom}_R(N, S^{-1}R/R).$$

On the other hand, there exists an exact sequence $R \to N \to 0$, which induces the exact sequence

$$0 \to \mathrm{Hom}_R(N, S^{-1}R/R) \to \mathrm{Hom}_R(R, S^{-1}R/R) \cong S^{-1}R/R.$$

Since $S^{-1}(S^{-1}R/R) = 0$, we have $S^{-1}(\mathrm{Hom}_R(N, S^{-1}R/R)) = 0$. Thus

$$\mathrm{Hom}_{S^{-1}R}(M, S^{-1}R) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}R \otimes_R N, S^{-1}R)$$

$$\cong \mathrm{Hom}_R(N, S^{-1}R) \cong S^{-1}\mathrm{Hom}_R(N, S^{-1}R) \cong S^{-1}\mathrm{Hom}_R(N, R).$$

Since $\mathrm{Hom}_R(N, R)$ is a finitely generated R-module by [19, Theorem 2.3], we have $\mathrm{Hom}_{S^{-1}R}(M, S^{-1}R)$ is a finitely generated $S^{-1}R$-module. So R/I is an AFG ring by [19, Theorem 2.3] again. □

At the end of this section, we consider several rings related to AFG rings.

Recall that R is said to be a *right AC ring* [18] if the right annihilator of each nonempty subset of R is a cyclic right ideal. R is called a *right* Π-*coherent ring* [4] in case every finitely generated torsionless right R-module is finitely presented. R is called a *right coherent ring* [5] if every finitely generated right ideal is finitely presented. R is called a *right pseudo-coherent ring* [3] if the right annihilator of each finite subset of R is a finitely generated right ideal.

Obviously, we have the following implications:

$$\begin{array}{ccccc} & & \text{right } \Pi-\text{coherent rings} & \Longrightarrow & \text{right coherent rings} \\ & & \Downarrow & & \Downarrow \\ \text{right } AC \text{ rings} & \Longrightarrow & \text{right } AFG \text{ rings} & \Longrightarrow & \text{right pseudo}-\text{coherent rings.} \end{array}$$

But these are not generally reversible as shown by the following examples.

2.8. Example. Let F be a field with an isomorphism $x \mapsto \bar{x}$ from F to a subfield $\bar{F} \neq F$. Let R denote the right F-space on a basis $\{1,\ c\}$ where $c^2 = 0$ and $cx = \bar{x}c$ for all $x \in F$. Then by [3, Example] or [28, Example 2.7], R is right Artinian, and so is right AFG. But R is not right AC. Otherwise, suppose that R is a right AC ring. Let $t \neq 0$ be an element of the Jacobson radical $J = Rc = Fc$, then $J \subseteq r(t) \neq R$. Since R is local, $J = r(t)$. Thus $J = aR$ and so $a = bc$ for some $b \in R$. Note that b is a unit since $b \notin J$. So $cR = b^{-1}aR = b^{-1}J = J = Fc$. But $cR = \bar{F}c$, and so $\bar{F}c = Fc$, which contradicts the fact that $\bar{F} \neq F$.

In fact, we have the following result.

2.9. Proposition. *R is a right AC ring if and only if R is a right AFG ring and $rl(S)$ is a cyclic right ideal for any finite subset S of R.*

Proof. " $\Leftarrow$ " Let $r(T)$ be a right annihilator in R for $T \subseteq R$. Then $r(T) = a_1R + a_2R + \cdots + a_nR$. By [1, Proposition 2.15], we have

$$r(T) = rl(r(T)) = rl\{a_1, a_2, \cdots, a_n\}$$

is a cyclic right ideal of R. So R is a right AC ring.

" $\Rightarrow$ " is trivial. □

2.10. Example. Let F be a field and R the subring of $F^{\mathbb{N}}$ consisting of "sequences" $(a_1, a_2, \cdots) \in F^{\mathbb{N}}$ that are eventually constant. Then R is a commutative von Neumann regular ring (see [16, Example 7.54]) and so is pseudo-coherent.

Let $e_i \in R$ denote the i^{th} unit vector $(0, \cdots, 1, 0, \cdots)$ and $S = \{e_1, e_3, e_5, \cdots\}$. Then $r(S)$ consists of sequences $(a_1, a_2, \cdots)$ that are eventually zero and such that $a_n = 0$ for n odd. Clearly, $r(S)$ is not a finitely generated ideal of R. Thus R is not an AFG ring.

Björk proved that R is a right AFG ring if R is a right pseudo-coherent left perfect ring (see [3, Proposition 4.3]).

2.11. Example. Let $x, y_1, y_2, \cdots$ be indeterminates over a field K, $S = K[x, y_i]$ and $R = K[x^2, x^3, y_i, xy_i]$. Then R is a subring of the commutative domain S. Hence R is also a commutative domain, and so is an AFG ring. But R is not a Π-coherent ring (see [12, p.110]).

It is known that R is a right Π-coherent ring if and only if every $n \times n$ matrix ring $\mathbb{M}_n(R)$ $(n \geq 1)$ is a right AFG ring (see [20, Corollary 2.5]). Although being right Π-coherent ring is Morita invariant, it is false for right AFG rings.

3. CTF rings

In [27], Xue introduced the concept of right CTF rings. He called a ring R *right CTF* if every cyclic torsionless right R-module embeds in a free module. This concept is a generalization of right $FGTF$ rings introduced by Faith [11]. Recall that a ring R is *right $FGTF$* if every finitely generated torsionless right R-module embeds in a free module.

3.1. Lemma. *The following are true:*

(1) *R is a right CTF ring if and only if every right annihilator in R is a right annihilator of a finite subset of R.*
(2) *A ring R is right $FGTF$ if and only if every $n \times n$ matrix ring $M_n(R)$ is right CTF for every $n \geq 1$.*

Proof. (1) " $\Rightarrow$ " Let I be a right annihilator in R. Then there is a monomorphism $f : R/I \to R^n, n \in \mathbb{N}$. Put $f(\overline{1}) = (a_1, a_2, \cdots, a_n)$. It is easy to check that $I = r\{a_1, a_2, \cdots, a_n\}$.

" $\Leftarrow$ " Let I be a right annihilator in R. Then $I = r\{b_1, b_2, \cdots, b_n\}$ by hypothesis. Define $g : R/I \to R^n$ by

$$g(\overline{r}) = (b_1 r, b_2 r, \cdots, b_n r).$$

It is easy to verify that g is a monomorphism. So R is a right CTF ring.

(2) follows from (1) and [11, Theorem 1.1]. □

3.2. Remark. (1) Although being right $FGTF$ is Morita invariant, being right CTF is not Morita invariant by Lemma 3.1(2).

(2) If R has the a.c.c. on left annihilators, then R is a right CTF ring by Lemma 3.1(1) and [10, Corollary 2].

(3) Clearly, any right CF ring is right CTF. But the converse is not true in general.

3.3. Example. Let k be a division ring and V_k be a right k-vector space of infinite dimension. Let $R = \mathrm{End}(V_k)$. Then R is a right self-injective von Neumann regular ring but not semisimple Artinian (see [16, Example 3.74B]). Note that R is a Baer ring, so R is a right CTF ring. Clearly R is not a right CF ring.

In fact, we have the following easy observation.

3.4. Proposition. *R is a right CF ring if and only if R is a right CTF right dual ring.*

Recall that a left R-module M is *H-finitely generated* [7] if there is a finitely generated submodule N of M such that $(M/N)^* = 0$.

R is called a *left f-injective ring* if $\mathrm{Ext}^1_R(R/I, R) = 0$ for any finitely generated left ideal I.

3.5. Theorem. *If the dual module of every cyclic torsionless right R-module is H-finitely generated, then R is a right CTF ring. The converse holds if R is a left f-injective ring.*

Proof. Let M be a cyclic torsionless right R-module. Then there exists a finitely generated submodule N of M^* such that $(M^*/N)^* = 0$ by hypothesis.

Let $N = Rf_1 + Rf_2 + \cdots + Rf_n$. Define $\alpha : M \to R^n$ by

$$\alpha(x) = (f_1(x), f_2(x), \cdots, f_n(x)), x \in M.$$

We next prove that α is a monomorphism.

Let $\alpha(x) = 0$, define $\beta : M^*/N \to R$ by

$$\beta(\overline{g}) = g(x), g \in M^*.$$

It is easy to check that β is well defined, and so $\beta = 0$. Thus $x \in \bigcap_{g \in M^*} \ker(g)$. Since M is torsionless, we have $x = 0$. So α is a monomorphism and hence R is a right CTF ring.

Conversely, suppose that R is a right CTF ring and R is left f-injective. For any cyclic torsionless right R-module M, there exists an exact sequence $0 \to M \xrightarrow{\gamma} R^n \to L \to 0$. Let $\pi_i : R^n \to R$ be the ith projection, $\varphi_i = \pi_i \gamma \in M^*$ and $N = R\varphi_1 + R\varphi_2 + \cdots + R\varphi_n$. We claim that $(M^*/N)^* = 0$. Otherwise, if there exists $0 \neq \xi \in (M^*/N)^*$, then there exists $\theta \in M^*$ such that $\xi(\overline{\theta}) \neq 0$. Write $\lambda : N \to R\theta + N$ and $\iota : R\theta + N \to M^*$ to be the inclusions. Since M is cyclic, there is an exact sequence $R \xrightarrow{\rho} M \to 0$, which induces the exact sequence $0 \to M^* \overset{\rho^*}{\to} R^*$. Since R is a left f-injective ring, the exact sequence $0 \to N \overset{\rho^*\iota\lambda}{\to} R^*$ induces the exact sequence $R^{**} \overset{\lambda^*\iota^*\rho^{**}}{\to} N^* \to 0$. Thus $\lambda^*\iota^*\sigma_M\rho = \lambda^*\iota^*\rho^{**}\sigma_R$ is epic, and so $\lambda^*\iota^*\sigma_M$ is epic. We next show that $\lambda^*\iota^*\sigma_M$ is also monic. In fact, if $\lambda^*\iota^*\sigma_M(x) = 0$, then $\sigma_M(x)\iota\lambda = 0$, and so $\sigma_M(x)\iota\lambda(\varphi_i) = 0, i = 1, 2, \cdots, n$. Thus $\varphi_i(x) = 0$, and so $\gamma(x) = 0$. Since γ is monic, $x = 0$. Hence $\lambda^*\iota^*\sigma_M$ is an isomorphism.

Similarly, the exact sequence $0 \to R\theta + N \overset{\rho^*\iota}{\to} R^*$ induces the exact sequence $R^{**} \overset{\iota^*\rho^{**}}{\to} (R\theta + N)^* \to 0$. Then $\iota^*\sigma_M\rho = \iota^*\rho^{**}\sigma_R$ is an epimorphism. So $\iota^*\sigma_M$ is an epimorphism. Also $\iota^*\sigma_M$ is a monomorphism. Thus $\iota^*\sigma_M$ is an isomorphism. Hence $\lambda^* : (R\theta + N)^* \to N^*$ is an isomorphism. Note that the exact sequence

$$0 \to N \overset{\lambda}{\to} R\theta + N \to (R\theta + N)/N \to 0$$

induces the exact sequence

$$0 \to ((R\theta + N)/N)^* \to (R\theta + N)^* \overset{\lambda^*}{\to} N^*.$$

So $((R\theta + N)/N)^* = 0$. But $\xi|_{(R\theta+N)/N} \neq 0$, a contradiction. Thus $(M^*/N)^* = 0$. Therefore M^* is H-finitely generated. □

3.6. Corollary. *R is a quasi-Frobenius ring if and only if R is a two-sided dual ring and the dual module of every cyclic right R-module is H-finitely generated.*

Proof. It follows from Theorem 3.5 and [13, Theorem 2.1]. □

Next we consider the relationships between AFG rings and CTF rings.

3.7. Lemma. *The following are true:*

(1) *If R is a left AFG ring, then R is a right CTF ring.*
(2) *If R is a right CTF right pseudo-coherent ring, then R is a right AFG ring.*

Proof. (1) By Theorem 2.1, the dual module of every cyclic torsionless right R-module is finitely generated and so is H-finitely generated. Thus R is a right CTF ring by Theorem 3.5.

(2) is clear by Lemma 3.1(1). □

In general, a right or left CTF ring need not be a left AFG ring.

3.8. Example. Let K be a field with a subfield L such that $\dim_L K = \infty$, and there exists a field isomorphism $\varphi : K \to L$ (for instance, $K = \mathbb{Q}(x_1, x_2, x_3, \cdots), L = \mathbb{Q}(x_2, x_3, \cdots)$). Let $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'), x, y, x', y' \in K.$$

Then it is easy to see that R has exactly three right ideals: 0, R and $(0, K)$. Therefore R has the a.c.c and the d.c.c on right annihilators and so has the a.c.c. on left annihilators. Thus R is a two-sided CTF ring by Remark 3.2(2).

On the other hand, let $a = (0, 1) \in R$. Then $l(a)$ is not finitely generated (see [16, Example 4.46 (e)]). Thus R is not a left AFG ring.

However we have the following result.

3.9. Proposition. *Let R be a two-sided pseudo-coherent ring. Then the following are equivalent:*

(1) *R is a left AFG ring.*
(2) *R is a right AFG ring.*
(3) *R is a left CTF ring.*
(4) *R is a right CTF ring.*

Proof. (1) ⇒ (4) and (2) ⇒ (3) follow from Lemma 3.7(1).
(4) ⇒ (2) and (3) ⇒ (1) hold by Lemma 3.7(2). □

3.10. Corollary. *The following are true for a ring R:*

(1) *R is a two-sided AFG ring if and only if R is a two-sided CTF two-sided pseudo-coherent ring.*
(2) *R is a two-sided Π-coherent ring if and only if R is a two-sided $FGTF$ two-sided coherent ring.*

Proof. (1) is an immediate consequence of Proposition 3.9.
(2) follows from (1), Lemma 3.1(2) and [20, Corollary 2.5]. □

Recall that R is a *right FP-injective ring* if R_R is an FP-injective right R-module. Clearly, any right FP-injective ring is right singly injective.

3.11. Proposition. *The following are true:*

(1) *R is a left AFG ring if and only if R is a right CTF ring and $lr(S)$ is a finitely generated left ideal for any finite subset S of R.*
(2) *A right singly injective ring R is left AFG if and only if R is right CTF.*
(3) [27, Corollary 3.4] *A right FP-injective ring R is left Π-coherent if and only if R is right $FGTF$.*

Proof. (1) By Lemma 3.7(1), it is enough to show the sufficiency.
Let $l(T)$ be a left annihilator in R for $T \subseteq R$. By Lemma 3.1(1), $rl(T) = r(S)$ for a finite subset S of R. So by [1, Proposition 2.15], $l(T) = lrl(T) = lr(S)$ is a finitely generated left ideal. Hence R is a left AFG ring.
(2) For any finite subset $S = \{r_1, r_2, \cdots, r_n\}$ of R, $Rr_1 + Rr_2 + \cdots + Rr_n = l(T)$ for some $T \subseteq R$ by [22, Proposition 2.8] since R is a right singly injective ring. So

$$lr(S) = lr(Rr_1 + Rr_2 + \cdots + Rr_n) = lrl(T) = l(T)$$

is a finitely generated left ideal. Thus the result holds by (1).
(3) By [23, Theorem 5.41 and Corollary 5.42], R is a right FP-injective ring if and only if every $n \times n$ matrix ring $M_n(R)$ is right singly injective for every $n \geq 1$. So (3) follows from (2), Lemma 3.1(2) and [20, Corollary 2.5]. □

3.12. Corollary. *The following are equivalent for a ring R:*

(1) *R is a two-sided AFG two-sided singly injective ring.*
(2) *R is a two-sided AFG two-sided FP-injective ring.*
(3) *R is a two-sided CTF two-sided FP-injective ring.*

Proof. (1) ⇒ (2) We first prove that R is a right coherent ring. Let I and J be two finitely generated right ideals of R. Then $I = r(X)$ and $J = r(Y)$ for some finitely generated left ideals X and Y of R by [22, Proposition 2.8] and Proposition 3.11. Thus $I \cap J = r(X + Y)$ is finitely generated. Also $r(a)$ is finitely generated for any $a \in R$. So R is a right coherent ring by [5, Theorem 2.2].
On the other hand, $l(I \cap J) = l(r(X) \cap r(Y)) = l(r(X + Y)) = X + Y = l(I) + l(J)$. Thus R is a right f-injective ring by [14, Theorem 1]. So R is a right FP-injective ring by [25, Lemma 3.1]. Similarly, R is a left FP-injective ring.
(2) ⇒ (3) ⇒ (1) follow from Proposition 3.11. □

ACKNOWLEDGEMENTS

This research was supported by NSFC (No. 11371187, 11171149), Jiangsu 333 Project, Jiangsu Six Major Talents Peak Project. The author would like to thank the referee for the very helpful comments and suggestions.

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