Classical completely prime submodules

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Abstract

We define and characterize classical completely prime submodules which are a generalization of both completely prime ideals in rings and reduced modules (as defined by Lee and Zhou in [18]). A comparison of these submodules with other "prime" submodules in literature is done. If $\text{Rad}(M)$ is the Jacobson radical of M and $\beta_{cl}^c(M)$ the classical completely prime radical of M, we show that for modules over left Artinian rings R, $\text{Rad}(M) \subseteq \beta_{cl}^c(M)$ and $\text{Rad}(_R R) = \beta_{cl}^c(_R R)$.

1. Introduction

All modules are left modules, the rings are associative but not necessarily unital. An ideal $\mathcal{P}$ of a ring R is completely prime (completely semiprime) if for any $a, b \in R$ ($a \in R$) such that $ab \in \mathcal{P}$ ($a^2 \in \mathcal{P}$) we have, $a \in \mathcal{P}$ or $b \in \mathcal{P}$ ($a \in \mathcal{P}$). A ring R is completely prime if the zero ideal is completely prime. A ring R is completely semiprime (or reduced) if and only if for all $a \in R$, $a^2 = 0 \Rightarrow a = 0$. An R-module M is reduced if for all $a \in R$ and every $m \in M$, $am = 0$ implies $\langle m \rangle \cap aM = 0$, where $\langle m \rangle = \mathbb{Z}m + Rm$ is the submodule of M generated by $m \in M$. It is worth noting that, if R is unital then $\langle m \rangle = Rm$, otherwise $Rm \subseteq \langle m \rangle$ but $\langle m \rangle \not\subseteq Rm$ in general. By $(P : N)$ (resp. $(P : m)$) where P, N are submodules of an R-module M and $m \in M$, we mean $\{r \in R \ : \ rN \subseteq P\}$ (resp. $\{r \in R \ : \ rm \in P\}$). If a is an element of a ring R, by $\langle a \rangle$ we denote the ideal of R generated by a. We write $N \leq M$ to mean N is a submodule of M. Our definition of a

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reduced module is a generalization of that in [18], where Rm is used in the place of $\langle m \rangle$. We state an equivalent but more handy definition for a reduced module.

1.1. Definition. An R-module M is reduced if for all $a \in R$ and every $m \in M$, $a^2 m = 0$ implies $a\langle m \rangle = 0$.

This definition of a reduced (completely semiprime) module motivates the following two definitions:

1.2. Definition. A proper submodule P of an R-module M is completely semiprime if for all $a \in R$ and every $m \in M$, $a^2 m \in P$ implies $a\langle m \rangle \subseteq P$.

1.3. Definition. A proper submodule P of an R-module M is classical completely prime if for all $a, b \in R$ and every $m \in M$, $abm \in P$ implies $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$.

An R-module M/P is a classical completely prime module if and only if P is a classical completely prime submodule of M. Thus, an R-module M is classical completely prime (completely semiprime) if and only if the zero submodule is a classical completely prime (completely semiprime) submodule of M. Although the phrase "completely prime" would seem suitable in the place of classical completely prime in Definition 1.3, we reserve it for a different meaning - one given by Tuganbaev in [24, p.1480] and discussed in [21] (in which it is most suitable).

1.1. Example. A free module M over a domain R is classical completely prime.

Proof. Suppose $abm = 0$ for some $a, b \in R$ and $m \in M$. Then

$$abm = ab \sum_{i=1}^{n} (r_i m_i) = \sum_{i=1}^{n} (abr_i) m_i = 0$$

for some $r_i \in R$ and $m_i \in M$. Since M is free $abr_i = 0$ for all $i \in \{1, \cdots n\}$. For $m \neq 0$, there is atleast one $j \in \{1, \cdots n\}$ such that $r_j \neq 0$. Now $abr_j = 0$ implies $a = 0$ or $b = 0$ (since R is a domain) such that $a\langle m \rangle = 0$ or $b\langle m \rangle = 0$. □

1.2. Example. A torsionfree module M over a domain R is classical completely prime. It follows that flat modules over domains (and hence projective modules over domains) are classical completely prime modules.

Proof. Suppose for $a, b \in R$ and $m \in M$, $abm = 0$. If $m = 0$, $a\langle m \rangle = 0$ and $b\langle m \rangle = 0$. Let $m \neq 0$, then $ab = 0$ since M is torsionfree. Hence, $a = 0$ or $b = 0$ since R is a domain. Therefore, $a\langle m \rangle = 0$ or $b\langle m \rangle = 0$. The last part is due to the fact that flat modules are torsionfree, see [23, Example 1, p.15] and projective modules are flat modules. □

1.3. Example. Every submodule P of a module M over a division ring R is a classical completely prime submodule.

Proof. Suppose $a, b \in R$ and $m \in M$ such that $abm \in P$. If $ab = 0$, $a = 0$ or $b = 0$ such that $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$. Suppose $ab \neq 0$, then, $m \in (ab)^{-1} P \subseteq P$.§ Thus, $a\langle m \rangle \subseteq P$ and $b\langle m \rangle \subseteq P$. □

1.4. Example. Any prime (sub)module over a commutative ring is a classical completely prime (sub)module.

Proposition 1.1 below and its corollaries provide more justification for our definition of classical completely prime submodules.

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§$(ab)^{-1}$ is here used to mean the inverse of ab in R

1.1. Proposition. If $1 \in R$ and $P \lhd R$, then P is a completely prime ideal of R if and only if P is a classical completely prime submodule of $_R R$.

Proof. Suppose P is a completely prime ideal of R and for any $a, b \in R$ and $m \in_R R$, $abm \in P$. By definition of a completely prime ideal, $a \in P$ or $b \in P$ or $m \in P$. Thus, $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$. Conversely, suppose the ideal P of R is a classical completely prime submodule of $_R R$. Let for any $a, b \in R$, $ab \in P$. Since $1 \in R$ by definition of classical completely prime submodule, $ab.1 \in P$ implies $aR \in P$ or $bR \in P$ such that $a \in P$ or $b \in P$. $\square$

1.1. Corollary. If $1 \in R$, then R is a domain if and only if $_R R$ is a classical completely prime module.

1.2. Corollary. If $1 \in R$ and $P \lhd R$, then P is a completely semiprime ideal of R if and only if it is a completely semiprime submodule of $_R R$.

1.3. Corollary. A unital ring R is reduced if and only if $_R R$ is a reduced module.

2. Investigation of properties

In this section, we investigate properties exhibited by classical completely prime (semiprime) submodules. First, we introduce notions of symmetric and IFP submodules that will prove useful later in the sequel. Lambek in [17, p.364] called a module M symmetric if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$. We call a submodule P of an R-module M symmetric if $abm \in P$ implies $bam \in P$ for $a, b \in R$ and $m \in M$. So, a module M is symmetric if its zero submodule is symmetric. From [8], a right (or left) ideal I of a ring R is said to have the insertion-of-factor-property (IFP) if whenever $ab \in I$ for $a, b \in R$, we have $aRb \subseteq I$. We call a submodule N of an R-module M an IFP submodule if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteq N$. A module is IFP if its zero submodule is IFP.

2.1. Proposition. For any submodule P of an R-module M,

$$\text{completely semiprime} \Rightarrow \text{symmetric} \Rightarrow \text{IFP.}$$

Proof. Let $abm \in P$. $(bab)^2 m \in P$ and P completely semiprime gives $bab\langle m \rangle \subseteq P$. Thus, $(ba)^2 m = bab(am) \in bab\langle m \rangle \subseteq P$ and again P completely semiprime gives $bam \in ba\langle m \rangle \subseteq P$. For the second implication, let $am \in P$ for $a \in R$ and $m \in M$. Then $Ram \subseteq P$ and P symmetric implies $aRm \subseteq P$. $\square$

2.1. Example. A module M over a left duo ring R (a ring whose all left ideals are two sided) is fully IFP (every submodule of M is IFP) but it need not be symmetric.

Proof. Let $P \leq M$, $a \in R$ and $m \in M$ such that $am \in P$, then $a \in (P : m)$. $(P : m)$ is a left ideal of R but since R is left duo, we have $(P : m) \lhd R$ and $aR \subseteq (P : m)$ such that $aRm \subseteq P$. Hence, P is IFP. $\mathbb{Z}_2$ is a left quasi duo ring (i.e., every maximal left ideal of $\mathbb{Z}_2$ is two sided). By [16, Prop. 2.1], any n-by-n upper triangular matrix ring R over $\mathbb{Z}_2$ is left quasi duo. Hence, every submodule of the module $_R R$ is IFP. We show that the zero submodule of $_R R$ is not symmetric. Take $m = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$, $a = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}$, and $b = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \in R$; $abm = 0$ but $bam \neq 0$. $\square$

2.2. Example. A submodule P of a module M over a commutative ring R is symmetric but it need not be completely semiprime.

2.1. Properties of underlying ring. We give information classical completely prime (completely semiprime) submodules of $_RM$ reveal about the underlying ring R. Propositions 2.2 and 2.3 indicate that there is a one to one correspondence between completely semiprime (classical completely prime) submodules P of the module $_RM$ and completely semiprime (completely prime) ideals of R of the form $(P : m)$ for all $m \in M \setminus P$.

2.2. Proposition. For $P \leq {}_RM$, the following statements are equivalent:

    (1) P is a completely semiprime submodule of M;

    (2) $(P : m) = (P : \langle m \rangle) = (\bar{0} : \bar{m})$ is a completely semiprime ideal of R for every $m \in M \setminus P$, where $\bar{m} = m + P$;

    (3) for all $m \in M \setminus P$, $(P : m) \triangleleft R$ and for all $a \in R$ if $a^2 m \in P$, then $am \in P$;

    (4) for all $m \in M \setminus P$, $(P : m) \triangleleft R$ and for all $a \in R$ if $\langle a^2 m \rangle \subseteq P$, then $\langle am \rangle \subseteq P$;

    (5) for all $a \in R$ and every $m \in M$, if $\langle a^2 m \rangle \subseteq P$, then $\langle a \langle m \rangle \rangle \subseteq P$.

Proof. $(1) \Rightarrow (2)$. Since $(P : m)$ is always a left ideal of R for all $m \in M \setminus P$, we show that if $a \in (P : m)$, then $aR \subseteq (P : m)$. Suppose $a \in (P : m)$, then $Ram \subseteq P$ and from Proposition 2.1, we have $aRm \subseteq P$ and therefore, $aR \subseteq (P : m)$ as required. Let $m \in M \setminus P$, $(P : m) = \{r \in R : rm \in P\} = \{r \in R : r\bar{m} = \bar{0}\} = (\bar{0} : \bar{m})$. The inclusion $(P : \langle m \rangle) \subseteq (P : m)$ is clear. Suppose $x \in (P : m)$, then $xR \subseteq (P : m)$. Hence, $x\langle m \rangle \subseteq P$ and we have $x \in (P : \langle m \rangle)$. Lastly, suppose $a^2 \in (P : m)$, i.e., $a^2 m \in P$. Then, $am \in a\langle m \rangle \subseteq P$ since P is a completely semiprime submodule of M. Thus, $a \in (P : m)$.

$(2) \Rightarrow (1)$. Let for all $a \in R$ and $m \in M$, $a^2 m \in P$. Then, $a^2 \in (P : m)$ which implies $a \in (P : m)$ by definition of a completely semiprime ideal of a ring R. Thus, $aR \subseteq (P : m)$ and $aRm \subseteq P$. Therefore, $a\langle m \rangle = \mathbb{Z}am + aRm \subseteq P$ and P is a completely semiprime submodule of M.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$ and $(5) \Leftrightarrow (1)$ are trivial. $\qquad\square$

2.1. Corollary. An R-module M is reduced if and only if for every $0 \neq m \in M$, $(0 : m)$ is a completely semiprime two sided-ideal of R.

2.3. Proposition. For a proper submodule P of an R-module M, the following statements are equivalent:

    (1) P is a classical completely prime submodule of M;

    (2) for every $m \in M \setminus P$, $(P : m) = (P : \langle m \rangle) = (\bar{0} : \bar{m})$ is a completely prime ideal of R;

    (3) for all $m \in M \setminus P$, $(P : m) \triangleleft R$ and if $a, b \in R$ such that $abm \in P$, then $am \in P$ or $bm \in P$;

    (4) for all $m \in M \setminus P$, $(P : m) \triangleleft R$ and if $a, b \in R$ such that $\langle abm \rangle \subseteq P$, then $\langle am \rangle \subseteq P$ or $\langle bm \rangle \subseteq P$;

    (5) for all $a, b \in R$ and every $m \in M$, if $\langle abm \rangle \subseteq P$, then $\langle a \langle m \rangle \rangle \subseteq P$ or $\langle b \langle m \rangle \rangle \subseteq P$.

Proof. $(1) \Rightarrow (2)$. Every classical completely prime submodule of M is a completely semiprime submodule of M. We have seen in Proposition 2.2 that $(P : m)$ is an ideal of R and $(P : m) = (P : \langle m \rangle) = (\bar{0} : \bar{m})$. Let $a, b \in R$ and $0 \neq m \in M$ such that $ab \in (P : m)$, i.e., $abm \in P$. Now, P classical completely prime submodule gives $am \in a\langle m \rangle \subseteq P$ or $bm \in b\langle m \rangle \subseteq P$. Hence, $a \in (P : m)$ or $b \in (P : m)$.

$(2) \Rightarrow (1)$. Let for $a, b \in R$ and $0 \neq m \in M$, $abm \in P$, i.e., $ab \in (P : m)$. $(P : m)$ a completely prime ideal of R gives $a \in (P : m)$ or $b \in (P : m)$. Hence, $(am \in P$ and $aRm \subseteq P)$ or $(bm \in P$ and $bRm \subseteq P)$ such that $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$ and $(5) \Leftrightarrow (1)$ are trivial. $\qquad\square$

The zero divisor set of $_R M$ [3, p.316] is the set

$$Zd(M) := \{r \in R \ : \ \text{there exists } 0 \neq m \in M, \ \text{with } rm = 0\}.$$

The following proposition provides us with two other ways of constructing completely prime ideals of a ring R from a submodule P of an R-module M.

2.4. Proposition. Let P be a classical completely prime submodule of an R-module M. Then,

(1) for any $m, n \in M \setminus P$ either $(P : n) \subseteq (P : m)$ or $(P : m) \subseteq (P : n)$;
(2) $Zd(M/P)$ is a completely prime ideal of R;
(3) for all submodules K and L of M not contained in P, $(P : L) \subseteq (P : K)$ or $(P : K) \subseteq (P : L)$;
(4) $(P : K)$ is a completely prime ideal of R for all submodules K of M such that $K \nsubseteq P$.

Proof. (1) Assume $n, m \in M \setminus P$. Then, $(P : n)(P : m) \subseteq (P : n) \cap (P : m) \subseteq (P : n + m)$. We know that, $(P : n + m)$ is a completely prime ideal of R and hence a prime ideal of R. So, we have $(P : n) \subseteq (P : n + m)$ or $(P : m) \subseteq (P : n + m)$. If $(P : n) \subseteq (P : n + m)$, then $(P : n) = (P : n) \cap (P : n + m) \subseteq (P : m)$. Similarly, if $(P : m) \subseteq (P : n + m)$, we get $(P : m) \subseteq (P : n)$.

(2) By definition, $Zd(M/P) = \bigcup\limits_{m \in M \setminus P} (P : m)$. But $\{(P : m)\}_{m \in M \setminus P}$ form a chain of completely prime ideals of R. We see that $Zd(M/P)$ is the largest of all the $(P : m)$'s and hence a completely prime ideal of R.

(3) $(P : K)(P : L) \subseteq (P : K) \cap (P : L) \subseteq (P : K + L)$. Hence, $(P : K) \subseteq (P : K + L) \subseteq (P : L)$ or $(P : L) \subseteq (P : K + L) \subseteq (P : K)$.

(4) To show that $(P : K)$ is a completely prime ideal of R, it is enough to show that it is both prime and completely semiprime as an ideal of R. If P is classical completely prime, by Theorem 3.1 it is classical prime (see definition 3.2) and hence $(P : K)$ is a prime ideal of R for all $K \leq M$ such that $K \nsubseteq P$. Suppose $a^2 \in (P : K)$ for $a \in R$ and $K \leq M$ with $K \nsubseteq P$, then $a^2 k \in P$ for all $k \in K$. By hypothesis, $a \langle k \rangle \subseteq P$ for all $k \in K$. Thus, $aK \subseteq P$ such that $a \in (P : K)$. $\qquad\square$

2.2. Homomorphic images.

2.5. Proposition. Let M be an R-module, $N, P \leq M$ such that $N \subseteq P$. If $f : M \to M/N$ is a canonical epimorphism, then P is a classical completely prime submodule of M if and only if $f(P)$ is a classical completely prime submodule of M/N.

The proof is elementary, if $N \nsubseteq P$, P classical completely prime submodule of M does not in general imply $f(P)$ is a classical completely prime submodule of M/N (and hence classical completely prime is not in general closed under homomorphic images).

2.3. Example. The $\mathbb{Z}$-module $\mathbb{Z}$ is a classical completely prime module by Corollary 1.1 and $N = 8\mathbb{Z}$ is a submodule of $M = {}_{\mathbb{Z}}\mathbb{Z}$. By [1, Example 2.5], M/N is not a reduced module (i.e., not a completely semiprime module) and hence not a classical completely prime module.

2.6. Proposition. Let $f : R \to A$ be a ring epimorphism and M an A-module, then M is an R-module and $_A M$ is classical completely prime if and only if $_R M$ is classical completely prime.

Proof. Define a function from $_RM$ to $_AM$ by $rm = f(r)m$. This function turns M into an R-module whenever M is an $_AM$ module. Suppose $_AM$ is classical completely prime and for all $r, s \in R$ and $m \in M$, $rsm = 0$. Then, $0 = rsm = f(r)f(s)m$. Since $_AM$ is classical completely prime, $f(r)\langle m \rangle_A = 0$ or $f(s)\langle m \rangle_A = 0$. Then by structure of R-module, it follows that $r\langle m \rangle_R = 0$ or $s\langle m \rangle_R = 0$. Thus, $_RM$ is classical completely prime. Assume $_RM$ is classical completely prime and for all $a, b \in R$ and $m \in M$, $abm = 0$. Then since f is an epimorphism, there exists $r, s \in A$ such that $a = f(r)$ and $b = f(s)$, i.e., $f(r)f(s)m = rsm = 0$. By assumption, $r\langle m \rangle_R = 0$ or $s\langle m \rangle_R = 0$. If $r\langle m \rangle_R = 0$ (resp. $s\langle m \rangle_R = 0$), the fact that f is onto leads to $a\langle m \rangle_A = 0$ (resp. $b\langle m \rangle_A = 0$). Hence, $_AM$ is classical completely prime. $\square$

2.3. Properties of submodules and direct summands.

2.7. Proposition. If M is a classical completely prime module, then any submodule N of M is also a classical completely prime module.

Proof. Elementary. $\square$

2.8. Proposition. For an R-module M, the following statements are equivalent:
   (1) M is a classical completely prime module,
   (2) Each direct summand of M is a classical completely prime submodule of M.

Proof. (1) $\Rightarrow$ (2). By Proposition 2.7 any submodule N of M is a classical completely prime module. If $M = K \oplus P$ where K and P are submodules, then M/K is isomorphic to P which is a classical completely prime module and so K is a classical completely prime submodule.

(2) $\Rightarrow$ (1). If each direct summand of M is a classical completely prime submodule, then so is the zero submodule and hence M is a classical completely prime module. $\square$

2.4. Classical multiplicative systems.

2.1. Definition. Let R be a ring and M an R-module. A nonempty set $S \subseteq M \setminus \{0\}$ is called a classical multiplicative system if, for all $a, b \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m \rangle) \cap S \neq \emptyset$ and $(K + b\langle m \rangle) \cap S \neq \emptyset$, then $(K + abm) \cap S \neq \emptyset$.

2.9. Proposition. Let M be an R-module. A proper submodule P of M is classical completely prime if and only if its complement $M \backslash P$ is a classical multiplicative system.

Proof. Suppose $S := M \backslash P$. Let $a, b \in R$, $m \in M$ and K be a submodule of M such that $(K + a\langle m \rangle) \cap S \neq \emptyset$ and $(K + b\langle m \rangle) \cap S \neq \emptyset$. If $(K + \{abm\}) \cap S = \emptyset$, then $abm \in P$. Since P is classical completely prime , $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$. It follows that $(K + a\langle m \rangle) \cap S = \emptyset$ or $(K + b\langle m \rangle) \cap S = \emptyset$, a contradiction. Therefore, S is a classical multiplicative system in M. For the converse, let $S := M \backslash P$ be a classical multiplicative system in M. Suppose for $a, b \in R$ and $m \in M$, $abm \in P$. If $a\langle m \rangle \not\subseteq P$ and $b\langle m \rangle \not\subseteq P$, then $a\langle m \rangle \cap S \neq \emptyset$ and $b\langle m \rangle \cap S \neq \emptyset$. Thus, $abm \in S$, a contradiction. Therefore, P is a classical completely prime submodule of M. $\square$

2.10. Proposition. Let M be an R-module, P be a proper submodule of M, and $S := M \backslash P$. Then, the following statements are equivalent:
   (1) P is a classical completely prime submodule of M;
   (2) S is a classical multiplicative system of M;
   (3) for all $a, b \in R$ and $m \in M$, if $a\langle m \rangle \cap S \neq \emptyset$ and $b\langle m \rangle \cap S \neq \emptyset$, then $abm \in S$;
   (4) for all $a, b \in R$ and $m \in M$, if $\langle a\langle m \rangle \rangle \cap S \neq \emptyset$ and $\langle b\langle m \rangle \rangle \cap S \neq \emptyset$, then $\langle abm \rangle \cap S \neq \emptyset$.

2.1. Lemma. Let M be an R-module, $S \subseteq M$ a classical multiplicative system of M and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is a classical completely prime submodule of M.

Proof. Suppose $a \in R$ and $m \in M$ such that $\langle abm \rangle \subseteq P$. If $\langle a \langle m \rangle \rangle \nsubseteq P$ and $\langle b \langle m \rangle \rangle \nsubseteq P$ then $(\langle a \langle m \rangle \rangle + P) \cap S \neq \emptyset$ and $(\langle b \langle m \rangle \rangle + P) \cap S \neq \emptyset$. By definition of a classical multiplicative system S of M, $(\langle abm \rangle + P) \cap S \neq \emptyset$. Since $\langle abm \rangle \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be a classical completely prime submodule. $\square$

2.2. Definition. Let R be a ring and M an R-module. For $N \leq M$, if there is a classical completely prime submodule of M containing N, we define clc.$\sqrt{N} := \{m \in M$: every classical multiplicative system containing m meets $N\}$. We write clc.$\sqrt{N} = M$ when there are no classical completely prime submodules of M containing N.

2.1. Theorem. Let M be an R-module and $N \leq M$. Then, either clc.$\sqrt{N} = M$ or clc.$\sqrt{N}$ equals the intersection of all classical completely prime submodules of M containing N, which is denoted by $\beta_{cl}^c(N)$.

Proof. Suppose clc.$\sqrt{N} \neq M$. Both clc.$\sqrt{N}$ and N are contained in the same classical completely prime submodules. By definition of clc.$\sqrt{N}$ it is clear that $N \subseteq$ clc.$\sqrt{N}$. Hence, any classical completely prime submodule of M which contains clc.$\sqrt{N}$ must necessarily contain N. Suppose P is a classical completely prime submodule of M such that $N \subseteq P$, and let $t \in$ clc.$\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a classical multiplicative system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence clc.$\sqrt{N} \subseteq P$ as we wished to show. Thus, clc.$\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume $s \notin$ clc.$\sqrt{N}$, then there exists a classical multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's Lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 2.1, P is a classical completely prime submodule of M and $s \notin P$. $\square$

2.5. Complete systems.

2.3. Definition. Let R be a ring and M an R-module. A nonempty set $T \subseteq M \setminus \{0\}$ is called a complete system if, for all $a \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m \rangle) \cap T \neq \emptyset$, then $(K + \{a^2 m\}) \cap T \neq \emptyset$.

2.2. Corollary. Let M be an R-module. A proper submodule P of M is completely semiprime if and only if $M \backslash P$ is a complete system.

2.11. Proposition. Let M be an R-module, P be a proper submodule of M, and $T := M \backslash P$. Then, the following statements are equivalent:

(1) P is completely semiprime;
(2) T is a complete system;
(3) for all $a \in R$ and $m \in M$, if $a\langle m \rangle \cap T \neq \emptyset$, then $a^2 m \in T$;
(4) for all $a \in R$ and $m \in M$, if $\langle a\langle m \rangle \rangle \cap T \neq \emptyset$, then $\langle a^2 m \rangle \cap T \neq \emptyset$.

2.1. Remark. Every classical multiplicative system is a complete system but not conversely.

2.2. Question. Is every completely semiprime submodule of a module an intersection of classical completely prime submodules?

3. Comparison with "primes" in literature

In this section we compare classical completely prime (resp. completely semiprime) submodules with prime (resp. semiprime) and classical prime (resp. classical semiprime) submodules.

3.1. Definition. [2], [11] $P \leq\ _R M$ with $RM \nsubseteq P$ is prime if for any $N \leq\ _R M$ and any $\mathcal{A} \triangleleft R$ such that $\mathcal{A}N \subseteq P$, then $\mathcal{A}M \subseteq P$ or $N \subseteq P$. P is a semiprime submodule of M if for $a \in R$ and $m \in M$ such that $aRam \subseteq P$, then $am \in P$.

3.2. Definition. [4, p.338] $P \leq\ _R M$ with $RM \nsubseteq P$ is classical prime if for any $N \leq\ _R M$ and any $\mathcal{A}, \mathcal{B} \triangleleft R$ such that $\mathcal{A}\mathcal{B}N \subseteq P$, then $\mathcal{A}N \subseteq P$ or $\mathcal{B}N \subseteq P$. $P \leq_R M$ is classical semiprime if for every $\mathcal{A} \triangleleft R$, and $N \leq M$ such that $\mathcal{A}^2 N \subseteq P$, then $\mathcal{A}N \subseteq P$.

Propositions 3.1 and 3.2 are modifications of [4, Proposition 1.1] and [4, Proposition 1.2] to suit a not necessarily unital module.

3.1. Proposition. Let $P \leq\ _R M$, the following statements are equivalent:
  (1) P is a classical prime submodule of M;
  (2) for all $a, b \in R$ and every $m \in M$, if $\langle a \rangle \langle b \rangle m \subseteq P$, then $\langle a \rangle m \subseteq P$ or $\langle b \rangle m \subseteq P$;
  (3) for all $a, b \in R$ and every $m \in M$ such that $aRb\langle m \rangle \subseteq P$, then $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$.

3.2. Proposition. Let $P \leq\ _R M$, the following statements are equivalent:
  (1) P is a classical semiprime submodule of M;
  (2) for all $a \in R$ and every $m \in M$, if $\langle a \rangle^2 m \subseteq P$, then $\langle a \rangle m \subseteq P$;
  (3) for all $a \in R$ and every $m \in M$, if $aRa\langle m \rangle \subseteq P$, then $a\langle m \rangle \subseteq P$.

3.1. Remark. In literature, classical prime is used interchangeably with weakly prime, cf., [3], [4], [5], [6]. We here use classical prime instead of weakly prime. In defense of our nomenclature, weakly prime modules exist in [13] when used in a totally different context - a context which generalizes the notion of weakly prime ideals for rings to modules. To the best of our knowledge, classical prime has never been used by other authors to mean something different. Our "classical semiprime" is what is called "semiprime" in [4], our nomenclature reflects that classical semiprime is derived from classical prime. Lastly, our "semiprime" is the semiprime in [11].

3.1. Theorem. For any submodule $P \leq\ _R M$, we have the following implications:

(i) in general			(ii)	P IFP	submodule
		prime	prime		
		$\Downarrow$	$\Downarrow$		
classical	$\Rightarrow$	classical	classical	$\Leftrightarrow$	classical
completely prime		prime	completely prime		prime

Proof. (i). By [22, Prop. 4.1.11], it is known that a prime submodule is classical prime. Now we show that a classical completely prime submodule is classical prime. Let $a, b \in R$ and $m \in M$ such that $\langle a \rangle \langle b \rangle m \subseteq P$. Then, $abm \in P$ and P classical completely prime in M implies $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$. Thus, $(am \in P$ and $aRm \subseteq P)$ or $(bm \in P$ and $bRm \subseteq P)$ so that $\langle a \rangle m = (\mathbb{Z}a + Ra + aR + RaR)m \subseteq P$ or $\langle b \rangle m = (\mathbb{Z}b + Rb + bR + RbR)m \subseteq P$. Hence, P is classical prime.

(ii). Suppose a classical prime submodule P is IFP, we show that P is classical completely prime. If $a, b \in R$ and $m \in M$ such that $abm \in P$, then $aRbm \subseteq P$ and $aRb(Rm) \subseteq P$ so that $aRb\langle m \rangle \subseteq P$. This implies, either $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$ by definition of classical prime submodule. So, P is classical completely prime. $\square$

3.2. Theorem. P is a classical completely prime submodule of an R-module M if and only if P is both a classical prime and a completely semiprime submodule of M.

Proof. Every classical completely prime submodule is completely semiprime. From Theorem 3.1, classical completely prime submodules are classical prime. For the converse, assume P is both a completely semiprime and a classical prime submodule of M. Now, let $a, b \in R$ and $m \in M$ such that $abm \in P$. By Proposition 2.1, P is IFP. Hence, $aRb\langle m \rangle \subseteq P$. P classical prime implies $a\langle m \rangle \subseteq P$ or $b\langle m \rangle \subseteq P$. $\qquad \square$

3.1. Example. Every maximal submodule P of an R-module M is a classical prime submodule but there exist modules with maximal submodules which are not classical completely prime. Let $\mathcal{A} \triangleleft R$ and $N \leq M$ such that $\mathcal{A}N \subseteq P$, where P is a maximal submodule of M. If $N \subseteq P$, we are through. Suppose $N \nsubseteq P$. Then, $M = N + P$ so that $\mathcal{A}M = \mathcal{A}N + \mathcal{A}P \subseteq P$. This shows P is a prime submodule and hence a classical prime submodule. We construct a maximal submodule which is not classical completely prime. Let $R = (M_2(\mathbb{Z}), +, .)$ be a ring of all 2-by-2 matrices with integral entries and $(M_2(\mathbb{Z}_2), +)$ be a group of all 2-by-2 matrices with entries from the ring $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Then, $M_2(\mathbb{Z}_2)$ is an $M_2(\mathbb{Z})$-module and

$$P = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$$

is a maximal submodule of $M_2(\mathbb{Z}_2)$. Now, let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $m = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$. $abm = 0 \in P$ but $a\langle m \rangle \nsubseteq P$ and $b\langle m \rangle \nsubseteq P$ since $am \notin P$ and $bm \notin P$. Therefore, P is a maximal submodule of $M_2(\mathbb{Z}_2)$ but not a classical completely prime submodule of $M_2(\mathbb{Z})$-module $M_2(\mathbb{Z}_2)$.

In regard to Example 3.1, we point out that, although it is not true in general, we can find noncommutative rings for which every maximal submodule is classical completely prime. To illustrate this, we use left (quasi) duo rings. A ring R is called left (quasi) duo if every left (maximal left) ideal of R is two sided. A ring R is called left quasi-duo, if every maximal left ideal of R is two sided.

3.3. Proposition. [20, Proposition 3.6] R is a left quasi-duo ring if and only if each simple R-module M is reduced.

3.4. Proposition. If R is a left quasi-duo ring, then each maximal submodule P of M is a classical completely prime submodule of M.

Proof. Let P be a maximal submodule of M and R a left quasi-duo ring. M/P is simple and from Proposition 3.3, it is reduced. Hence, P is a completely semiprime submodule of M. Since every maximal submodule of M is classical prime, it follows from Theorem 3.2 that P is a classical completely prime submodule of M. $\qquad \square$

3.2. Remark. It is not possible to get an example like Example 3.1 for a ring R which is a collection of all upper triangular matrices over $\mathbb{Z}$. This is because, upper triangular matrix rings are left quasi-duo and from Proposition 3.4, maximal submodules are always classical completely prime.

It is clear from Example 3.1 that simple modules are not always classical completely prime. We give another example to show that simple modules are not always classical completely prime. It makes use of Lemma 3.1.

3.1. Lemma. For a simple and reduced module $_RM$, $am = 0$ implies $aM = 0$ for all $a \in R$ and $0 \neq m \in M$.

Proof. Suppose $am = 0$. Since M is simple and reduced, we have $0 = aM \cap \langle m \rangle = aM \cap M = aM$. $\square$

3.2. Example. Let $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ where entries of matrices in M are from $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and $R = M_2(\mathbb{Z})$. $_R M$ is a simple module which is not classical completely prime.

Proof. Let $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$,

$$rM = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} a & a \\ c & c \end{pmatrix}, \begin{pmatrix} b & b \\ d & d \end{pmatrix}, \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \right\} \subseteq M$$

for any $a, b, c, d \in \mathbb{Z}$. The would be nontrivial proper submodules, namely; $N_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$, $N_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ and

$N_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ are not closed under multiplication by R since, for a and c odd, $rN_1 \not\subseteq N_1$, for b and d odd, $rN_2 \not\subseteq N_2$ and for a odd but b, c, d even, $rN_3 \not\subseteq N_3$. Take $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in R$ and $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$, $am = 0$ but $aM \neq 0$ since $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq 0$. By Lemma 3.1, M is not reduced and hence not classical completely prime. $\square$

3.3. Example. If P is a classical prime submodule of an R-module M, $(P : M)$ is a prime ideal of R which is not necessarily a completely prime ideal of R. On the other hand, if P is a classical completely prime submodule of an R-module M, then $(P : M)$ is a completely prime ideal of R. This shows that a classical prime submodule need not be classical completely prime.

Since over commutative rings classical completely prime submodules and classical submodules are indistinguishable, we have:

3.4. Example. [6, Example 1] Assume that R is a unital commutative domain and $\mathcal{P}$ is a non-zero prime ideal in R. $\mathcal{P} \oplus 0$ and $0 \oplus \mathcal{P}$ are classical completely prime submodules in the free module $M = R \oplus R$, but they are not prime submodules.

4. Comparison of "semiprimes"

4.1. Theorem. For any submodule P of an R-module M,

$$\text{completely semiprime} \Rightarrow \text{semiprime} \Rightarrow \text{classical semiprime}.$$

Proof. Suppose for $a \in R$ and $m \in M$, $aRam \subseteq P$, then $(a^2)^2 m \in P$ and P completely semiprime implies $a^2 m \in a^2 \langle m \rangle \subseteq P$. Hence, $am \in a\langle m \rangle \subseteq P$ and P is semiprime. Now, suppose $aRa\langle m \rangle \subseteq P$ but $a\langle m \rangle \not\subseteq P$. Then, there exists $m_1 \in \langle m \rangle$ such that $am_1 \notin P$. By definition of semiprime submodules, $aRam_1 \not\subseteq P$ and so $aRa\langle m \rangle \not\subseteq P$ which is a contradiction. Therefore, whenever $aRa\langle m \rangle \subseteq P$, we have $a\langle m \rangle \subseteq P$ and semiprime $\Rightarrow$ classical semiprime. $\square$

The reverse implications in Theorem 4.1 are not true in general. The simple module M constructed in Examples 3.2 is semiprime (because all simple modules are prime) but it is not completely semiprime. For the second implication, a counter example was constructed by Hongan in [15, p.119].

4.1. Corollary. If P is an IFP submodule of M, then

$$\text{completely semiprime} \iff \text{semiprime} \iff \text{classical semiprime}.$$

Proof. It is enough to show that classical semiprime $\Rightarrow$ completely semiprime, the rest follows from Theorem 4.1. Let $a^2 m \in P$, where $a \in R$ and $m \in M$. For P IFP, $aRa\langle m \rangle \subseteq P$. By definition of classical semiprime, $a\langle m \rangle \subseteq P$ and P is completely semiprime. $\square$

A ring R is left (right) permutable [10, p.258], if for all $a, b, c \in R$, $abc = bac$ ($abc = acb$). R is permutable if it is both left and right permutable. Commutative rings and nilpotent rings of index ≤ 3 are left (right) permutable. A ring R is medial [10], if for all $a, b, c, d \in R$, $abcd = acbd$. A left (right) permutable ring is medial but not conversely. A unital medial ring is indistinguishable from a commutative ring. A ring R is left self distributive, denoted by LSD (resp. right self distributive, denoted by RSD) if for all $a, b, c, d \in R$, the identity: $abc = abac$ (resp. $abc = acbc$) holds. LSD rings are left permutable, see [14, Corollary 2.2]. Left (right) permutable rings and medial rings exist in abundancy; according to Birkenmeier and Heatherly in [10, p.258], they are a special type of PI-rings and also exist as special subrings of every ring. Furthermore, if R is a noncommutative medial (left permutable, right permutable or permutable) ring, then the ring of polynomials (resp. formal power series or formal Laurent series) over R is a medial (left permutable, right permutable or permutable) ring which is not commutative, see [10, p.262-263].

4.2. Theorem. If P is a classical semiprime submodule of $_R M$ and R is a medial (left permutable, right permutable or LSD) ring then each of the following statements implies P is a completely semiprime submodule of $_R M$:

(1) M is finitely generated,
(2) M is free,
(3) M is cyclic.

Proof. We prove only the case for M cyclic, the proofs for other cases are similar. Suppose $a^2 m \in P$ for $a \in R$ and $m \in M$, $R^2 a^2 m \subseteq P$. R medial implies $RaRam \subseteq P$. Since M is cyclic, $m = rm_0$ for some $r \in R$ and $m_0 \in M$. $RaRarm_0 \subseteq P$ and $R^2 aRarm_0 \subseteq P$. Again, R medial leads to $RaRaRm \subseteq P$. It follows that $RaRa\langle m \rangle \subseteq P$. Since P is classical semiprime, $Ra\langle m \rangle \subseteq P$, i.e., $Ra \subseteq (P : \langle m \rangle)$. P classical semiprime implies $(P : \langle m \rangle)$ is a semiprime ideal of R and hence $a \in (P : \langle m \rangle)$, i.e., $a\langle m \rangle \subseteq P$. $\square$

4.2. Corollary. If P is a prime (semiprime, classical prime) submodule of $_R M$ with R medial (left permutable, right permutable or LSD), then each of the following statements implies P is completely prime and hence classical completely prime.

(1) M is finitely generated,
(2) M is free,
(3) M is cyclic.

5. The radicals $\beta^c_{cl}(M)$ and $\beta_{co}(R)$

Let $\mathcal{M}^c$ be the class of all completely prime rings, i.e., rings which have no non-zero divisors. Then $\mathcal{M}^c_R$ is the class of all classical completely prime R-modules. We have $\mathcal{R}_c = \mathcal{N}_g$, the generalized nil radical which we shall call the completely prime radical of R (denoted by $\beta_{co}(R)$) with

$$\beta_{co}(R) := \cap \{ I \triangleleft R \ : \ I \text{ is a completely prime ideal} \}.$$

The corresponding classical completely prime radical for the R-module M will be denoted by
$$\beta_{cl}^c(M) := \cap\{N \leq M : M/N \in \mathcal{M}_R^c\}.$$
Since each classical completely prime submodule of an R-module M is also classical prime submodule, we have $\beta_{cl}(M) \subseteq \beta_{cl}^c(M)$ where $\beta_{cl}(M)$ is the classical prime radical (the intersection of all classical prime submodules of M). If M is an R-module over a commutative ring, then the two radicals coincide.

5.1. Proposition. For any ring R, $\beta_{cl}^c(_RR) \subseteq \beta_{co}(R)$.

Proof. Follows from [21, Lemma 4.1] and the fact that any completely prime module is classical completely prime. $\square$

5.1. Lemma. For any R-module M, we have
$$\beta_{co}(R) \subseteq (\beta_{cl}^c(M) : M)_R.$$

Proof. We have $(\beta_{cl}^c(M) : M) = (\bigcap_{S \leq M} S : M)_R = \bigcap_{S \leq M} (S : M)_R$ where M/S is a classical completely prime module. Since $(S : M)_R$ is a completely prime ideal, we get $\beta_{co}(R) \subseteq (\beta_{cl}^c(M) : M)$. $\square$

5.1. Remark. The containment in Lemma 5.1 is in general strict. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. Now $\beta_{cl}^c(M) = \mathbb{Z}_{p^\infty}$ and $\beta_{co}(R) = 0$, i.e., $\beta_{co}(R)M = (0)$.

5.2. Lemma. For any ring R, we have $\beta_{co}(R) = (\beta_{cl}^c(_RR) : R)_R$.

Proof. Follows from [12, Proposition 4.6]. $\square$

Recall that for an R-module M, we have the Jacobson radical $\mathrm{Rad}(M)$ of the module M defined as:
$$\mathrm{Rad}(M) = \cap\{K \leq M \ : \ K \text{ is a maximal submodule of } M\}.$$

5.1. Theorem. Let M be a module over a left Artinian ring R. Then
$$\mathrm{Rad}(M) \subseteq \beta_{cl}^c(M) \text{ and } \mathrm{Rad}(_RR) = \beta_{cl}^c(_RR).$$

Proof. From [9, Cor. 4.3.17, p.178], $\mathrm{Rad}(M) = \mathrm{Jac}(R)M = \beta_{co}(R)M$ and from the fact that $\beta_{co}(R) \subseteq (\beta_{cl}^c(M) : M)_R$ we get $\mathrm{Rad}(M) \subseteq \beta_{cl}^c(M)$. Again from [9, Cor. 4.3.17, p.178], and Lemma 5.2, $\mathrm{Rad}(_RR) = \mathrm{Jac}(R)R = \beta_{co}(R)R = \beta_{cl}^c(_RR)$. $\square$

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