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A new efficient multi-parametric homotopy approach for two-dimensional Fredholm integral equations of the second kind

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Abstract

In this paper, a new multi-parametric homotopy approach is proposed to find the approximate solution of linear and non-linear two-dimensional Fredholm integral equations of the second kind. In this framework, convergence of the proposed approach for these types of equations is investigated. This homotopy contain two auxiliary parameters that provide a simple way of controlling the convergence region of series solution. The results of present method are compared with Adomian decomposition method (ADM) results which provide confirmation for the validity of proposed approach. Two examples are presented to illustrate the accuracy and effectiveness of the proposed approach.

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1. Introduction

Homotopy analysis method (HAM) has been presented by Liao [1, 2] to obtain the analytical solutions for various nonlinear problems. There are many eminent researchers that deal with the Homotopy analysis method such as, Alomari et al. [3,4] applied the HAM to study the delay differential equations and the hyperchaotic Chen system, Turkyilmazoglu [5] constructed the convergent series solutions of strongly nonlinear problems via HAM, Gupta [6] implemented the HAM to obtain the approximate analytical solution of nonlinear fractional diffusion equation, Abbasbandy applied the uni-parametric homotopy method to solve the Fredholm integral equations [7], Marinca and Herisanu [8, 9] proposed the OHAM for fluid mechanics problem and nonlinear differential equations. Recently Turkyilmazoglu [10] constructed the explicit analytic solution of the Thomas-Fermi equation thorough a new kind of homotopy analysis technique. He used new base functions

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and auxiliary linear operator to form a better homotopy method.The basic motivation of the present study is proposed a new multi-parametric homotopy approach to develop an approximate solution for the linear and nonlinear two-dimensional Fredholm integral equations. The present method is much easier to implement as compared with the decomposition method where huge complexities are involved. Moreover, we prove the convergence of the solution for two-dimensional Fredholm integral equations. It is shown that the approximate solutions given by the proposed approach are more accurate than the numerical solution given by the traditional homotopy analysis method (THAM) and the Adomian decomposition method (ADM) [11, 12].

2. Description of approach

To illustrate the procedure, consider the following second kind of two-dimensional Fredholm integral equation:

$$F(u(t,x)) = u(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) N(u(s,\xi)) d\xi ds, \ (t,x) \in D, \tag{2.1}$$

where $f(t,x)$ and $K(t,s,x,\xi)$ are analytical functions on $D = L^2([a,b] \times [c,d])$ and $E = D \times D$, respectively.
We choose $u_0(t,x) = f(t,x)$ as initial approximation guess for simplicity, in order to obtain convergent series solutions to two-dimensional Fredholm integral equation (2.1), we first construct the zeroth order deformation equation

$$\begin{aligned} (1 - A(q;\varpi_1))[\varphi(t,x;q) &- u_0(t,x)] = B(q;\hbar)[\varphi(t,x;q) - f(t,x) \\ &- \int_a^b \int_c^d K(t,s,x,\xi) N(\varphi(s,\xi;q)) d\xi ds], \end{aligned} \tag{2.2}$$

where

$$A(q;\varpi) = (1-\varpi)\sum_{j=1}^{\infty} \varpi^{j-1} q^j, \ \ |\varpi| < 1, B(q;\hbar) = q\hbar, \ \hbar \neq 0. \tag{2.3}$$

Due to Taylor's theorem, we can write

$$\varphi(t,x;q) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x) q^j, \tag{2.4}$$

where

$$u_j(t,x) = \frac{1}{j!} \frac{\partial^j \varphi(t,x;q)}{\partial q^j}\Big|_{q=0}. \tag{2.5}$$

The convergence of series (2.4) depends upon $\hbar$ and ϖ. Assume that $\hbar$ and ϖ are properly chosen so that the power series of (2.4) converges at $q = 1$, then we have under these assumption the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x). \tag{2.6}$$

By differentiating (2.2) m times with respect to q, then dividing the equation by $m!$ and setting $q = 0$, the mth-order deformation equation is formulated as follows

$$u_m(t,x) - \sum_{k=1}^{m-1} (1-\varpi)\varpi^{m-k-1} u_k(t,x) = \hbar H_m(u_0(t,x), ..., u_{m-1}(t,x)), \tag{2.7}$$

where

$$H_m = u_{m-1}(t,x) - (1-\chi_m)f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) \frac{\partial^{m-1} N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}} \mid_{q=0} d\xi ds, \tag{2.8}$$

$$u_m(t,x) = \frac{\partial^m \varphi(t,x;q)}{m!\partial q^m} \mid_{q=0}, \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \tag{2.10}$$

The mth-order deformation equations (2.7) are linear in principle. The code is developed by using symbolic computation software MAPLE. Then, the Nth-order approximate solution of (2.7) can be written as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x). \tag{2.11}$$

If $\varpi = 0$ the mth-order deformation equation defined by (2.7) becomes

$$u_1(t,x) = \hbar[u_0(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) N(u_0(s,\xi;q)) d\xi ds], \tag{2.12}$$

and

$$u_m(t,x) - u_{m-1}(t,x) = \hbar[u_{m-1}(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) \frac{\partial^{m-1} N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}} \mid_{q=0} d\xi ds], \tag{2.13}$$

Then, we can derive the following remarks instantly.

Remark1: The value $\varpi = 0$ reduces the present approach to the traditional HAM.

Remark2: The values $\hbar = -1$ and $\varpi = 0$ reduce the present approach to the ADM.

2.1. Convergence theorems.

Theorem 1. If the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{2.14}$$

is convergent, then we have the following statement

$$\sum_{m=1}^{\infty} H_m = 0. \tag{2.15}$$

Proof.
Since the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{2.16}$$

is convergent, we have

$$\lim_{m \to \infty} u_m(t,x) = 0. \tag{2.17}$$

Using the the left-hand side of (2.6) satisfies

$$\begin{aligned}\sum_{m=1}^{\infty}\big[u_m(t,x) &- \sum_{k=1}^{m-1}(1-\varpi)\varpi^{m-k-1}u_k(t,x)\big]\\ &= u_1(t,x)\\ &+ u_2(t,x)-(1-\varpi)u_1(t,x)\\ &+ u_3(t,x)-(1-\varpi)\varpi u_1(t,x)-(1-\varpi)u_2(t,x)\\ &\vdots\\ &= (1-(1-\varpi)\sum_{j=0}^{\infty}\varpi^j)\sum_{j=0}^{\infty}u_j(t,x)=0.\end{aligned} \tag{2.18}$$

Then, from (2.7) and (2.18) we have

$$\sum_{m=1}^{\infty}H_m=0. \tag{2.19}$$

Theorem 2. Assume that the operator $N[u(t,x)]$ be contraction and the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \tag{2.20}$$

is convergent, it must be the solution of two-dimensional Fredholm integral equation.

Proof.
Let

$$\varepsilon(t,x;q)=\varphi(t,x;q)-f(t,x)-\int_a^b\int_c^d K(t,s,x,\xi)N(\varphi(s,\xi;q))d\xi ds. \tag{2.21}$$

Using Taylor's series around $q=0$ for $\varepsilon(t,x;1)$, we have

$$\begin{aligned}\varepsilon(t,x;1) &= \sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m\varphi(t,x;q)}{\partial q^m}|_{q=0}-f(t,x)\\ &- \int_a^b\int_c^d K(t,s,x,\xi)\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m N(\varphi(s,\xi;q))}{\partial q^m}|_{q=0}d\xi ds.\end{aligned} \tag{2.22}$$

If the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \tag{2.23}$$

is convergent, then the series

$$\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m N(\varphi(t,x;q))}{\partial q^m}|_{q=0}, \tag{2.24}$$

will converge to $N[u(t,x)]$ (see [12]).
Now, by using theorem 1 we have

$$\varepsilon(t,x;1)=u(t,x)-f(t,x)-\int_a^b\int_c^d K(t,s,x,\xi)N(u(s,\xi))d\xi ds=0. \tag{2.25}$$

This completes the proof.

3. Main results

The Nth-order approximation of the solution $u(t,x)$ can be expressed as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x), \tag{3.1}$$

which is mathematically dependent upon the convergence-control parameters $\hbar$ and ϖ. In our work for optimal values of $\hbar$ and ϖ, we use a technique that has been shown to produce a fast converging approximation. In principle, the technique seeks to minimize the exact residual error (ERE) of (2.1) at the Nth-order approximation. The ERE is given by

$$\widehat{E}_M(\hbar,\varpi) = \int_a^b \int_c^d (F(U_N(s,\xi)))^2 d\xi ds, \tag{3.2}$$

In practice, however, the evaluation of $\widehat{E}_M(\hbar,\varpi)$ tends to be time-consuming. A simpler alternative consists of calculating the averaged residual error (ARE). We use here the ARE defined by

$$E_M^n(\hbar,\varpi) = \frac{(b-a)(d-c)}{n^2} \sum_{j=0}^{n} \sum_{i=0}^{n} (F(U_N(t_i,x_j)))^2, \tag{3.3}$$

where

$$t_i = \frac{(b-a)i}{n},\ i = 1,2,...,n,\ \ x_j = \frac{(d-c)j}{n},\ j = 1,2,...,n. \tag{3.4}$$

At the Nth-order of approximation, the ARE contain two unknown convergence-control parameters, whose "optimal" values are determined by solving the nonlinear algebraic equations

$$\frac{\partial E_M^n}{\partial \hbar} = 0,\ \frac{\partial E_M^n}{\partial \varpi} = 0. \tag{3.5}$$

In this section, two different examples of two-dimensional integral equations are employed to illustrate the validity of present approach which is described in Section 2. The convergence, accuracy and efficiency of this approach are investigated by comparing it with the THAM and the ADM.

3.1. Example 1. Consider the following nonlinear two-dimensional integral equation

$$u(t,x) = x\sin(\pi t) - \frac{x}{6} + \int_0^1 \int_0^1 (x + \cos(\pi s))u^2(s,\xi) d\xi ds, \tag{3.6}$$

with the exact solution

$$u(t,x) = x\sin(\pi t). \tag{3.7}$$

For $\hbar \neq 0$ and $\varpi = 0$, our approach gives the "optimal" value of the convergence-control parameter $\hbar \neq 0$ by solving the equation $\frac{dE_4^{20}}{dh} = 0$, which leads to $\hbar = -1.456$ with the corresponding minimum ARE $E_4^{20} = 5.483E-7$. For $\hbar \neq 0$ and $\varpi \neq 0$, we obtain the "optimal" values of $\hbar = -1.521$ and $\varpi = -0.148$ by solving the algebraic equations $\frac{dE_4^{20}}{dh} = 0$ and $\frac{dE_4^{20}}{d\varpi} = 0$, which gives with the corresponding minimum ARE $E_4^{20} = 4.458E-22$.
For comparison the solution series given by the present approach with the exact solution, we report the absolute error which is defined by

$$|e_N(t,x)| = |u(t,x) - U_N(t,x)|.$$

In Table 1, we compared the present approach with the ADM. The approximate solutions given by the present approach are more accurate than the solution given by the ADM, as shown in Table 1.

Table 1. Absolute errors of the proposed approach and ADM (example 1).

convergence-control parameters $(\hbar, \varpi)$			
$t = x$	$(-1.521, -0.148)$ *with seven terms*	$(-1.456, 0)$ *with seven terms*	*ADM* *with seven terms*
1	$1.531E-4$	$3.996E-4$	$2.503E-3$
$\frac{1}{2}$	$7.651E-5$	$1.998E-4$	$1.252E-3$
$\frac{1}{2^2}$	$3.826E-5$	$9.999E-5$	$6.258E-4$
$\frac{1}{2^3}$	$1.913E-5$	$4.995E-5$	$3.129E-4$
$\frac{1}{2^4}$	$9.567E-6$	$2.497E-5$	$1.565E-4$
$\frac{1}{2^5}$	$4.783E-6$	$1.249E-5$	$7.823E-5$
$\frac{1}{2^6}$	$2.392E-6$	$6.244E-6$	$3.911E-5$

Table 2. Absolute errors of the proposed approach and ADM (example 2).

convergence-control parameters $(\hbar, \varpi)$			
$t = x$	$(-0.756, -0.755)$ *with nine terms*	$(-1.581, 0)$ *with nine terms*	*ADM* *with nine terms*
1	$3.712E-6$	$1.109E-3$	$1.699E-2$
$\frac{1}{2}$	$2.520E-6$	$9.600E-4$	$1.161E-2$
$\frac{1}{2^2}$	$1.929E-6$	$8.953E-4$	$8.914E-3$
$\frac{1}{2^3}$	$1.625E-6$	$8.629E-4$	$7.567E-3$
$\frac{1}{2^4}$	$1.481E-6$	$8.467E-4$	$6.894E-3$
$\frac{1}{2^5}$	$1.401E-6$	$8.386E-4$	$6.557E-3$
$\frac{1}{2^6}$	$1.366E-6$	$8.345E-4$	$6.389E-3$

3.2. Example 2. Consider the following linear two-dimensional integral equation

$$u(t,x) = xe^{-t} + (\frac{e^{-2}}{4} - \frac{1}{4})x + \frac{e^{-2}}{6} - \frac{1}{6} + \int_0^1 \int_0^1 (x+\xi)e^{-(2t-s)}u(s,\xi)d\xi ds, \tag{3.8}$$

with the exact solution

$$u(t,x) = xe^{-t}. \tag{3.9}$$

For $\hbar \neq 0$ and $\varpi = 0$, the present approach reduces to traditional HAM and E_8^{20} has the minimum $1.393E-7$ at the "optimal" value $\hbar = -1.581$. For $\hbar \neq 0$ and $\varpi \neq 0$, the optimal convergence occurs at $\hbar = -1.155$ and $\varpi = -0.306$ and has a ARE of $E_8^{20} = 9.892E-22$.
The approximate solutions given by the present approach are more accurate than the solution given by the ADM and the THAM, as shown in Table 2.

4. Concluding remarks

In this paper, we have proposed a method for solving two-dimensional Fredholm integral equations. The results have been compared with the THAM and ADM solutions to show the efficiency of our technique. By introducing this method for two-dimensional Fredholm integral equations, the following observations have been made:
(i) This approach contains two convergence-control parameters which provide us a simple way to adjust and control the convergence region and rate of the obtained series solution.
(ii) The obtained results elucidate the very fast convergence of present approach, which does not need higher-order of approximation.
(iii) All the given examples reveal that the multi-parametric homotopy yields a very effective and convenient approach to the approximate solutions of two-dimensional Fredholm integral equations.

(iv) The ADM cannot give better results than the present approach.
(v) In fact, the THAM and ADM are special cases of present method.
In conclusion, a new multi-parametric homotopy approach may be considered as a nice refinement in existing numerical techniques.

References

[1] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PHD thesis, Shanghai Jiao Tong University (1992).
[2] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/ CRC Press, Boca Raton (2003).
[3] A.K. Alomari, M.S.M. Noorani and R. Nazar, Solution of delay differential equation by means of homotopy analysis method, Acta Applicandae Mathematicae. 108 (2009) 395-412.
[4] A.K. Alomari, M.S.M. Noorani and R. Nazar, Homotopy approach for the hyperchaotic Chen system, Physica Scripta 81 (2010) 045005.
[5] M. Turkyilmazoglu, A note on the homotopy analysis method, Appl. Math. Letts. 23 (2010) 1226-1230.
[6] P. K. Gupta, An approximate analytical solution of nonlinear fractional diffusion equation by homotopy analysis method, Int. J. Phys. Sci. 6 (2011) 7721-7728.
[7] S. Abbasbandy, E. Shivanian, A new analytical technique to solve Fredholms integral equations, Numer. Algorithms. 56 (2011) 27-43.
[8] V. Marinca, N. Herisanu, C. Bota, B. Marinca, An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate, Appl. Math. Lett. 22 (2009)245-251.
[9] V. Marinca, N. Herisanu, Comments on "A one-step optimal homotopy analysis method for nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3735-3739.
[10] M. Turkyilmazoglu, Solution of the Thomas Fermi equation with a convergent approach, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012) 4097-4103.
[11] P. J. Rebelo, An approximate solution to an initial boundary value problem: Rakib-Sivashinsky equation, Int. J. Comput. Math. 89 (2012) 881-889
[12] Y. Cherruault, Convergence of Adomians method, Kybernetes 18 (1989) 31-38.