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Oscillation criteria for solutions to nonlinear dynamic equations of higher order

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Abstract

In this paper using some new dynamic inequalities we present some oscillation results for higher order dynamic equation

$$\left\{r_{n-1}(t)\ \phi_{\alpha_{n-1}}\left[(r_{n-2}(t)(...(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}...)^{\Delta})^{\Delta}\right]\right\}^{\Delta} + p(t)\,\phi_{\gamma}\left(x\left(g\left(t\right)\right)\right) = 0,$$

on an unbounded time scale $\mathbb{T}$. Some new oscillation criteria are obtained using comparison techniques. Some applications illustrating our results are included.

Keywords: Asymptotic behavior, oscillation, higher order, dynamic equations, dynamic inequality, time scales.

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1. Introduction

This paper considers the oscillatory behavior of the higher order dynamic equation

(1.1)

$$\left\{r_{n-1}(t)\ \phi_{\alpha_{n-1}}\left[(r_{n-2}(t)(...(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}...)^{\Delta})^{\Delta}\right]\right\}^{\Delta} + p(t)\,\phi_{\gamma}\left(x\left(g\left(t\right)\right)\right) = 0,$$

on an unbounded time scale $\mathbb{T}$, where $\phi_\alpha(u) := |u|^{\alpha-1} u$, γ, $\alpha_i > 0$, $i = 1, 2, \ldots, n-1$, r_i, $i = 1, 2, \ldots, n-1$, are positive rd-continuous functions on $\mathbb{T}$, p is a positive rd-continuous function on $\mathbb{T}$, and $g : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\lim_{t\to\infty} g(t) = \infty$.

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We recall that a solution x of equation (1.1) is said to be nonoscillatory if there exists $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$; otherwise, it is said to be oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the literature many papers discuss the behavior of solutions for certain classes of dynamic equations; we refer the reader to [1, 3, 5, 9, 11, 12, 13, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 29] and the references cited therein. In particular these papers present oscillatory criteria and asymptotic behavior for first, second and third order dynamic equations on time scales and some interesting results were obtained for special cases of (1.1); see [10, 14, 16, 17, 28].

The aim of this paper is to present some new criteria for equation (1.1) . Our approach is to reduce the problem so that specific oscillation results for first, second and third order dynamic equations can be used for the arbitrary higher order case.

The paper will have four sections. In section 2, we state and prove some new dynamic inequalities. Section 3 uses comparison ideas to discuss (1.1). The last section illustrates the main results of our paper.

The theory of time scales was introduced by Stefan Hilger in his Ph. D. Thesis in 1988 in order to unify continuous and discrete analysis, see [22]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [6]). This new theory of these so-called "dynamic equations" not only unifies the corresponding theories for the differential equations and difference equations cases, but it also extends these classical cases to cases "in between". That is, we are able to treat the so-called $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$ and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the set of the harmonic numbers. The books on the subject of time scales by Bohner and Peterson [6], [7] summarizes and organizes much of time scale calculus.

For completeness, we recall some concepts on time scales. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided that h is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. Let $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \qquad \text{for all } s \in U,$$

then f is said to be differentiable at t, and we call α the delta derivative of f at t and denote it by $f^{\Delta}(t)$.

2. Dynamic Inequalities

In this section we state and prove some dynamic inequalities which will be used in the next section. Throughout this paper, we let

$$x \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \text{ for some } t_0 \in [0, \infty)_{\mathbb{T}},$$

and

$$x^{[i]} \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}),\ i = 1, ..., n,$$

where

$$x^{[i]}(t) := r_i(t)\ \phi_{\alpha_i}\left[\left(x^{[i-1]}(t)\right)^{\Delta}\right] \text{ with } r_n = \alpha_n = 1 \text{ and } x^{[0]} = x, \tag{2.1}$$

and $\phi_{\alpha_i}(u) := |u|^{\alpha_i - 1}\, u$, $\alpha_i > 0$, $i = 1, \ldots, n-1$, are constants, and $r_i \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ $i = 1, ..., n-1$, such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(s)\Delta s = \infty,\ i = 1, ..., n-1. \tag{2.2}$$

2.1. Lemma. *Let*

$$x(t) > 0 \quad \textit{and} \quad x^{[n]}(t) < 0 \tag{2.3}$$

eventually. Then there exists an integer $m \in \{0, \ldots, n\}$ *with* $m + n$ *odd such that*

$$x^{[k]}(t) > 0 \quad \textit{for} \quad k = 0, \ldots, m, \tag{2.4}$$

and

$$(-1)^{m+k}\, x^{[k]}(t) > 0 \quad \textit{for} \quad k = m, ..., n, \tag{2.5}$$

eventually.

Proof. Let

$$x(t) > 0 \quad \text{and} \quad x^{[n]}(t) < 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.6}$$

This implies that $x^{[i]}(t),\ i = 1, ..., n-1$, are eventually monotone and hence are of one sign. There are two possibilities:

(a) $x^{[k]}(t)$ and $x^{[k-1]}(t)$ have opposite signs eventually for $k = 1, \ldots, n$; or
(b) there exists a largest $m \in \{1, \ldots, n\}$ such that $x^{[m]}(t)x^{[m-1]}(t) > 0$ eventually.

If (a) holds, then (2.4) and (2.5) hold with $m = 0$ (note that for this case from (2.6) n must be odd).

Assume that (b) holds with $x^{[m]}(t) < 0$ and $x^{[m-1]}(t) < 0$ for $t \geq t_1$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then

$$\begin{aligned} x^{[m-2]}(t) &= x^{[m-2]}(t_1) + \int_{t_1}^{t} \phi^{-1}_{\alpha_{m-1}}\left[x^{[m-1]}(s)\right] r_{m-1}^{-1/\alpha_{m-1}}(s)\,\Delta s \\ &< x^{[m-2]}(t_1) + \phi^{-1}_{\alpha_{m-1}}\left[x^{[m-1]}(t_1)\right] \int_{t_1}^{t} r_{m-1}^{-1/\alpha_{m-1}}(s)\,\Delta s. \end{aligned}$$

From (2.2) with $i = m-1$, $\lim_{t\to\infty} x^{[m-2]}(t) = -\infty$. Hence $x^{[m-2]}(t) < 0$ eventually. By the same reasoning we see that $x^{[k]}(t) < 0$ eventually for $k = m-2, \ldots, 0$. This contradicts the assumption that $x(t)$ is eventually positive.

Assume that (b) holds with $x^{[m]}(t) > 0$ and $x^{[m-1]}(t) > 0$ eventually. Using an argument similar to the above, we see that $x^{[k]}(t) > 0$ eventually for $k = m-2, \ldots, 0$. Therefore, (2.4) and (2.5) hold with this m (From (2.5) (with $k = n$) we find that $m + n$ is an odd number). □

Let

$$\alpha[h, k] := \begin{cases} \alpha_h \cdots \alpha_k & h \leq k, \\ 1, & h > k, \end{cases}$$

and for a fixed $m \in \{0, \ldots, n-1\}$ and an integer $k \in \{m, ..., n-1\}$, define the functions $R_{i,j}(v,u)$, $j = 0, ..., k$ by the recurrence formula:

$$R_{k,j}(v,u) := \begin{cases} 1, & j = 0, \\ \int_u^v \left[\frac{R_{k,j-1}(v,s)}{r_{k-j+1}(s)}\right]^{1/\alpha_{k-j+1}} \Delta s, & j = 1, ..., k-m+1, \\ \int_u^v \left[\frac{R_{k,j-1}(s,u)}{r_{k-j+1}(s)}\right]^{1/\alpha_{k-j+1}} \Delta s, & j = k-m+2, ..., k. \end{cases}$$

2.2. Lemma. *Assume that (2.2) and (2.3) hold and $m \in \{0, \ldots, n\}$ is given in Lemma 2.1 such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then the following hold for $v \geq u \in [t_1, \infty)_{\mathbb{T}}$:*

(a) *for $j = m, \ldots, k$,*

$$(2.7) \qquad (-1)^{m+j} x^{[j]}(u) \geq (-1)^{m+k} \phi^{-1}_{\alpha[j+1,k]} \left(x^{[k]}(v)\right) R_{k,k-j}(v,u);$$

(b) *if $m \geq 1$, then for $j = 0, ..., m-1$,*

$$(2.8) \qquad x^{[j]}(v) \geq (-1)^{m+k} \phi^{-1}_{\alpha[j+1,k]} \left(x^{[k]}(v)\right) R_{k,k-j}(v,u),$$

where $k \in \{m, ..., n-1\}$.

Proof. (a) From (2.5), we have that $(-1)^{m+k} x^{[k]}$, $k = m, ..., n-1$, are positive, decreasing on $[t_1, \infty)_{\mathbb{T}}$. This shows that (2.7) holds for $j = k$. Then for $v \geq u \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} (-1)^{m+k} x^{[k]}(u) &\geq (-1)^{m+k} x^{[k]}(v) \\ &= (-1)^{m+k} \phi^{-1}_{\alpha[k+1,k]} \left(x^{[k]}(v)\right) R_{k,0}(v,u), \end{aligned}$$

which implies

$$(-1)^{m+k} \left(x^{[k-1]}(u)\right)^{\Delta} \geq (-1)^{m+k} \phi^{-1}_{\alpha[k,k]} \left(x^{[k]}(v)\right) \left(\frac{R_{k,0}(v,u)}{r_k(u)}\right)^{1/\alpha_k}.$$

Replacing u by s in the above inequality and then integrating it from u to $v \in [u, \infty)_{\mathbb{T}}$, we obtain that

$$\begin{aligned} & (-1)^{m+k} x^{[k-1]}(v) - (-1)^{m+k} x^{[k-1]}(u) \\ \geq\ & (-1)^{m+k} \phi^{-1}_{\alpha[k,k]} \left(x^{[k]}(v)\right) \int_u^v \left(\frac{R_{k,0}(v,s)}{r_k(s)}\right)^{1/\alpha_k} \Delta s \\ =\ & (-1)^{m+k} \phi^{-1}_{\alpha[k,k]} \left(x^{[k]}(v)\right) R_{k,1}(v,u). \end{aligned}$$

From (2.5), we obtain

$$\begin{aligned} (-1)^{m+k-1} x^{[k-1]}(u) &\geq (-1)^{m+k} x^{[k-1]}(v) - (-1)^{m+k} x^{[k-1]}(u) \\ &\geq (-1)^{m+k} \phi^{-1}_{\alpha[k,k]} \left(x^{[k]}(v)\right) R_{k,1}(v,u). \end{aligned}$$

This shows that (2.7) holds for $j = k-1$. Assume that (2.7) holds for some $j \in \{m+1, \ldots, k-1\}$. Then for $v \geq u \in [t_1, \infty)_{\mathbb{T}}$,

$$(-1)^{m+j} x^{[j]}(u) \geq (-1)^{m+k} \phi^{-1}_{\alpha[j+1,k]} \left(x^{[k]}(v)\right) R_{k,k-j}(v,u),$$

which implies

$$(-1)^{m+j} \left(x^{[j-1]}(u)\right)^{\Delta} \geq (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}(v)\right) \left(\frac{R_{k,k-j}(v,u)}{r_j(u)}\right)^{1/\alpha_j}.$$

Replacing u by s in the above inequality and then integrating it from u to $v \in [u,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
& (-1)^{m+j}\, x^{[j-1]}(v) - (-1)^{m+j}\, x^{[j-1]}(u) \\
\geq\ & (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) \int_u^v \left(\frac{R_{k,k-j}(v,s)}{r_j\,(s)} \right)^{1/\alpha_j} \Delta s \\
=\ & (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) R_{k,k-j+1}(v,u).
\end{aligned}$$

Then from (2.5), we have

$$(-1)^{m+j-1}\, x^{[j-1]}(u) \geq (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) R_{k,k-j+1}(v,u).$$

This shows that (2.7) holds for $j-1$. By induction, (2.7) holds for all $j = m, m+1, \ldots, k$.

(b) From Part (a) we have that for $j = m$

$$x^{[m]}(u) \geq (-1)^{m+k} \phi^{-1}_{\alpha[m+1,k]} \left(x^{[k]}\,(v) \right) R_{k,k-m}(v,u),$$

which implies

$$\left(x^{[m-1]}(u) \right)^{\Delta} \geq (-1)^{m+k} \phi^{-1}_{\alpha[m,k]} \left(x^{[k]}\,(v) \right) \left(\frac{R_{k,k-m}(v,u)}{r_m(u)} \right)^{1/\alpha_m}.$$

Replacing u by s in the above inequality and then integrating it from u to $v \in [u,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
x^{[m-1]}(v) &\geq x^{[m-1]}(v) - x^{[m-1]}(u) \\
&\geq (-1)^{m+k} \phi^{-1}_{\alpha[m,k]} \left(x^{[k]}\,(v) \right) \int_u^v \left(\frac{R_{k,k-m}(v,s)}{r_m(s)} \right)^{1/\alpha_m} \Delta s \\
&= (-1)^{m+k} \phi^{-1}_{\alpha[m,k]} \left(x^{[k]}\,(v) \right) R_{k,k-m+1}(v,u).
\end{aligned}$$

This shows that (2.8) holds for $j = m-1$. Assume that (2.8) holds for some $j \in \{1, \ldots, m-1\}$. Then for $v \geq u \in [t_1,\infty)_{\mathbb{T}}$,

$$\left(x^{[j-1]}(v) \right)^{\Delta} \geq (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) \left(\frac{R_{k,k-j}(v,u)}{r_j\,(v)} \right)^{1/\alpha_j}.$$

Replacing v by s in the above inequality and then integrating it from u to $v \in [u,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
x^{[j-1]}(v) &\geq x^{[j-1]}(v) - x^{[j-1]}(u) \\
&\geq \int_u^v (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(s) \right) \left(\frac{R_{k,k-j}(s,u)}{r_j\,(s)} \right)^{1/\alpha_j} \Delta s \\
&\geq (-1)^{m+k} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) \int_u^v \left(\frac{R_{k,k-j}(s,u)}{r_j\,(s)} \right)^{1/\alpha_j} \Delta s \\
&= (-1)^{m+k-1} \phi^{-1}_{\alpha[j,k]} \left(x^{[k]}\,(v) \right) R_{k,k-j+1}(v,u).
\end{aligned}$$

This shows that (2.8) holds for $j-1$. By induction, (2.8) holds for all $j = 0, 1, \ldots, m-1$. $\square$

3. Main Results

In this section we consider the asymptotic behavior of solutions of the nth-order nonlinear dynamic equation (1.1). From (2.1), Eq. (1.1) can be written as

$$x^{[n]}(t) + p\,(t)\,\phi_\gamma\,(x\,(g\,(t))) = 0. \tag{3.1}$$

3.1. Theorem. *Assume that* $n \in 2\mathbb{N}$ *and* (2.2) *holds. If for an integer* $k \in \{m, ..., n-1\}$,

$$(-1)^{m+k}\{r_n\phi_{\alpha_n}[(r_{n-1}(...(r_{k+1}\phi_{\alpha_{k+1}}[z^{\Delta}])^{\Delta}...)^{\Delta})^{\Delta}]\}(t) \\ +P_k\,(t)\,\phi_{\gamma/\alpha[1,k]}(z(g(t))) \leq 0, \tag{3.2}$$

where for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, $P_k(t) := p\,(t)\,R^{\gamma}_{k,k}(g(t), T)$, *has no eventually positive solution, then every solution of Eq.* (3.1) *is oscillatory.*

Proof. Assume that Eq. (3.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume that $x\,(g\,(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. From (3.1), we have that for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$x^{[n]}(t) = -p\,(t)\,\phi_\gamma\,(x\,(g\,(t))) < 0.$$

This implies that $x^{[i]}(t),\ i = 1, 2, ..., n-1$, are eventually monotone and hence are of one sign. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1, \ldots, n\}$ such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. From Lemma 2.2, Part (b) with $j = 0$, we get for $v \geq u \in [t_1, \infty)_{\mathbb{T}}$,

$$x(v) \geq (-1)^{m+k}\phi^{-1}_{\alpha[1,k]}\left(x^{[k]}\,(v)\right) R_{k,k}(v, u).$$

Setting $v = g(t)$ and $u = t_1$ gives

$$x(g(t)) \geq (-1)^{m+k}\phi^{-1}_{\alpha[1,k]}\left(x^{[k]}\,(g(t))\right) R_{k,k}(g(t), t_1).$$

Therefore (3.1) becomes

$$\begin{aligned} -x^{[n]}(t) &= p\,(t)\,\phi_\gamma\,(x\,(g\,(t))) \\ &\geq p\,(t)\,R^{\gamma}_{k,k}(g(t), t_1)\ \phi_{\gamma/\alpha[1,k]}\left((-1)^{m+k}x^{[k]}\,(g(t))\right) \\ &= P_k\,(t)\ \ \phi_{\gamma/\alpha[1,k]}\left((-1)^{m+k}x^{[k]}\,(g(t))\right), \end{aligned}$$

or

$$(-1)^{m+k}\{r_n\phi_{\alpha_n}[(r_{n-1}(...(r_{k+1}\phi_{\alpha_{k+1}}[z^{\Delta}])^{\Delta}...)^{\Delta})^{\Delta}]\}(t) \\ +P_k\,(t)\,\phi_{\gamma/\alpha[1,k]}(z(g(t))) \leq 0,$$

where $z(t) := (-1)^{m+k}x^{[k]}\,(t) > 0$, for an integer $k \in \{m, ..., n-1\}$. Thus (3.2) has an eventually positive solution, a contradiction. □

3.2. Theorem. *Assume that* $n \in 2\mathbb{N} - 1$ *and* (2.2) *holds. If* (3.2) *for an integer* $k \in \{m, ..., n-1\}$ *has no eventually positive solution and there is a function* τ *such that* $g(t) \leq \tau(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$ *and*

$$(-1)^{k}\{r_n\phi_{\alpha_n}[(r_{n-1}(...(r_{k+1}\phi_{\alpha_{k+1}}[z^{\Delta}])^{\Delta}...)^{\Delta})^{\Delta}]\}(t) \\ +Q_k(t)\phi_{\gamma/\alpha[1,k]}(z(\tau(t))) \leq 0, \tag{3.3}$$

for an integer $k \in \{0, ..., n-1\}$, *where* $Q_k(t) := p\,(t)\,R^{\gamma}_{k,k}(\tau(t), g(t))$, *has no eventually positive solution, then every solution of Eq.* (3.1) *is oscillatory.*

Proof. Assume that Eq. (3.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume that $x\left(g\left(t\right)\right) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. From (3.1), we have that for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$x^{[n]}(t) = -p\left(t\right)\phi_{\gamma}\left(x\left(g\left(t\right)\right)\right) < 0. \tag{3.4}$$

This implies that $x^{[i]}(t),\ i = 1, 2, ..., n-1$, are eventually monotone and hence are of one sign. It follows from Lemma 2.1 that there exists an even integer $m \in \{0, \dots, n\}$ such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(i) Assume that $m \geq 1$. Then the same argument as in the proof of Theorem 3.1 leads to a contradiction.

(ii) Assume that $m = 0$. From Lemma 2.2, Part (a) with $j = m = 0$, we get for $v \geq u \in [t_1, \infty)_{\mathbb{T}}$,

$$x(u) \geq (-1)^k \phi^{-1}_{\alpha[1,k]}\left(x^{[k]}\left(v\right)\right) R_{k,k}(v, u).$$

Setting $u = g(t)$ and $v = \tau(t)$ gives

$$x(g(t)) \geq \phi^{-1}_{\alpha[1,k]}\left((-1)^k x^{[k]}\left(\tau(t)\right)\right) R_{k,k}(\tau(t), g(t)).$$

Therefore (3.1) becomes

$$\begin{aligned} -x^{[n]}(t) &= p\left(t\right)\phi_{\gamma}\left(x\left(g\left(t\right)\right)\right) \\ &\geq p\left(t\right) R^{\gamma}_{k,k}(\tau(t), g(t))\phi_{\gamma/\alpha[1,k]}\left((-1)^k x^{[k]}\left(\tau(t)\right)\right) \\ &= Q_k(t)\phi_{\gamma/\alpha[1,k]}\left((-1)^k x^{[k]}\left(\tau(t)\right)\right), \end{aligned}$$

or

$$\begin{aligned} &(-1)^k \{r_n \phi_{\alpha_n}[(r_{n-1}(...(r_{k+1}\phi_{\alpha_{k+1}}[z^{\Delta}])^{\Delta}...)^{\Delta})^{\Delta}]\}(t) \\ &\qquad\qquad\qquad\qquad + Q_k(t)\phi_{\gamma/\alpha[1,k]}(z(\tau(t))) \leq 0, \end{aligned}$$

where $z(t) := (-1)^k x^{[k]}(t) > 0$, for an integer $k \in \{0, ..., n-1\}$. Thus (3.3) has an eventually positive solution, a contradiction. □

For further discussion, we introduce the following notation: For any $t \in \mathbb{T}$, define

$$p_j(t) := \begin{cases} p\left(t\right), & j = 0, \\ \left[\dfrac{1}{r_{n-j}(t)} \int_t^{\infty} p_{j-1}(s)\Delta s\right]^{1/\alpha_{n-j}}, & j = 1, 2, \dots, n-1, \end{cases}$$

provided that the improper integrals involved are convergent.

3.3. Theorem. *Assume that* $n \in 2\mathbb{N} - 1$, (2.2) *and*

$$\int_{t_0}^{\infty} p_{n-1}(s)\Delta s = \infty, \tag{3.5}$$

hold. If (3.2) *for an integer* $k \in \{m, ..., n-1\}$ *has no eventually positive solution, then every solution of Eq.* (3.1) *is oscillatory or tends to zero eventually.*

Proof. Assume that Eq. (3.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume that $x\left(g\left(t\right)\right) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 and Theorem 3.2 that there exists an odd integer $m \in \{1, \dots, n\}$ such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(i) Assume that $m \geq 1$. Then the same argument as in the proof of Theorem 3.1 leads to a contradiction.

(ii) Assume that $m = 0$. Since $x^{\Delta} < 0$ eventually, then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume that $l > 0$. Then

$$x(t),\ x(g(t)) > l_1 \quad \text{for } t \geq t_2 \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating (3.1) from t to $v \in [t, \infty)_{\mathbb{T}}$ and using (2.5) we get that

$$\begin{aligned} x^{[n-1]}(t) &\geq -x^{[n-1]}(v) + x^{[n-1]}(t) \\ &= \int_t^v p(s)\, \phi_\gamma(x(g(s)))\, \Delta s \geq c \int_t^v p(s)\, \Delta s. \end{aligned}$$

By taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \geq c \int_t^\infty p(s)\, \Delta s = c \int_t^\infty p_0(s)\, \Delta s.$$

Thus

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq c^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^\infty p_0(s)\, \Delta s\right]^{1/\alpha_{n-1}} = c^{1/\alpha_{n-1}}\ p_1(t). \tag{3.6}$$

Integrating the inequality (3.6) from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking limits as $v \to \infty$ and using the fact $x^{[n-2]} < 0$ eventually, we get

$$\begin{aligned} -x^{[n-2]}(t) &\geq c^{1/\alpha_{n-1}} \int_t^\infty p_1(s) \Delta s \\ &= c^{1/\alpha[n-1,n-1]} \int_t^\infty p_1(s) \Delta s. \end{aligned}$$

Continuing this process, we get

$$-x^{[1]}(t) \geq c^{1/\alpha[2,n-1]} \int_t^\infty p_{n-2}(s) \Delta s,$$

which implies

$$-x^{\Delta}(t) > c^{1/\alpha[1,n-1]} \left[\frac{1}{r_1(t)} \int_t^\infty p_{n-2}(s) \Delta s\right]^{1/\alpha_1} = c^{1/\alpha[1,n-1]} p_{n-1}(t).$$

Again, integrating the above inequality from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$ and noting that $x > 0$ eventually, we get

$$x(t_2) - x(t) \geq c^{1/\alpha[1,n-1]} \int_{t_2}^t p_{n-1}(s) \Delta s.$$

Using (3.5), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Therefore $\lim_{t\to\infty} x(t) = 0$. This completes the proof. $\square$

4. Applications

As direct consequences of Theorems 3.1, 3.2 and 3.3, we obtain the following comparison criteria for Eq. (3.1) when $k = n - 1$.

4.1. Corollary. *Assume that* (2.2) *holds and the first order dynamic inequality*

$$z^{\Delta}(t) + P_{n-1}(t)\, \phi_{\gamma/\alpha[1,n-1]}(z(g(t))) \leq 0, \tag{4.1}$$

where for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, $P_{n-1}(t) := p(t)\, R^{\gamma}_{n-1,n-1}(g(t), T)$, has no eventually positive solution.

(i) *If $n \in 2\mathbb{N}$, then every solution of Eq.* (3.1) *is oscillatory.*

(ii) *If* $n \in 2\mathbb{N}-1$ *and the first order dynamic inequality*

$$z^{\Delta}(t) + Q_{n-1}(t)\phi_{\gamma/\alpha[1,n-1]}(z(\tau(t))) \leq 0, \tag{4.2}$$

where $Q_n(t) := p(t)\, R^{\gamma}_{n-1,n-1}(\tau(t), g(t))$, *has no eventually positive solution, then every solution of Eq.* (3.1) *is oscillatory.*

4.2. Corollary. *Assume that* (2.2) *holds and the first order dynamic inequality* (4.1) *has no eventually positive solution.*

(i) *If* $n \in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*
(ii) *If* $n \in 2\mathbb{N}-1$ *and* (3.5) *holds, then every solution of Eq.* (3.1) *is oscillatory or tends to zero eventually.*

Using the main results of [29, 5] we get the following oscillation criteria of Eq. (3.1).

4.3. Corollary. *Let* $\gamma = \alpha[1, n-1]$, *and* $g(t) < t$ *and* $\tau(t) < t$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Assume that* (2.2) *holds and*

$$\limsup_{t\to\infty} \sup_{\lambda\in E_1} \{\lambda\, e_{-\lambda P_{n-1}}(t, g(t))\} < 1, \tag{4.3}$$

where

$$E_1 = \{\lambda : \lambda > 0,\ 1 - \lambda\, P_{n-1}(t)\mu(t) > 0,\ t \in \mathbb{T}\}.$$

(i) *If* $n \in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*
(ii) *If* $n \in 2\mathbb{N}-1$ *and*

$$\limsup_{t\to\infty} \sup_{\lambda\in E_2} \{\lambda\, e_{-\lambda Q_{n-1}}(t, \tau(t))\} < 1,$$

where

$$E_2 = \{\lambda : \lambda > 0,\ 1 - \lambda\, Q_{n-1}(t)\mu(t) > 0,\ t \in \mathbb{T}\},$$

then every solution of Eq. (3.1) *is oscillatory.*

4.4. Corollary. *Let* $\gamma = \alpha[1, n-1]$, *and* $g(t) < t$ *and* $\tau(t) < t$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Assume that* (2.2) *and* (4.3) *hold.*

(i) *If* $n \in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*
(ii) *If* $n \in 2\mathbb{N}-1$ *and* (3.5) *holds, then every solution of Eq.* (3.1) *is oscillatory or tends to zero eventually.*

4.5. Corollary. *Let* $\mathbb{T} = \mathbb{R}$, $\gamma = \alpha[1, n-1]$, *and* $g(t) < t$ *and* $\tau(t) < t$ *on* $[t_0, \infty)$. *Assume that* (2.2) *holds and*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} P_{n-1}(s)ds > \frac{1}{e}, \tag{4.4}$$

(i) *If* $n \in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*
(ii) *If* $n \in 2\mathbb{N}-1$ *and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} Q_{n-1}(s)ds > \frac{1}{e}$$

then every solution of Eq. (3.1) *is oscillatory.*

4.6. Corollary. *Let* $\mathbb{T} = \mathbb{R}$, $\gamma = \alpha[1, n-1]$, *and* $g(t) < t$ *and* $\tau(t) < t$ *on* $[t_0, \infty)$. *Assume that* (2.2) *and* (4.4) *hold.*

(i) *If* $n \in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*
(ii) *If* $n \in 2\mathbb{N}-1$ *and* (3.5) *holds, then every solution of Eq.* (3.1) *is oscillatory or tends to zero eventually.*

4.7. Corollary. *Let* $\mathbb{T}=\mathbb{Z}$, $\gamma=\alpha\,[1,n-1]\,,$ *and* $g(n)=\tau(n)=n-k$ *for* $n\in\mathbb{Z}$ *and* $k\in\mathbb{N}$. *Assume that* (2.2) *holds and*

$$(4.5)\qquad \liminf_{t\to\infty}\sum_{i=n-k}^{n-1}P_{n-1}(i)>\left(\frac{k}{k+1}\right)^{k+1},$$

(i) *If* $n\in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*

(ii) *If* $n\in 2\mathbb{N}-1$ *and*

$$\sum_{i=n-k}^{n-1}Q_{n-1}(i)>\left(\frac{k}{k+1}\right)^{k+1},$$

then every solution of Eq. (3.1) *is oscillatory.*

4.8. Corollary. *Let* $\mathbb{T}=\mathbb{Z}$, $\gamma=\alpha\,[1,n-1]\,,$ *and* $g(n)=\tau(n)=n-k$ *for* $n\in\mathbb{Z}$ *and* $k\in\mathbb{N}$. *Assume that* (2.2) *and* (4.5) *hold.*

(i) *If* $n\in 2\mathbb{N}$, *then every solution of Eq.* (3.1) *is oscillatory.*

(ii) *If* $n\in 2\mathbb{N}-1$ *and* (3.5) *holds, then every solution of Eq.* (3.1) *is oscillatory or tends to zero eventually.*

4.9. Remark. (1) For more oscillation criteria, see [4, 5, 8, 29].

(2) When $n=3$, the result in Corollary 4.3 is related to a problem posed in [3, Remark 3.3] when $\tau(t)<t$ for $t\geq t_0\in\mathbb{T}$.

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References

[1] Adıvar, M., Akın E. and Higgins R. *Oscillatory behavior of solutions of third-order delay and advanced dynamic equations*, Journal of Inequalities and Applications **2014**, 2014:95, 16 pp.

[2] Grover, L.K. and Kaur, P. *An improved estimator of the finite population mean in simple random sampling*, Model Assisted Statistics and Applications **6** (1), 47-55, 2011.

[3] Agarwal, R.P., Bohner, M., Li, T. and Zhang, C. *Hille and Nehari type criteria for third-order delay dynamic equations*, Journal of Difference Equations and Applications **19** (10), 1563-1579, 2013.

[4] Binggen, Z., Xinzhou, Y. and Xueyan, L. *Oscillation criteria of certain delay dynamic equations on time scales*, Journal of Difference Equations and Applications **11** (10), 933-946, 2005.

[5] Bohner, M. *Some oscillation criteria for first order delay dynamic equations*, Far East J. Appl. Math. **18** (3), 289-304, 2005.

[6] Bohner, M. and Peterson, A. Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[7] Bohner, M. and Peterson, A., editors, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[8] Bohner, M., Karpuz, B. and Öcalan, Ö. *Iterated oscillation criteria for delay dynamic equations of first order*, Advances in Difference Equations Volume **2008**, Article ID 458687, 12 pp.

[9] Chen, D. *Oscillation and asymptotic behavior of solutions of certain third-order nonlinear delay dynamic equations*, Theoretical Mathematics & Applications **3**, 19-33, 2013.

[10] Erbe, L., Baoguo and Peterson, A. Oscillation of nth order superlinear dynamic equations on time scales, Rocky Mountain Journal of Mathematics **41** (2), 471-491, 2011.

[11] Erbe, L., Hassan, T.S., Peterson, A. and Saker, S.H. *Interval oscillation criteria for forced second-order nonlinear delay dynamic equations with oscillatory potential*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **17**(4), 533-542, 2010.

[12] Erbe, L., Hassan, T.S. and Peterson, A. *Oscillation of second order functional dynamic equations*, International Journal of Difference Equations **5**(2), 175-193, 2010.

[13] Erbe, L., Karpuz, B. and Peterson, A. *Kamenev-type oscillation criteria for higher order neutral delay dynamic equations*, International Journal Difference Equations **6** (1), 1-16, 2011.

[14] Erbe, L., Mert, R., Peterson, A. and Zafer, A. *Oscillation of even order nonlinear delay dynamic equations on time scales*, Czechoslovak Mathematical Journal **63** (138), 265-279, 2013.

[15] Erbe, L. and Hassan, T.S. *New oscillation criteria for second order sublinear dynamic equations*, Dynamic Systems and Applications **22** (1), 49-63, 2013.

[16] Grace, S.R. and Hassan, T.S. *Oscillation criteria for higher order nonlinear dynamic equations*, Mathematische Nachrichten **287** (14-15), 1659-1673, 2014.

[17] Grace, S.R., Agarwal, R.P. and Zafer, A. *Oscillation of higher order nonlinear dynamic equations on time scales*, Advances in Difference Equations **2012**, 2012:67, 18 pp.

[18] Han, Z., Li, T., Sun S. and Zhang, M. *Oscillation behavior of solutions of third-order nonlinear delay dynamic equations on time scales*, Communications of the Korean Mathematical Society **26** (3), 499-513, 2011.

[19] Hassan, T.S. *Oscillation criteria for second-order nonlinear dynamic equations*, Advances in Difference Equations **2012:171**, 13 pp, 2012.

[20] Hassan, T.S. *Interval oscillation for second order nonlinear differential equations with a damping term*, Serdica Mathematical Journal **34** (4), 715-732, 2008.

[21] Hassan, T.S. *Asymptotic behavior of solutions of second order nonlinear dynamic equations*, Dynamic Systems and Applications **23** (2-3), 179-187, 2014.

[22] Hilger, S. *Analysis on measure chains — a unified approach to continuous and discrete calculus*, Results in Mathematics **18** (1-2), 18-56, 1990.

[23] Karpuz, B. *Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients*, Electronic Journal of Qualitative Theory of Differential Equations **2009**, No. 34, 14 pp.

[24] Kiguradze, I. T. *On oscillatory solutions of some ordinary differential equations*, Soviet Mathematics Doklady 144, 33-36, 1962.

[25] Mert, R. *Oscillation of higher order neutral dynamic equations on time scales*, Advances in Difference Equations **2012**, 2012:68, 11 pp.

[26] Saker, S.H. *Oscillation criteria of second-order half-linear dynamic equations on time scales*, Journal of Computational and Applied Mathematics **177** (2), 375-387, 2005.

[27] Sun, Y. and Hassan, T.S. *Comparison criteria for odd order forced nonlinear functional neutral dynamic equations*, Applied Mathematics and Computation, **251**, 387–395, 2015.

[28] Sun, T., Yu, W. and Xi, H. *Oscillatory behavior and comparison for higher order nonlinear dynamic equations on time scales*, Journal of Applied Mathematics & Informatics **30** (1-2), 289-304, 2012.

[29] Zhang, B. G. and Deng, X. *Oscillation of delay differential equations on time scales*, Mathematical and Computer Modeling **36** (11-13), 1307–1318, 2002.