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Comparison of near sets by means of a chain of features

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Abstract

If the number of features of objects in a perceptual system, is large, then the objects can be known better and comparable. In this paper basically, we form a chain of feature sets that describe objects and then by means of this chain of feature sets, we investigate the nearness of sets and near sets in a perceptual system.

Keywords: Near set, Feature chain, Indiscernibility , Nearness.

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1. Introduction

Near sets were introduced by J.F. Peters [11], which are indeed a form of generalization of rough sets proposed by Z. Pawlak [6]. The algebraic properties of near sets are described in [9]. Recent work has considered near soft sets [20], soft nearness approximation spaces [4], near groups [3], isometries in proximity spaces [18], and applications of near sets [17,19]. The fundamental idea of near set theory is object description and classification according to perceptual knowledge. It is supposed that perceptual knowledge about objects is always given with respect to probe functions, i.e., real-valued functions which represent features of a physical object. Some well known examples of probe functions are the colour, size or weight of an object [1,2,9-16,21].

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2. Preliminiaries

In this section, we present the basic definitions of near set theory [9,11]. More detailed explanations related to near sets and rough sets can be found in [1,2,9-16,21] and [5-8], respectively.

2.1. Definition. [9] (*Perceptual Object*) A perceptual object is something perceivable that has its origin in the physical world.

2.2. Definition. [9] (*Probe Function*) A probe function is a real-valued function representing a feature of a perceptual object.Simple examples of probe functions are the colour, size or weight of an object.

2.3. Definition. [9] (*Perceptual System*) A perceptual system $\langle O, F\rangle$ consists of a non-empty set O of sample perceptual objects and a non-empty set F of real-valued functions $\phi \in F$ such that $\phi : O \to \mathbb{R}$.

2.4. Definition. [9] (*Object Description*) Let $\langle O, F\rangle$ be a perceptual system, and let $B \subseteq F$ be a set of probe functions. Then, the description of a perceptual object $x \in O$ is a feature vector given by

$$\phi_B(x) = (\phi_1(x), \phi_2(x), ..., \phi_i(x), ..., \phi_l(x))$$

where l is the length of the vector ϕ_B, and each $\phi_i(x)$ in $\phi_B(x)$ is a probe function value that is part of the description of the object $x \in O$.

2.5. Definition. $[2, 6]$ (*Indiscernibility relation*) Let $\langle O, F\rangle$ be a perceptual system. For every $B \subseteq F$ the indiscernibility relation $\sim_B$ is defined as follows:

$$\sim_B = \{(x, y) \in O \times O \mid \forall \phi_i \in B, \phi_i(x) = \phi_i(y)\}.$$

If $B = \{\phi\}$ for some $\phi \in F$, instead of $\sim_{\{\phi\}}$ we write $\sim_\phi$.

The indiscernibility relation $\sim_B$ is an equivalence relation on object descriptions.

2.6. Lemma. [9] *Let $\langle O, F\rangle$ be a perceptual system. For every $B \subseteq F$,*

$$\sim_B = \bigcap_{\phi \in B} \sim_\phi .$$

2.7. Definition. (*Equivalence Class*) Let $\langle O, F\rangle$ be a perceptual system and let $x \in O$. For a set $B \subseteq F$ an equivalence class is defined as $x_{/\sim_B} = \{y \in O \mid y \sim_B x\}$.

2.8. Definition. (*Quotient Set*)Let $\langle O, F\rangle$ be a perceptual system.For a set $B \subseteq F$ a quotient set is defined as

$$O_{/\sim_B} = \{x_{/\sim_B} \mid x \in O\}.$$

2.9. Definition. [9] Let $\langle O, F\rangle$ be a perceptual system. Then

$$\prod(O, F) := \bigcup_{B \subseteq F} O_{/\sim_B},$$

i.e., $\prod(O, F)$ is the family of equivalence classes of all indiscernibility relations determined by a perceptual information system $\langle O, F\rangle$.

2.10. Definition. [9] (*Nearness relation*). Let $\langle O, F\rangle$ be a perceptual system and let $X, Y \subseteq O$. A set X is near to a set Y within the perceptual system $\langle O, F\rangle$ ($X \bowtie_F Y$) iff there are $F_1, F_2 \subseteq F$ and $f \in F$ and there are $A \in O_{/\sim_{F_1}}, B \in O_{/\sim_{F_2}}, C \in O_{/\sim_f}$ such that $A \subseteq X, B \subseteq Y$ ve $A, B \subseteq C$.If a perceptual system is understood, then we say briefly that a set X is near to a set Y.

2.11. Definition. [9] (*Perceptual near sets*) Let $\langle O, F\rangle$ be a perceptual system and let $X \subseteq O$. A set X is a perceptual near set iff there is $Y \subseteq O$ such that $X \bowtie_F Y$. The family of near sets of a perceptual system $\langle O, F\rangle$ is denoted by $Near_F(O)$.

2.12. Example. Let $\langle O, F\rangle$ be a perceptual system such that $O = \{x_1, x_2, ..., x_6\}$, $F = \{\phi_1, \phi_2\}, \phi_1(x_1) = \phi_1(x_2) = \phi_1(x_3), \phi_1(x_4) = \phi_1(x_5) = \phi_1(x_6), \phi_1(x_1) \neq \phi_1(x_4)$ and $\phi_2(x_1) = \phi_2(x_2), \phi_2(x_3) = \phi_2(x_4), \phi_2(x_5) = \phi_2(x_6), \phi_2(x_1) \neq \phi_2(x_4) \neq \phi_2(x_5)$.

Thus $O_{/\sim_{\phi_1}} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$, $O_{/\sim_{\phi_2}} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$, $O_{/\sim_{\{\phi_1,\phi_2\}}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}\}$.

Let $X = \{x_1, x_2, x_3, x_5\}, Y = \{x_2, x_4, x_5, x_6\}$.Thus there are $A = \{x_4\} \in O_{/\sim_{\{\phi_1,\phi_2\}}}$, $B = \{x_5, x_6\} \in O_{/\sim_{\phi_2}} C = (A \cup B) \in O_{/\sim_{\phi_1}}$ such that $A \subseteq X, B \subseteq Y$. Therefore $X \bowtie_F Y$.

2.13. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system, $B \subseteq F$ and $x_{/\sim_B} \in O_{/\sim_B}$, where $|x_{/\sim_B}| \geq 2$. All elements belonging to a class $x_{/\sim_B}$ are near each other.*

2.14. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system. For any $B \subseteq F$, every equivalence class of an indiscernibility relation $\sim_B$ is a near set .*

3. Some New Properties of Near Sets

In this section, we give some new propositions which are related to some propositions in [9].

3.1. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system. For every $X \subseteq O$, the following conditions are equivalent:*

(1) $X \in Near_F(O)$,
(2) there is $A \in \prod(O, F)$ such that $A \subseteq X$,
(3) there is $A \in O_{/\sim_F}$ such that $A \subseteq X$.

3.2. Proposition. *Let $\langle O, F\rangle$ be a perceptual system and $X, Y \subseteq O$. Then*

$$X \bowtie_F Y \Rightarrow X, Y \in Near_F(O) .$$

Proof. Let $X \bowtie_F Y$. From Definition 2.11, there are $A, B \in \prod(O, F)$ such that $A \subseteq X, B \subseteq Y$.Thus, from Proposition 3.1, $X, Y \in Near_F(O)$. □

3.3. Remark. From Proposition 3.2, two near sets may not be near to each other. We can see this in the following example.

3.4. Example. Let $\langle O, F\rangle$ be a perceptual system such that $O = \{x_1, x_2, ..., x_6\}$,for simplicity $F = (\phi)$ *and* $\phi(x_2) = \phi(x_3), \phi(x_4) = \phi(x_5) = \phi(x_6), \phi(x_1) \neq \phi(x_2) \neq \phi(x_4)$ .Thus $O_{/\sim_\phi} = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}\}$. Let $X = \{x_1, x_2\}, Y = \{x_2, x_3, x_6\}$.There are $A = \{x_1\} \in O_{/\sim\phi}, B = \{x_2, x_3\} \in O_{/\sim\phi}$ such that $A \subseteq X, B \subseteq Y$, so $X, Y \in Near_F(O)$. But there is no $C \in O_{/\sim\phi}$ such that $A, B \subseteq C$.Therefore X and Y are not near to each other.

3.5. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system and $X, Y \subseteq O$. Then*

$$X, Y \in Near_F(O) \Rightarrow X \cup Y \in Near_F(O),$$

i.e., the family of near sets of a perceptual system $\langle O, F\rangle$ is closed for the union of sets.

3.6. Proposition. *Let $\langle O, F\rangle$ be a perceptual system and $X, Y \subseteq O$. Then*

$$X \bowtie_F Y \Rightarrow X \cup Y \in Near_F(O).$$

Proof. It is clear from Proposition 3.2 and Proposition 3.5 . □

3.7. Proposition. [9] *Let $\langle O, F\rangle$ be a . Then*

$$X \in \prod (O, F) \Rightarrow X \bowtie_F X,$$

i.e., the relation $\bowtie_F$ is reflexive within the family $\prod (O, F)$.

3.8. Proposition. *Let $\langle O, F\rangle$ be a perceptual system. Then*

$$X \bowtie_F X \ \Leftrightarrow \text{there is } A \in \prod (O, F) \text{ such that } \ A \subseteq X.$$

That is, a set $X \subseteq O$ to be near to itself need not be a equivalence class or need not be a union of equivalence classes. But at least it has to contain an equivalence class.

Proof. It is clear. □

3.9. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system . For any $X, Y \subseteq O$, if there is $A \in \prod (O, F)$ such that $A \subseteq X \cap Y$, then $X \bowtie_F Y$.*

3.10. Proposition. *Let $\langle O, F\rangle$ be a perceptual system and let $X, Y \subseteq O$ and F is a singleton set. Then*

$$X \bowtie_F Y \Leftrightarrow \text{there is } A \in \prod (O, F) \text{ such that } \ A \subseteq X \cap Y.$$

Proof. It is enough to prove the implication ($\Rightarrow$). From Definition 2.10, there are $A \in O_{/\sim_F}, B \in O_{/\sim_F}, C \in O_{/\sim_F}$ such that $A \subseteq X, B \subseteq Y$ and $A, B \subseteq C$. Since F is a singleton set and $A, B \subseteq C$, then $A = B = C$. Therefore $A \subseteq X \cap Y$. □

3.11. Proposition. [9] *Let $\langle O, F\rangle$ be a perceptual system and let $X, Y, Z \subseteq O$. Then the following conditions hold:*

(1) $X \bowtie_F Y \;\&Y \subseteq Z \Rightarrow X \bowtie_F Z,$
(2) $X \subseteq Y \;\&\; X \bowtie_F Z \Rightarrow Y \bowtie_F Z$.

3.12. Proposition. *Let $\langle O, F\rangle$ be a perceptual system and $A_1, A_2, B_1, B_2 \subseteq O$.Then the following conditions hold:*

(1) $A_1 \bowtie_F A_2 \;\&B_1 \bowtie_F B_2 \Rightarrow (A_1 \cup B_1) \bowtie_F (A_2 \cup B_2)$ or $(A_1 \cup B_2) \bowtie_F (A_2 \cup B_1)$,
(2) $(A_1 \cap A_2) \bowtie_F (B_1 \cap B_2) \Rightarrow A_1 \bowtie_F B_1$ or $A_1 \bowtie_F B_2$ or $A_2 \bowtie_F B_1$ or $A_2 \bowtie_F B_2$.

Proof. Let $\langle O, F\rangle$ be a perceptual system and let $A_1, A_2, B_1, B_2 \subseteq O$.

Case (1). Let $A_1 \bowtie_F A_2$ and $B_1 \bowtie_F B_2$. So $A_1 \bowtie_F A_2$, $A_2 \subseteq A_2 \cup B_2$ and $B_1 \bowtie_F B_2, B_2 \subseteq (A_2 \cup B_2)$ then from Proposition 3.11 (1) $A_1 \bowtie_F (A_2 \cup B_2)$ and $B_1 \bowtie_F (A_2 \cup B_2)$. Since $A_1 \bowtie_F (A_2 \cup B_2)$ and $B_1 \bowtie_F (A_2 \cup B_2)$, $(A_1 \cup B_1) \bowtie_F (A_2 \cup B_2)$. Similarly it can be shown that $(A_1 \cup B_2) \bowtie_F (A_2 \cup B_1)$.

Case (2). Let $(A_1 \cap A_2) \bowtie_F (B_1 \cap B_2)$. Since $(A_1 \cap A_2) \subseteq A_1$ and from Proposition 3.11 (2) $A_1 \bowtie_F (B_1 \cap B_2)$. Since $A_1 \bowtie_F (B_1 \cap B_2)$ and from Proposition 3.11 (1), then $A_1 \bowtie_F B_1$. Similarly it can be shown that $A_2 \bowtie_F B_1$ or $A_2 \bowtie_F B_1$ or $A_2 \bowtie_F B_2$. □

The fact that the reverse of the implication reversed in Proposition 3.12 (1) does not hold is shown by example . Similarly it can be shown that the Proposition 3.12 (2) does not hold always.

3.13. Example. Let $\langle O, F\rangle$ be a perceptual system such that $O = \{x_1, x_2, ..., x_8\}$,so $O_{/\sim_F} = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_6, x_7, x_8\}\}$ .Let $A_1 = \{x_2, x_3, x_4\}, A_2 = \{x_1, x_2, x_3, x_5\}$, $B_1 = \{x_1, x_3, x_4, x_7\}, B_2 = \{x_2, x_4, x_6, x_8\}$,so $A_1 \cup B_1 = \{x_1, x_2, x_3, x_4, x_7\}$ and $A_2 \cup B_2 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_8\}$. Since $\{x_1, x_2, x_3\} \in O_{/\sim_F}$ and $\{x_1, x_2, x_3\} \subseteq A_1 \cup B_1, A_2 \cup B_2$ $A_1 \cup B_1 \bowtie_F A_2 \cup B_2$.But there is no $X_{/\sim_F} \in O_{/\sim_F}, Y_{/\sim_F} \in O_{/\sim_F}, Z_{/\sim_F} \in O_{/\sim_F}$ such that $X_{/\sim_F} \subseteq A_1$, $Y_{/\sim_F} \subseteq A_2$ and $X, Y \subseteq Z$. Therefore, from Definition 2.10, A_1 and A_2 are not near to each other. For same reason, B_1 and B_2 are not near to each other.

4. Chain of Features, Nearness and Near Sets

In this section basically, a nested chain of probe functions (features) is formed and corresponding indiscernibility relation, nearness relation and near sets in $\langle O, F\rangle$ perceptual system are investigated.

4.1. Definition. *Let $\langle O, F\rangle$ be a perceptual system. Then*

$\prod(O, \sim_F) := \{\sim_B | \ B \subseteq F\}$,

i.e. $\prod(O, \sim_F)$ is the family of indiscernibility relations of all probe functions determined by a perceptual information system $\langle O, F\rangle$.

4.2. Lemma. *Let $\langle O, F\rangle$ be a perceptual system, $\prod(O, F)$ is the family of equivalence classes of all indiscernibility relations and $\prod(O, \sim_F)$ is the family of indiscernibility relations of all probe functions. Then for all $B \subseteq F$, the function*

$$\begin{aligned} f : \prod(O, \sim_F) &\to \prod(O, F) \\ \sim_B &\mapsto O_{/\sim_B} \end{aligned}$$

is one-to-one and onto.

4.3. Proposition. *Let $\langle O, F\rangle$ be a perceptual system and $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$. Then for all $B_i \subseteq F$, $1 \leq j, i \leq n$,*

$$B_j \subseteq B_i \Leftrightarrow \sim_{B_i} \subseteq \sim_{B_j} .$$

Proof. Let $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$. Since $\bigcap\limits_{\phi\in B_j} \sim_\phi \subseteq \bigcap\limits_{\phi\in B_i} \sim_\phi$ and, from Lemma 2.6, $\sim_{B_i} \subseteq \sim_{B_j}$. □

4.4. Corollary. *Let $\langle O, F\rangle$ be a perceptual system and $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$. Then for all $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$,*

$$\sim_{B_i} \subseteq \sim_{B_j} \Leftrightarrow \bigcap\limits_{\phi\in B_i} \sim_\phi \subseteq \bigcap\limits_{\phi\in Bj} \sim_\phi .$$

4.5. Proposition. *Let $\langle O, F\rangle$ be a perceptual system , $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$. Then*

$\sim_{B_i} \subseteq \sim_{B_j} \Rightarrow$ For all $A \in O_{/\sim_{B_i}}$ there is a unique $C \in O_{/\sim_{B_j}}$ such that $A \subseteq C$.

Proof. Let $\sim_{B_i} \subseteq \sim_{B_j}$, $x \in O$, $A = x_{/\sim_{B_i}}$ and $C = x_{/\sim_{B_j}}$ .Since $\sim_{B_i} \subseteq \sim_{B_j}$, then $x_{/\sim_{B_i}} \subseteq x_{/\sim_{B_j}}$. □

4.6. Proposition. *Let $\langle O, F\rangle$ be a perceptual system , $X \subseteq O, F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$. Then the following conditions hold:*

$$\begin{aligned} &(1) \prod\left(O, \sim_{B_j}\right) \subseteq \prod(O, \sim_{B_i}), \\ &(2) \prod(O, B_j) \subseteq \prod(O, B_i). \end{aligned}$$

Proof. Let $\langle O, F\rangle$ be a perceptual system , $X \subseteq O$, $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$.

(1) Since $B \subseteq B_j$ then $B \subseteq B_i$. Thus from Definition 4.1 $\prod\left(O, \sim_{B_j}\right) \subseteq \prod(O, \sim_{B_i})$.

(2) Since $B_j \subseteq B_i$, from Definition 2.9 $\prod(O, B_j) \subseteq \prod(O, B_i)$. □

4.7. Proposition. *Let $\langle O, F\rangle$ be a perceptual system ,$F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$. Then*

$$Near_{B_j}(O) \subseteq Near_{B_i}(O).$$

Proof. Let $X \subseteq O$ and $X \in Near_{B_j}(O)$. Since $X \in Near_{B_j}(O)$ there is $A \in \prod\left(O, \sim_{B_j}\right)$ such that $A \subseteq X$. From Proposition 4.6 (1) $A \in \prod\left(O, \sim_{B_i}\right)$. Therefore $X \in Near_{B_i}(O)$. □

The fact that the reverse of the implication reversed in Proposition 4.7 does not hold. We can see this in the next example.

4.8. Example. Let $\langle O, F\rangle$ be perceptual system in Example 2.12. Thus $O = \{x_1, x_2, ..., x_6\}$, $F = \{\phi_1, \phi_2\}$. Recall also that $O_{/\sim_{\phi_1}} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$, $O_{/\sim_{\phi_2}} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}$, $O_{/\sim_{\{\phi_1,\phi_2\}}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}\}$. Let $X \subseteq O$, $B_1, B_2 \subseteq F$ be defined as: $X = \{x_1, x_2, x_4\}$, $B_1 = \{\phi_1\}$, $B_2 = \{\phi_1, \phi_2\}$. Since $\{x_1, x_2\} \in O_{/\sim_{\{\phi_1,\phi_2\}}}$ and $\{x_1, x_2\} \subseteq X$, then $X \in Near_{B_2}(O)$. But there is no $A \in O_{/\sim_{\phi_1}}$ such that $A \subseteq X$, therefore $X \notin Near_{B_1}(O)$.

4.9. Proposition. *Let $\langle O, F\rangle$ be a perceptual system ,$F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$, $X, Y \subseteq O$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$ .Then*

$$X \bowtie_{B_j} Y \Rightarrow X \bowtie_{B_i} Y.$$

Proof. Let $X \bowtie_{B_j} Y$. From Definition 2.10 there are $A, B, C \in \prod(O, B_j)$ such that $A \subseteq X, B \subseteq Y$ *and* $A, B \subseteq C$. Since $A, B, C \in \prod(O, B_j)$, then from Proposition 4.6 (2) $A, B, C \in \prod(O, B_i)$. Again from Definition 2.10, $X \bowtie_{B_i} Y$. □

4.10. Definition. Let $\langle O, F\rangle$ be a perceptual system ,$X, Y \subseteq O, F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$. Then the expression

$X \bowtie_{\sim_{B_i}} Y$ means that: A set X is near to a set Y within the perceptual system $\langle O, F\rangle$ only for the $\sim_{B_i}$ relation.

4.11. Proposition. *Let $\langle O, F\rangle$ be a perceptual system , $X, Y \subseteq O, F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_i \subseteq F$,$B_j \subseteq B_i$, $1 \leq j, i \leq n$. Then*

$$X \bowtie_{\sim_{B_j}} Y \Rightarrow X \bowtie_{\sim_{B_i}} Y.$$

Proof. Let $X \bowtie_{\sim_{B_j}} Y$. From Proposition 3.10 and Proposition 3.1, respectively, then $X \cap Y \in Near_{B_j}(O)$. Thus from Proposition 4.7, $X \cap Y \in Near_{B_i}(O)$. Therefore, from Proposition 3.10, then $X \bowtie_{\sim_{B_i}} Y$. □

4.12. Example. Let $\langle O, F\rangle$ be perceptual system in the Example 2.12. Recall also that $O_{/\sim_{\phi_2}} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$, $O_{/\sim_{\{\phi_1,\phi_2\}}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}\}$. Let sets $X, Y \subseteq O$, $B_1, B_2 \subseteq F$ be defined as: $X = \{x_2, x_3, x_4\}$, $Y = \{x_3, x_4, x_6\}$ $B_1 = \{\phi_2\}$, $B_2 = \{\phi_1, \phi_2\}$. Since $\{x_3, x_4\} \in O_{/\sim_{\{\phi_2\}}}$ and $\{x_3, x_4\} \subseteq X, Y$ then $X \bowtie_{\sim_{B_1}} Y$. Since $\{x_4\} \in O_{/\sim_{\{\phi_1,\phi_2\}}}$ and $\{x_4\} \subseteq \{x_3, x_4\} \subseteq X, Y$ then $X \bowtie_{\sim_{B_2}} Y$.

4.13. Definition. Let $\langle O, F\rangle$ be a perceptual system and $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$.

$$B_1 \subseteq B_2 \subseteq ... \subseteq B_n \tag{4.1}$$

Then the ascending subsets (4.1) is called as a chain of probe function sets or briefly a feature sets chain.

From Proposition 4.6, we can give following proposition.

4.14. Proposition. *Let $\langle O, F\rangle$ be a perceptual system , $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $B_1 \subseteq B_2 \subseteq ... \subseteq B_n$ be a feature chain. Then the followings hold:*

$$(1) \prod (O, \sim_{B_1}) \subseteq \prod (O, \sim_{B_2}) \subseteq ... \subseteq \prod (O, \sim_F)$$
$$(2) \prod (O, B_1) \subseteq \prod (O, B_2) \subseteq ... \subseteq \prod (O, F).$$

4.15. Definition. Let $\langle O, F\rangle$ be a perceptual system and $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$.

(4.2) $\quad \bowtie_{B_1} \subseteq \bowtie_{B_2} \subseteq ... \subseteq \bowtie_F$

The relation (4.2) corresponding to (4.1) is called as chain of a perceptual nearness or briefly nearness chain.

From Proposition 4.7 and Proposition 4.9 we can give following proposition.

4.16. Proposition. *Let $\langle O, F\rangle$ be a perceptual system ,$F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$, $X, Y \subseteq O$ and $\bowtie_{B_1} \subseteq \bowtie_{B_2} \subseteq ... \subseteq \bowtie_F$ a nearness chain .Then the following conditions hold:*

$$(1)\ X \bowtie_{B_1} Y \Rightarrow X \bowtie_{B_2} Y \Rightarrow ... \Rightarrow X \bowtie_F Y$$
$$(2)\ Near_{B_1}(O) \subseteq Near_{B_2}(O) \subseteq ... \subseteq Near_F(O).$$

4.17. Definition. Let $\langle O, F\rangle$ be a perceptual system and $F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$.

(4.3) $\quad \sim_F \subseteq \sim_{B_{n-1}} \subseteq ... \subseteq \sim_{B_1}$

The relation (4.3) corresponding to (4.1) is called a chain of indiscernibility relations or briefly indiscernibility chain.

4.18. Remark. By using Definition 4.15 and Definition 4.17, we obtain $\bowtie_{\sim_{B_1}} \subseteq \bowtie_{\sim_{B_2}} \subseteq ... \subseteq \bowtie_{\sim_F}$. In fact, more than one indiscernibility chain can be formed. We can imagine this indiscernibility chain as a tree, i.e., a branching model which is formed by trunk, branch, thinner branch and so on, respectively. So we get a tree which has n features in the trunk and 1 feature in the thinnest branch.

From Proposition 4.11 we can give following proposition.

4.19. Proposition. *Let $\langle O, F\rangle$ be a perceptual system , $X, Y \subseteq O, F = B_n = \{\phi_1, \phi_2, ..., \phi_n\}$ and $\bowtie_{\sim_{B_1}} \subseteq \bowtie_{\sim_{B_2}} \subseteq ... \subseteq \bowtie_{\sim_F}$ nearness chain .Then ,*
$X \bowtie_{\sim_{B_1}} Y \subseteq X \bowtie_{\sim_{B_2}} Y \subseteq ... \subseteq X \bowtie_{\sim_F} Y$.

4.20. Remark. There is a nuance between $X \bowtie_F Y$ and $X \bowtie_{\sim_F} Y$. $X \bowtie_{\sim_F} Y$ implies that the sets X and Y near to each other with respect to only the $\sim_F$ indiscernebility relation in $\langle O, F\rangle$ perceptual system. However, $X \bowtie_F Y$ implies that the sets X and Y near to each other by means of Definition 2.10.

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