Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 83 – 91

Non existence of totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds

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Abstract

In present paper, after finding the conditions for the integrability of various distributions of a GCR-lightlike submanifold of indefinite Kenmotsu manifolds, we prove that there do not exist totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds other than totally contact geodesic GCR-lightlike submanifolds and moreover it is a totally geodesic GCR-lightlike submanifold.

2000 AMS Classification: 53C15, 53C40, 53C50.

Keywords: Indefinite Kenmotsu manifolds, GCR-lightlike submanifolds, totally contact umbilical GCR-lightlike submanifolds.

Received 16/03/2012 : Accepted 13/12/2013 Doi :10.5672/HJMS.20154490086

1. Introduction

Theory of lightlike submanifolds of semi-Riemannian manifolds is one of the most important topic of differential geometry since in this theory, the normal vector bundle intersects with the tangent bundle, contrary to classical theory of submanifolds. Therefore the theory of lightlike (degenerate) submanifolds becomes more interesting and remarkably different from the theory of non-degenerate submanifolds. In the development of the theory of lightlike submanifolds, Duggal and Bejancu [6] played a very crucial role. Since there is a significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnold [1], Maclane [11], Nazaikinskii et al. [12]). Therefore Duggal and Sahin [7] introduced contact CR-lightlike submanifolds and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. But there does not exist any inclusion relation between invariant and screen real submanifolds so a new

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class of submanifolds called, Generalized Cauchy-Riemann GCR-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds) was derived by Duggal and Sahin [8]. Recently Gupta and Sharfuddin [10], defined GCR-lightlike submanifold of indefinite Kenmotsu manifolds.

In present paper we further elaborate the theory of GCR-lightlike submanifold of indefinite Kenmotsu manifolds. In section 3, we find the conditions for the integrability of various distributions and for the distributions to define totally geodesic foliation in submanifold. In section 4, we study totally contact umbilical GCR-lightlike submanifolds and prove that there do not exist totally contact umbilical GCR-lightlike submanifolds of indefinite Kenmotsu manifolds other than totally contact geodesic GCR-lightlike submanifolds and moreover it is a totally geodesic GCR-lightlike submanifold.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [6] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\},$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM,$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

2.1. Theorem. *([6]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall \quad X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall \quad X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is linear a operator on M, known as a shape operator.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then using (2.2), the Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.8}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.9}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.10}$$

where $X \in \Gamma(TM), \quad N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (2.2)-(2.3) and (2.7)-(2.10), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Let P be the projection morphism of TM on $S(TM)$. Then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called structure vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \quad \bar{g}(X,V) = \eta(X), \tag{2.14}$$

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{2.15}$$

for $X, Y \in \Gamma(T\bar{M})$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Kenmotsu manifold if (see [4]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(\phi X, Y)V + \eta(Y)\phi X, \; \textit{and} \; \bar{\nabla}_X V = -X + \eta(X)V, \tag{2.16}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

3. Genralized Cauchy-Riemann (GCR)-Lightlike Submanifold

Calin[5], proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume characteristic vector V is tangent to M throughout this paper.

3.1. Definition. Let $(M, g, S(TM), S(TM^{\perp}))$ be a real lightlike submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM). \tag{3.1}$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S. \tag{3.2}$$

where D_0 is invariant non degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2). \tag{3.3}$$

A GCR-lightlike submanifold of indefinite Kenmotsu manifold is said to be proper if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_1 \neq \{0\}$.

Let Q, P_1, P_2 be the projection morphism on D, ϕL, ϕS respectively, therefore any $X \in \Gamma(TM)$ can be written as

$$X = QX + V + P_1 X + P_2 X. \tag{3.4}$$

Applying ϕ to (3.4), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{3.5}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(L)$ and $\omega P_2 X \in \Gamma(S)$, or, we can write (3.5), as

$$\phi X = fX + \omega X, \tag{3.6}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively. Similarly,

$$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)), \tag{3.7}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (3.5) and using (2.7)-(2.10) and (3.7), we have

$$D^l(X, \omega P_2 Y) = -\nabla^l_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y) + \eta(Y)\omega P_1 X, \tag{3.8}$$

$$D^s(X, \omega P_1 Y) = -\nabla^s_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y) + \eta(Y)\omega P_2 X, \tag{3.9}$$

for all $X, Y \in \Gamma(TM)$. By using Kenmotsu property of $\bar{\nabla}$ with (2.7) and (2.8), we have the following lemmas.

3.2. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

$$(\nabla_X f)Y = A_{\omega Y} X + Bh(X, Y) - g(\phi X, Y)V + \eta(Y) fX, \tag{3.10}$$

and

$$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY) + \eta(Y)\omega X, \tag{3.11}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \tag{3.12}$$

$$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \tag{3.13}$$

3.3. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

$$(\nabla_X B)U = A_{CU}X - fA_U X - g(\phi X, U)V, \tag{3.14}$$

and

$$(\nabla^t_X C)U = -\omega A_U X - h(X, BU), \tag{3.15}$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \tag{3.16}$$

$$(\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{3.17}$$

3.4. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then*

(A) *The distribution $D \oplus \{V\}$ is integrable, if and only if*

$$h(X, fY) = h(Y, fX), \quad \forall \quad X, Y \in \Gamma(D \oplus \{V\}). \tag{3.18}$$

(B) *The distribution $\bar{D}$ is integrable, if and only if*

$$A_{\phi Z}U = A_{\phi U}Z, \quad \forall \quad Z, U \in \Gamma(\bar{D}). \tag{3.19}$$

Proof: Using (3.8) and (3.9), we have $\omega\nabla_X Y = h(X, fY) - Ch(X, Y)$, for any $X, Y \in \Gamma(D \oplus \{V\}$. Here replacing X by Y and subtracting the resulting equation from this equation, we get $\omega[X, Y] = h(X, fY) - h(Y, fX)$, which proves (A).

Next from (3.10) and (3.12), we have $-f(\nabla_Z U) = A_{\omega U}Z + Bh(Z, U)$, for all $Z, U \in \Gamma(\bar{D})$. Then, similarly as above, we obtain $f[Z, U] = A_{\phi Z}U - A_{\phi U}Z$, which completes the proof of (B).

3.5. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof: Since $\bar{D} = \phi(L \perp S)$, therefore $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0,$$

for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$.
Using (2.7) and (2.16), we have

$$g(\nabla_X Y, \phi\xi) = -\bar{g}(\bar{\nabla}_X \phi Y, \xi) = -\bar{g}(h^l(X, fY), \xi), \tag{3.20}$$

$$g(\nabla_X Y, \phi W) = -\bar{g}(\bar{\nabla}_X \phi Y, W) = -\bar{g}(h^s(X, fY), W). \tag{3.21}$$

Hence, from (3.20) and (3.21) the assertion follows.

3.6. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution $\bar{D}$ does not define a totally geodesic foliation in M.*

Proof: We know that $\bar{D}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0,$$

for $X, Y \in \Gamma(\bar{D})$, $N \in \Gamma(ltr(TM))$, $Z \in \Gamma(D_0)$ and $N_1 \in \Gamma(L)$. But using (2.5) and (2.16), we obtain $g(\nabla_X Y, V) = g(\bar{\nabla}_X Y, V) = -g(Y, \bar{\nabla}_X V) = -g(Y, X)$, which may be non zero because $\phi S \subset \bar{D}$ is non degenrate. Hence the assertion follows.

3.7. Theorem. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the induced connection ∇ is metric connection, if and only if*

$$A^*_{\phi\xi}X - \nabla^{*t}_X\phi\xi \in \Gamma(\phi D_2 \perp D_1), \quad for \quad \xi \in \Gamma(D_1),$$
$$\nabla^*_X\phi\xi + h^*(X,\phi\xi) \in \Gamma(\phi D_2 \perp D_1), \quad for \quad \xi \in \Gamma(D_2),$$
$$h(X,\phi\xi) \in \Gamma(L \perp S)^{\perp} \quad and \quad A^*_\xi X \in \Gamma(\bar{D} \perp D_0 \perp \phi D_2),$$

for $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$.

Proof: For any $X \in \Gamma(TM)$ and $\xi \in \Gamma Rad(TM)$, using (2.16), we have

$$\phi\bar{\nabla}_X\xi = \bar{\nabla}_X\phi\xi + g(\phi X,\xi)V,$$

applying ϕ to both sides of above equation and then using (2.13) and (2.15), we obtain

$$\nabla_X\xi + h(X,\xi) = -\phi(\nabla_X\phi\xi + h(X,\phi\xi)) + g(A^*_\xi X, V)V. \tag{3.22}$$

Let $\xi \in \Gamma(D_1)$ then again using (2.13) in (3.22), we obtain

$$\nabla_X\xi + h(X,\xi) = -\phi(-A^*_{\phi\xi}X + \nabla^{*t}_X\phi\xi) - Bh(X,\phi\xi) - Ch(X,\phi\xi) + g(A^*_\xi X, V)V.$$

Equating tangential components of above equation both sides, we get

$$\nabla_X\xi = fA^*_{\phi\xi}X - f\nabla^{*t}_X\phi\xi - Bh(X,\phi\xi) + g(A^*_\xi X, V)V,$$

therefore $\nabla_X\xi \in \Gamma(RadTM)$, if and only if, $Bh(X,\phi\xi) = 0$, $fA^*_{\phi\xi}X - f\nabla^{*t}_X\phi\xi \in \Gamma(RadTM)$ and $g(A^*_\xi X, V) = 0$ or, if and only if,

$$h(X,\phi\xi) \in \Gamma(L \perp S)^{\perp}, \quad A^*_{\phi\xi}X - \nabla^{*t}_X\phi\xi \in \Gamma(\phi D_2 \perp D_1), \tag{3.23}$$

and

$$A^*_\xi X \in \Gamma(\bar{D} \perp D_0 \perp \phi D_2). \tag{3.24}$$

Similarly, let $\xi \in \Gamma(D_2)$ then using (2.12) in (3.22) and then compare the tangential components of the resulting equation, we obtain

$$\nabla_X\xi = -f\nabla^*_X\phi\xi - fh^*(X,\phi\xi) - Bh(X,\phi\xi) + (A^*_\xi X, V)V,$$

therefore $\nabla_X\xi \in \Gamma(RadTM)$, if and only if, $Bh(X,\phi\xi) = 0$, $f\nabla^*_X\phi\xi + fh^*(X,\phi\xi) \in \Gamma(RadTM)$ and $g(A^*_\xi X, V) = 0$ or, if and only if,

$$h(X,\phi\xi) \in \Gamma(L \perp S)^{\perp}, \quad \nabla^*_X\phi\xi + h^*(X,\phi\xi) \in \Gamma(\phi D_2 \perp D_1), \tag{3.25}$$

and

$$A^*_\xi X \in \Gamma(\bar{D} \perp D_0 \perp \phi D_2). \tag{3.26}$$

Hence from (3.23) to (3.26), the assertion follows.

4. Totally Contact Umbilical GCR-Lightlike Submanifolds

4.1. Definition. ([13]). If the second fundamental form h of a submanifold tangent to characteristic vector field V, of a Sasakian manifold $\bar{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical submanifold. M is called a totally contact geodesic submanifold if $\alpha = 0$ and a totally geodesic submanifold if $h = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{4.2}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{4.3}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^{\perp}))$.

4.2. Lemma. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\nabla_X X \in \Gamma(D \oplus \{V\})$, for any $X \in \Gamma(D)$.*

Proof: Since $\bar{D} = \phi(L \perp S)$ therefore $\nabla_X X \in \Gamma(D \oplus \{V\})$, if and only if,

$$g(\nabla_X X, \phi\xi) = g(\nabla_X X, \phi W) = 0,$$

for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Since M is totally contact umbilical GCR-lightlike submanifold therefore for any $X \in \Gamma(D)$, using (2.5), (2.7), (2.16) and (4.2), we obtain

$$\begin{aligned} g(\nabla_X X, \phi\xi) &= \bar{g}(\bar{\nabla}_X X, \phi\xi) = -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, \xi) = -\bar{g}(h^l(X, \phi X), \xi) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_L, \xi) = 0. \end{aligned} \tag{4.4}$$

Also

$$\begin{aligned} g(\nabla_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, \phi W) = -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, W) = -\bar{g}(h^s(X, \phi X), W) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_S, \xi) = 0. \end{aligned} \tag{4.5}$$

Hence using (4.4) and (4.5), the assertion follows.

4.3. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha \in \Gamma(L \perp S)$.*

Proof: Using (3.9), for any $X \in \Gamma(D_0)$, we obtain $h^s(X, fX) = \omega P_2 \nabla_X X + Ch^s(X, X)$, then using (4.3) we get $g(X, \phi X)\alpha_S = \omega P_2 \nabla_X X + g(X, X)C\alpha_S$. By virtue of the Lemma (4.2), we get $g(X, X)C\alpha_S = 0$, then the non degeneracy of the distribution D_0 implies that $C\alpha_S = 0$. Hence $\alpha_S \in \Gamma(S)$.

Similarly by using (3.8) and (4.2) we can prove $\alpha_L \in \Gamma(L)$. Hence $\alpha \in \Gamma(L \perp S)$.

4.4. Remark. Since $\alpha \in \Gamma(L \perp S)$ therefore for any $X \in D_0$ with (4.1), we have $h(X, X) = g(X, X)\alpha$, this implies that $h(X, X) \in \Gamma(L \perp S)$.

4.5. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha_L = 0$.*

Proof: Since M is a totally contact umbilical GCR-lightlike submanifold then, by direct calculations, using (2.7), (2.8) and (2.16) and then taking tangential parts of the resulting equation, we obtain

$$A_{\phi Z} Z + f\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) = 0,$$

where $Z \in \phi(S)$. Hence for $\xi \in \Gamma(D_2)$, we obtain

$$\bar{g}(A_{\phi Z} Z, \phi\xi) + \bar{g}(h^l(Z, Z), \xi) = 0,$$

then using (2.11), we get $\bar{g}(h^s(Z, \phi\xi), \phi Z) + \bar{g}(h^l(Z, Z), \xi) = 0$. Therefore using (4.2) and (4.3), we obtain$g(Z, Z)\bar{g}(\alpha_L, \xi) = 0$, then the non degeneracy of ϕS implies that $\alpha_L = 0$, which completes the proof.

4.6. Lemma. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\nabla_X \phi X = \phi \nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof: For any $X \in \Gamma(D_0)$ using (3.11) and (3.13), we have $\omega\nabla_X X = h(X, fX) - Ch(X, X)$. Since M be a totally contact umbilical therefore using (4.1), we have $\omega\nabla_X X = g(X, \phi X)\alpha - Ch(X, X)$, then using remark (4.4), we get $\omega\nabla_X X = 0$. Hence $\nabla_X X \in \Gamma(D)$. Let $Y \in \Gamma(D_0)$ then using(2.14) to (2.16), we obtain

$$g(\nabla_X \phi X, Y) = \bar{g}(\bar{\nabla}_X \phi X, Y) = \bar{g}(\phi\bar{\nabla}_X X, Y) = -g(\nabla_X X, \phi Y) = g(\phi\nabla_X X, Y),$$

this implies $g(\nabla_X \phi X - \phi\nabla_X X, Y) = 0$, then the non degeneracy of the distribution D_0 gives the result.

4.7. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $\alpha_S = 0$.*

Proof: Let $W \in \Gamma(S(TM^{\perp}))$ and $X \in \Gamma(D_0)$ the using (2.16), (4.1) and the Lemma (4.6), we have

$$\begin{aligned} \bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\ &= \bar{g}(\nabla_X \phi X, \phi W) + \bar{g}(h(X, \phi X), \phi W) \\ &= \bar{g}(\phi\nabla_X X, \phi W) \\ &= \bar{g}(\nabla_X X, W) \\ &= 0. \end{aligned} \tag{4.6}$$

Also using (4.3), we have

$$\begin{aligned} \bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, W) - \eta(W)\eta(\phi\bar{\nabla}_X X) \\ &= \bar{g}(\nabla_X X + h^s(X, X) + h^l(X, X), W) \\ &= \bar{g}(\nabla_X X, W) + \bar{g}(h^s(X, X), W) \\ &= g(X, X)g(\alpha_s, W). \end{aligned} \tag{4.7}$$

Therefore using (4.6) and (4.7), we get $g(X, X)g(\alpha_S, W) = 0$, then the non degeneracy of D_0 and $S(TM^{\perp})$ implies that $\alpha_S = 0$.

4.8. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then M is a totally contact geodesic GCR-lightlike submanifold.*

Proof: The result follows from the Theorems (4.5) and (4.7).

4.9. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then the induced connection ∇ is a metric connection on M.*

Proof: Using the Theorem (4.5), we have $\alpha_L = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$ therefore this implies that $h^l(X, V) = 0$, hence using (4.2) we obtain

$$h^l = 0. \tag{4.8}$$

Thus using the Theorem 2.2 in [6] at page 159, the induced connection ∇ becomes a metric connection on M.

4.10. Theorem. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then M is totally geodesic GCR-lightlike submanifold.*

Proof: Using the Theorem (4.7), we have $\alpha_S = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$ therefore this implies that $h^s(X, V) = 0$, hence using (4.3), we obtain

$$h^s = 0. \tag{4.9}$$

Thus using (4.8) and (4.9), the assertion follows.

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