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Convergence processes of approximating operators

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Abstract

The aim of this paper is to present Korovkin type theorems on approximatin of continuous functions with the use of $A-$statistical convergence and matrix summability method which includes both convergence and almost convergence. Since statistical convergence and almost convergence methods are incompatible, we conclude that these methods can be used alternatively to get some approximation results.

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1. Introduction

The so-called Bohman-Korovkin theorem on approximation of continuous functions on a compact interval provides conditions in order to make a decision whether a sequence of positive linear operators converges to the identity operator [2],[7],[14], and so on many proofs have appeared in a variety of settings of this result (see[15],[18], [20],[27]). In [27], Uchiyama have given an alternate proof of it by using inequalities related to variance. If the sequence of positive linear operators does not convergence to the identity operator then it might be benefical to use summability methods ([1],[13],[16],[22],[26],[28]).

The main point of using summability theory has always been to make a nonconvergent sequence to converge. This was the motivation behind Fèjer's famous theorem showing Cesàro method being effective in making the Fourier series of a continuous periodic function to converge [29]. In this paper, using Uchiyama's idea [27], we give quite simple proofs of the Korovkin type approximation theorems studied in ([9],[13],[23]). And also we develop some Korovkin type results with the use of summation process and statistical convergence methods respectively.

We pause to collect some notation.

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Let $C[a,b]$ be the vector space of all real-valued continuous functions on $[a,b]$ and let L be a linear operator on $C[a,b]$. We say that L is positive if $Lf \geq 0$ whenever $f \geq 0$ on $[a,b]$. Note that $C[a,b]$ is a Banach space with norm $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$ and we denote norm of L operator by $\|L\| = \max\{\|Lf\| : \|f\| \leq 1\}$.

A subsequence $\mathcal{B}$ of $C[a,b]$ is called a subalgebra if $f.g$ belongs to $\mathcal{B}$ whenever f and g are members of $\mathcal{B}$.

We first recall the following lemma introduced in [27], which is useful in proving our results.

[A] **Lemma.** Let $\mathcal{B}$ be a norm-closed subalgebra of $C[a,b]$ that contains 1. If L is a positive linear operator on $\mathcal{B}$ with $L(1) \leq 1$, then

$$V(h) := L(h^2) - (L(h))^2 \geq 0$$

for every h in $\mathcal{B}$. Morever, for f, g and k in $\mathcal{B}$:

$$|L(fg) - L(f)L(g)|^2 \leq V(f)V(g) \tag{1.1}$$

$$\|L(fg) - L(f)L(g)\| \leq \|V(f)\|^{\frac{1}{2}} \|V(g)\|^{\frac{1}{2}} \tag{1.2}$$

$$\|L(fg) - L(f)L(g)\| \leq \|V(f)\|^{\frac{1}{2}} \|V(g) + V(k)\|^{\frac{1}{2}} \tag{1.3}$$

We now turn our attention to matrix summability method.

Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. A sequence $\{L_j\}$ of positive linear operators of $C[a,b]$ into $C[a,b]$ is called an $\mathcal{A}$ - summation process on $C[a,b]$ if $\{L_j(f)\}$ is $\mathcal{A}$- convergent to f for every $f \in C[a,b]$, i.e.,

$$\lim_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\| = 0, \text{ uniformly in } n, \tag{1.4}$$

where it is assumed that the series in (1.4) converges for each k, n and f. Recall that a sequence of real numbers $\{x_j\}$ is said to be $\mathcal{A}-$convergent (or $\mathcal{A}-$summable) to L if $\lim_k \sum\limits_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$, (uniformly in n), ([19],[25]).

If $A^{(n)} = A$ for some matrix A, then $\mathcal{A}-$summability is the ordinary matrix summability by A. If $a_{kj}^{(n)} = 1/k$ for $n \leq j < k+n$, $(n = 1, 2, ...)$, and $a_{kj}^{(n)} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence method [18]. Let $\{L_j\}$ be a sequence of positive linear operators of $C\ [a,b]$ into $C\ [a,b]$ such that for each $k, n \in \mathbb{N}$

$$\sum_{j=1}^{\infty} a_{kj}^{(n)} \|L_j(1)\| < \infty. \tag{1.5}$$

Furthermore, for each $k, n \in \mathbb{N}$ and $f \in C[a,b]$, let

$$B_k^{(n)}(f;x) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x)$$

which is well defined by (1.5), and belongs to $B[a,b]$. Observe that $\left\|B_k^{(n)}\right\| = maks\left\{\left\|B_k^{(n)}(f)\right\| : \|f\| \leq 1\right\}$.Hence $\left\|B_k^{(n)}\right\| = \left\|B_k^{(n)}(1)\right\|$.Some unification on Korovkin-type results through the use of a summability method may be found in ([3],[4],[5],[6],[8]).

2. Korovkin Type Approximation Theorems via Summation Process

This section is motivated by that of Uchiyama [27]. We give quite simple proof of a Korovkin type theorem which has been developed by Nishishiraho via $\mathcal{A}-$ summation process, with the use of inequalities related to variance. And also we obtain Korovkin type results for positive linear operators over $C_{2\pi}$ and $C(D)$ respectively with the use of $\mathcal{A}-$ summation process which includes both convergence and almost convergence.

2.1. Theorem. Let $\mathcal{A} := \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C\,[a,b]$ into $C\,[a,b]$ for which (1.5) holds. If,

$$\lim_k \left\|B_k^{(n)}h - h\right\| = 0, \text{ uniformly in } n, \tag{2.1}$$

for all $h = 1,\ x,\ x^2$ then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C\,[a,b]$ i.e., for every $f \in C\,[a,b]$,

$$\lim_k \left\|B_k^{(n)}f - f\right\| = 0, \text{ uniformly in } n.$$

Proof. We proceed as in [27]. Since $\lim_k \left\|B_k^{(n)}1 - 1\right\| = 0$, (uniformly in n), we have $\lim_k \left\|B_k^{(n)}1\right\| = 1$, (uniformly in n). Without loss of generality we may assume that $\left\|B_k^{(n)}\right\| \neq 0$ for all n and k.By considering $\frac{B_k^{(n)}}{\left\|B_k^{(n)}\right\|}$ in place of $B_k^{(n)}$, without loss of generality we assume that $B_k^{(n)}(1) \leq 1$ for all n, k. This implies that $\left\|B_k^{(n)}\right\| \leq 1$ for all n, k. Using (1.2), for every f in $C\,[a,b]$ and for all n, k,

we can write

$$\left\|B_k^{(n)}(xf) - B_k^{(n)}(x).B_k^{(n)}(f)\right\|^2$$

$$\leq \left\|B_k^{(n)}(x^2) - (B_k^{(n)}(x))^2\right\| \left\|B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2\right\|. \tag{2.2}$$

Since $\left\|B_k^{(n)}\right\| \leq 1$, we get

$$\left\|B_k^{(n)}(f^2) - (B_k^{(n)}(f))^2\right\| \leq \left\|B_k^{(n)}\right\| \left\{\left\|f^2\right\| + \|f\|^2\right\} \leq 2\,\|f\|^2.$$

Considering hypothesis we conclude that

$$\lim_k B_k^{(n)}(x^2) = x^2 = \lim_k (B_k^{(n)}(x))^2, \text{ uniformly in } n,$$

this implies that the right-hand side of (2.2) tends to zero (uniformly in n). We see that

$$\left\|B_k^{(n)}(xf) - xf\right\| \leq \left\|B_k^{(n)}(xf) - B_k^{(n)}(x).B_k^{(n)}(f)\right\| \left\|B_k^{(n)}(x).B_k^{(n)}(f) - xf\right\|.$$

If $\lim_k \left\|B_k^{(n)}f - f\right\| = 0$, uniformly in n, then it follows from (2.2) that

$$\lim_k \left\|B_k^{(n)}(xf) - xf\right\| = 0, \text{ uniformly in } n.$$

Here by taking $h = 1, x, x^2$ instead of f. We obtain (2.1) holds for $h = x^m$ for $m = 0, 1, 2, ...$.Since $B_k^{(n)}$ is linear, (2.1) holds for every polynomial p. From the Weierstrass theorem asserting the norm-density of polynomials in $C\,[a,b]$ (see [24],p.159), we have for every $f \in C\,[a,b]$

$$\begin{aligned}\left\|B_k^{(n)}f-f\right\| &\le \left\|B_k^{(n)}f-B_k^{(n)}p\right\|+\left\|B_k^{(n)}p-p\right\|+\|f-p\|\\ &\le 2\,\|f-p\|+\left\|B_k^{(n)}p-p\right\|.\end{aligned}$$

Taking supremum over n and letting $k\to\infty$, result follows. □

Let $C_{2\pi}$ be the space of real-valued continuous functions f on $[-\pi,\pi]$ such that $f(-\pi)=f(\pi)$. Then $C_{2\pi}$ is closed subalgebra of $C\,[-\pi,\pi]$ and 1 belongs to $C_{2\pi}$.

In the following theorem, we extend Korovkin type approximation theorem for a sequence of positive linear operators over $C_{2\pi}$ via $\mathcal{A}-$ summation process.

2.2. Theorem. Let $\mathcal{A}:=\left\{A^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C_{2\pi}$ into $C_{2\pi}$ for which (1.5) holds. If,

$$\lim_k \left\|B_k^{(n)}h-h\right\|=0, \text{ uniformly in } n,$$

for all $h=1$, $\sin x$, $\cos x$, then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C_{2\pi}$ i.e., for every $f\in C_{2\pi}$,

$$\lim_k \left\|B_k^{(n)}f-f\right\|=0, \text{ uniformly in } n.$$

Proof. As in the proof of Theorem 2.1, without loss generality we assume that $B_k^{(n)}(1)\le 1$. By (1.3), for every f in $C_{2\pi}$,we have

$$\begin{aligned}&\left\|B_k^{(n)}(f\sin x)-B_k^{(n)}(f).B_k^{(n)}(\sin x)\right\|^2\\ &\le \left\|B_k^{(n)}(f^2)-(B_k^{(n)}(f))^2\right\|\left\|B_k^{(n)}(\sin^2 x)-(B_k^{(n)}(\sin x))^2+B_k^{(n)}(\cos^2 x)-(B_k^{(n)}(\cos x))^2\right\|\\ &= \left\|B_k^{(n)}(f^2)-(B_k^{(n)}(f))^2\right\|\left\|B_k^{(n)}(1)-(B_k^{(n)}(\sin x))^2-(B_k^{(n)}(\cos x))^2\right\|\\ &\le 2\,\|f\|^2.\left\|B_k^{(n)}(1)-(B_k^{(n)}(\sin x))^2-(B_k^{(n)}(\cos x))^2\right\|.\end{aligned}$$

Considering hypothesis we conclude that

$$\lim_k \left\|B_k^{(n)}(f\sin x)-B_k^{(n)}(f).B_k^{(n)}(\sin x)\right\|=0, \text{ uniformly in } n.$$

Observe now that

$$\begin{aligned}&\left\|B_k^{(n)}(f\sin x)-f\sin x\right\|\\ &\le \left\|B_k^{(n)}(f\sin x)-B_k^{(n)}(f).B_k^{(n)}(\sin x)\right\|\left\|B_k^{(n)}(f).B_k^{(n)}(\sin x)-f\sin x\right\|.\end{aligned}$$

If $\lim_k \left\|B_k^{(n)}f-f\right\|=0$,(uniformly in n) then $\lim_k \left\|B_k^{(n)}(f\sin x)-f\sin x\right\|=0$, (uniformly in n). We obtain similarly that $\lim_k \left\|B_k^{(n)}(f\cos x)-f\cos x\right\|=0$, (uniformly in n). By taking $h=1,\sin x,\cos x$ instead of f then (2.1) holds for $h=\sin^m x\cos^t x$ for all nonnegative integers m and t, which ensures that it is valid for $h=\sin mx.\cos tx$ for all such m and t. Thus (2.1) holds for every trigonometric polynomial p, and since the latter functions are dense in $C_{2\pi}$. (see [24],p:190) we have for every f in $C_{2\pi}$ that $\lim_k \left\|B_k^{(n)}f-f\right\|=0$, uniformly in n. □

We next consider the space $C(D)$ of complex-valued continuous functions f on the closed unit disk $D=\{z:|z|\leq 1\}$ in the complex plane.

In what follows we require the following

[B] **Lemma.** [27] If L is a positive linear operators on $C(D)$ with $L(1)\leq 1$, then

$$V(h):=L(|h|^2)-|L(h)|^2\geq 0$$

for every h in $C(D)$. Morever, for f and g in $C(D)$ it is the case that

$$|L(fg)-L(f)L(g)|^2\leq V(f)V(g)$$

$$\|L(fg)-L(f)L(g)\|\leq \|V(f)\|^{\frac{1}{2}}\,\|V(g)\|^{\frac{1}{2}} \tag{2.4}$$

We now give a Korovkin type approximation theorem for a sequence of positive linear operators defined on $C(D)$ via $\mathcal{A}-$ summation process.

2.3. Theorem. Let $\mathcal{A}:=\left\{A^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\{L_j\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$ for which (1.5) holds. If,

$$\lim_k \left\|B_k^{(n)}h-h\right\|=0,\ \text{uniformly in } n,$$

for all $h=1,\ z,\ |z|^2$, then $\{L_j\}$ is $\mathcal{A}-$ summation process on $C(D)$ i.e., for every $f\in C(D)$,

$$\lim_k \left\|B_k^{(n)}f-f\right\|=0,\text{uniformly in } n.$$

Proof. We may assume that $B_k^{(n)}(1)\leq 1$ for all n,k. Since (2.4), we have

$$\begin{aligned}\left\|B_k^{(n)}(zf)-B_k^{(n)}(z).B_k^{(n)}(f)\right\|^2 &\leq \left\|B_k^{(n)}(\,|z|^2\,)-\left|B_k^{(n)}(z)\right|^2\right\|\left\|B_k^{(n)}(\,|f|^2\,)-\left|B_k^{(n)}(f)\right|^2\right\|\\ &\leq 2\,\|f\|^2\left\|B_k^{(n)}(\,|z|^2\,)-\left|B_k^{(n)}(z)\right|^2\right\|.\end{aligned}$$

By the hypothesis we get

$$\lim_k B_k^{(n)}(|z|^2)=|z|^2=\lim_k \left|B_k^{(n)}(z)\right|^2,\ \text{uniformly in } n,$$

this implies that

$$\lim_k \left\|B_k^{(n)}(zf)-B_k^{(n)}(z).B_k^{(n)}(f)\right\|=0,\ \text{uniformly in } n.$$

We can write

$$\left\|B_k^{(n)}(zf)-zf\right\|\leq \left\|B_k^{(n)}(zf)-B_k^{(n)}(z).B_k^{(n)}(f)\right\|\left\|B_k^{(n)}(z).B_k^{(n)}(f)-zf\right\|$$

if $\lim_k \left\|B_k^{(n)}f-f\right\|=0$, (uniformly in n) then $\lim_k \left\|B_k^{(n)}(zf)-zf\right\|=0,$ (uniformly in n). We obtain that (2.1) holds for $\overline{h}$ whenever it holds for h. Here by taking $h=1,z,|z|^2$ instead of f, (2.1) holds for $h=\overline{z}^m.z^k$ for all non-negative integers m and k, hence for every polynomial in z and $\overline{z}$. By Stone's theorem (see [24],p:165) the set of all such polynomials is dense in $C(D)$, so (2.1) holds for every f in $C(D)$.

□

3. A Korovkin Type Approximation Theorem via Statistical Convergence

In this section we give simple proofs for statistical analog of Korovkin's theorems considered in [9] and [13], also using $A-$statistically convergence, we extend a Korovkin type result for positive linear operators over the space $C(D)$.
First we recall the concept of $A-$statistical convergence. Let $A := (a_{jn})$, $j, n = 1, 2, ...$, be an infinite summability matrix. For a given sequence $x := (x_n)$, the $A-$transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j := \sum\limits_n a_{jn}x_n$, provided the series converges for each j. The matrix A is said to be regular if $\lim_j (Ax)_j = L$ whenever $\lim x = L$ [12].Suppose that A is a non-negative regular summability matrix. Then x is $A-$statistically convergent to L if for every $\varepsilon > 0$

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $st_A - \lim x = L$ ([11],[17]). The case in which $A = C_1$, the Cesaro matrix of order one, reduces to the statistical convergence ([10],[11]). Also if $A = I$, the identity matrix, then it reduces to the ordinary convergence.
Note that, if $A = (a_{jn})$ is a non-negative regular matrix such that $\lim_j \max_n \{a_{jn}\} = 0$, then $A-$statistical convergence is stonger than convergence [17].

3.1. Theorem.Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If,

$$st_A - \lim_n \|L_n h - h\| = 0. \tag{3.1}$$

for all $h = 1$, x, x^2, then, for every $f \in C[a, b]$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

Proof. As in the proof of Theorem 2.1 , without loss of generality we assume that in $L_n(1) \leq 1$ for all n. By (1.2) , for every f in $C[a, b]$ and for all n , we can write

$$\|L_n(xf) - L_n(x).L_n(f)\|^2 \leq \left\|L_n(x^2) - (L_n(x))^2\right\| \left\|L_n(f^2) - (L_n(f))^2\right\|. \tag{3.2}$$

Since $\|L_n\| \leq 1$, we get

$$\left\|L_n(f^2) - (L_n(f))^2\right\| \leq \|L_n\| \left\{\left\|f^2\right\| + \|f\|^2\right\} \leq 2\|f\|^2$$

by hypothesis we obtain that

$$st_A - \lim_n L_n(x^2) = x^2 = st_A - \lim_n (L_n(x))^2$$

this implies that the right-hand side of (3.2) is $A-$statistically convergent to zero. Observe that

$$\|L_n(xf) - xf\| \leq \|L_n(xf) - L_n(x).L_n(f)\| . \|L_n(x).L_n(f) - xf\| .$$

If $st_A - \lim\limits_n \|L_n f - f\| = 0$, then it follows from (3.2) that

$$st_A - \lim_n \|L_n(xf) - xf\| = 0.$$

Here by taking $h = 1, x, x^2$ instead of f. We see that (3.1) holds for $h = x^m$ for $m = 0, 1, 2, ...$Since L_n is linear, (3.1) holds for every polynomial p. Since $\|L_n\| \leq 1$ for every n, theorem follows from the Weierstrass theorem asserting the norm-density of polynomials in $C[a, b]$. $\square$

3.2. Theorem. Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C_{2\pi}$ into $C_{2\pi}$. If,

$$st_A - \lim_n \|L_n h - h\| = 0.$$

for all $h = 1, \sin x, \cos x$, then, for every $f \in C_{2\pi}$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

Proof. As in the proof of Theorem 2.2, there is no loss of generality in assuming that $L_n(1) \leq 1$. By (1.3), we have for every f in $C_{2\pi}$,

$$\begin{aligned} &\|L_n(f \sin x) - L_n(f).L_n(\sin x)\|^2 \\ &\leq \left\|L_n(f^2) - (L_n(f))^2\right\| \left\|L_n(\sin^2 x) - (L_n(\sin x))^2 + L_n(\cos^2 x) - (L_n(\cos x))^2\right\| \\ &= \left\|L_n(f^2) - (L_n(f))^2\right\| \left\|L_n(1) - (L_n(\sin x))^2 - (L_n(\cos x))^2\right\| \\ &\leq 2\|f\|^2 . \left\|L_n(1) - (L_n(\sin x))^2 - (L_n(\cos x))^2\right\| \end{aligned}$$

By the hypothesis we have

$$st_A - \lim_n \|L_n(f \sin x) - L_n(f)L_n(\sin x)\| = 0.$$

This implies that $st_A - \lim_n \|L_n(f \sin x) - f \sin x\| = 0$ whenever $st_A - \lim_n \|L_n f - f\| = 0$.We see similarly that $st_A - \lim_n \|L_n(f \cos x) - f \cos x\| = 0$ in this situtation. Thus (3.1) holds for $h = \sin^m x \cos^t x$ for all nonnegative integers m and t, which ensures that it is valid for $h = \sin mx. \cos tx$ for all such m and t.Thus (3.1) holds for every trigonometric polynomial p, and since the latter functions are dense in $C_{2\pi}$.(see [24],p:190) we have for every $f \in C_{2\pi}$ that

$$st_A - \lim_n \|L_n f - f\| = 0.$$

□

3.3. Theorem. Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$. If,

$$st_A - \lim_n \|L_n h - h\| = 0.$$

for all $h = 1, z, |z|^2$, then, for every $f \in C(D)$

$$st_A - \lim_n \|L_n f - f\| = 0.$$

Proof. We may assume that $L_n(1) \leq 1$ for all n. It is evident that (3.1) holds for $\overline{h}$ whenever it holds for h. The estimate (2.4) guarantees that (3.1) holds for $h = \overline{z}^m.z^k$ for all nonnegative integers m and k, hence for every polynomial in z and $\overline{z}$. By Stone's theorem the set of all such polynomials is dense in $C(D)$, so (3.1) holds for every f in $C(D)$.

□

Note that if we replace A by the identity matrix we get the complex Korovkin theorem.

3.1. Remark. Now we exhibit two examples of sequences of positive linear operators. The first one shows that Theorem 3.3 does not work, so the classical Korovkin theorem does not work either; but Theorem 2.3 works. The second one gives that Theorem 2.3 does not work but Theorem 3.3 does work. In order to see this let $\{L_j\}$ be a sequence of positive linear operators from $C(D)$ into $C(D)$ satisfying the hypothesis of the classical complex Korovkin theorem. Assume now that $\mathcal{A} = \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ is a sequence of

infinite matrices defined by $a_{kj}^{(n)} = 1/k$ if $n \leq j < n+k$, and $a_{kj}^{(n)} = 0$ otherwise. In this case $\mathcal{A}-$summability method reduces to almost convergence.. We also take $A = C_1$ in Theorem 3.3. In this case $A-$ statistical convergence reduces to the statistical convergence. Then consider the following two examples.
(a) Take $(u_j) = \{(-1)^j\}$. Note that u is almost convergent to zero [19], but it is not statistically convergent [11]. Now define

$$T_j(f;x) = (1+u_j)L_j(f;x) \text{ for all } f \in C(D).$$

Then observe that $\{T_j\}$ satisfies Theorem 2.3, but it satisfies neither the classical Korovkin theorem nor the present Theorem 3.3.
(b) Consider a non-negative sequence (u_j) which is statistically convergent to zero but not almost convergent. Such an example may be found in [21]. Proceeding exactly as in the case (a) we can construct a sequence of positive linear operators so that it is statistically convergent to the identity operator but not almost convergent. These two methods are incompatible [21].

The examples given above suggest that if the sequence of positive linear operators does not converge then we can use alternatively either almost convergence method or statistical convergence method to get some Korovkin type approximation results.

References

[1] O.Agratini, A-Statistical Convergence of a Class of İntegral Operators. Appl. Math. Inf. Sci. 6, (2), 325-328, 2012.

[2] F. Altomare and M. Campiti, Korovkin type Approximation Theory and its Application. (Walter de Gruyter Studies in Math. 17, de Gruyter&Co., Berlin, 1994).

[3] G. A. Anastassiou and O. Duman, Towards Intelligent Modelling: Statistical Approximation Theory,(Intelligent Systems Reference Library, Vol.14, Springer, Berlin, 2011).

[4] Ö. G. Atlihan and C. Orhan, Summation process of positive linear operators, Comp.Math. Appl., 56, 1188-1195, 2008.

[5] Ö.G.Atlihan, H.Gül İnce, C. Orhan, Some variations of the Bohman-Korovkin Theorem, Math.Comp. Modelling. 50, 1205-1210, 2009.

[6] H. T. Bell, Order summability and almost convergence, Proc. Amer. Math. Soc. 38, 548-552.,1973.

[7] H. Bohman, On approximation of continuous and analytic functions. Ark. Mat. 2, 43-56, 1952.

[8] J. Boos, Classical and Modern Methods in Summability. (Oxford Sci. Publ., London. 2000).

[9] O. Duman, Statistical approximation for periodic functions, Demonstratio Math. 36, 873–878 , 2003.

[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2, 241-244 ,1951.

[11] J. A. Fridy, On statistical convergence, Analysis. 5, 301-313., 1985.

[12] A. R. Freedman, J. J. Sember, Densities and summability, Pacific J. Math. 95, 293-305 ,1981.

[13] A. D. Gadjiev and C.Orhan, Some aprproximation theorems via statistical convergence, Rocky Mountain Journal of Math. 32 (1) 129-137 ,2002.

[14] P. P. Korovkin, On convergence of linear positive operators in the space of continuos functions, Doklady Akad. NaukSSSR. 90, 961-964 ,1953.

[15] P.P. Korovkin, Linear operators and Theory of Approximation.(Hindustan Publ. Co., Delhi, 1960).

[16] J. P. King and J. J. Swetits, Positive linear operators and summability, J. Austral. Math. Soc. 11, 281-290, 1970.

[17] E. Kolk, Matrix summability of statistically convergent sequences, Analysis. 13, 77-83 ,1993.

[18] G.G. Lorentz, Approximation of Functions. (Chelsa Publ. Company. New York, 1986).

[19] G.G. Lorentz, A contribution to the theory of divergent sequence, Acta Math. 80, 167-190 ,1948.

[20] H. E. Lomelí and C. L. García, Variations on a Theorem of Korovkin, The Amer. Math. Monthly, 113 (8) 744-750 ,2006.
[21] H. I. Miller and C. Orhan, On almost convergent and statistically convergent subsequences, Acta Math. Hungar. 93, 135–151 ,2001.
[22] R. N. Mohapatra, Quantitative results on almost convergence of a sequence of positive linear operators, J.of Approx. Theory, 20, 239-250 ,1977.
[23] T. Nishishiraho, Convergence of positive linear approximation processes, Tôhoku Math. J., 35, 441-458 ,1983.
[24] W. Rudin, Principles of Mathematical Analysis, (McGraw-Hill, New York, 1976).
[25] M. Stieglitz, Eine verallgenmeinerung des begriffs festkonvergenz, Math. Japonica. 18, 53-70 ,1973.
[26] J. J. Swetits, On summability and positive linear operators, J. Approx. Theory. 25, 186-188 ,1979.
[27] M. Uchiyama, Proofs of Korovkin's theorems via inequalities, The Amer. Math. Monthly, 110, 334-336 ,2003.
[28] T.Yurdakadim, E.Tas, and İ.Sakaoğlu, Approximation of functions by the sequence of integral operators. Appl. Math. Comput., Vol. 219(8), 3863-3871., 2012.
[29] A. Zygmund, Trigonometric Series. (Cambridge University Press. 1979).