Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 823 – 837

# Restricted hom-Lie algebras

Baoling Guan * and Liangyun Chen †

**Abstract**

The paper studies the structure of restricted hom-Lie algebras. More specifically speaking, we first give the equivalent definition of restricted hom-Lie algebras. Second, we obtain some properties of $p$-mappings and restrictable hom-Lie algebras. Finally, the cohomology of restricted hom-Lie algebras is researched.

*2000 AMS Classification:* 17A60, 17B56, 17B50

**Keywords:** Restricted hom-Lie algebras; restrictable hom-Lie algebras; $p$-mapping; cohomology.

Received 25/09/2013 : Accepted 21/04/2014 Doi : 10.15672 /HJMS.2015449439

## 1. Introduction

The concept of a restricted Lie algebra is attributable to N. Jacobson in 1943. It is well known that the Lie algebras associated with algebraic groups over a field of characteristic $p$ are restricted Lie algebras [14]. Now, restricted theories attract more and more attentions. For example: restricted Lie superalgebras[6], restricted Lie color algebras[2], restricted Leibniz algebras[4], restricted Lie triple systems[8] and restricted Lie algebras [5] were studied, respectively.

However, The notion of hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov in [7] as part of a study of deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Some q-deformations of the Witt and the Virasoro algebras have the structure of a hom-Lie algebra [7]. Because of close relation to discrete and deformed vector fields and differential calculus [7, 9, 10], hom-Lie algebras are widely studied recently [1, 3, 11, 12, 16, 17, 18]. As a natural generalization of a restricted Lie algebra, it seems

*School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email:
†College of Sciences, Qiqihar University, Qiqihar 161006, China
Email:chenly640@nenu.edu.cn Corresponding author.
Supported by NNSF of China (No.11171055), Natural Science Foundation of Jilin province (No. 201115006), Scientific Research Foundation for Returned Scholars, Ministry of Education of China and the Fundamental Research Funds for the Central Universities(No. 11SSXT146).

desirable to investigate the possibility of establishing a parallel theory for restricted hom-Lie algebras. As is well known, restricted Lie algebras play predominant roles in the theories of modular Lie algebras [15]. Analogously, the study of restricted hom-Lie algebras will play an important role in the classification of the finite-dimensional modular simple hom-Lie algebras.

The paper study the structure of restricted hom-Lie algebras. Let us briefly describe the content and setup of the present article. In Sec. 2, the equivalent definition of restricted hom-Lie algebras is given. In Sec. 3, we obtain some properties of $p$-mappings and restrictable hom-Lie algebras. In Sec. 4, we research the cohomology of restricted hom-Lie algebras.

In the paper, $\mathbb{F}$ is a field of prime characteristic. Let $L$ denote a finite-dimensional restricted hom-Lie algebra over $\mathbb{F}$.

**1.1. Definition.** [14] Let $L$ be a Lie algebra over $\mathbb{F}$. A mapping $[p] : L \to L, a \mapsto a^{[p]}$ is called a $p$-mapping, if

(1) $\mathrm{ad}a^{[p]} = (\mathrm{ad}a)^p,\ \forall a \in L,$

(2) $(ka)^{[p]} = k^p a^{[p]},\ \forall a \in L, k \in \mathbb{F},$

(3) $(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum\limits_{i=1}^{p-1} s_i(a,b),$

where $(\mathrm{ad}(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum\limits_{i=1}^{p-1} i s_i(a,b) \otimes X^{i-1}$ in $L \otimes_{\mathbb{F}} \mathbb{F}[X], \forall a, b \in L$, The pair $(L, [p])$ is referred to as a restricted Lie algebra.

**1.2. Definition.** [13] (1) A hom-Lie algebra is a triple $(L, [\cdot,\cdot]_L, \alpha)$ consisting of a linear space $L$, a skew-symmetric bilinear map $[\cdot,\cdot]_L : \Lambda^2 L \to L$ and a linear map $\alpha : L \to L$ satisfying the following hom-Jacobi identity:

$$[\alpha(x), [y,z]_L]_L + [\alpha(y), [z,x]_L]_L + [\alpha(z), [x,y]_L]_L = 0$$

for all $x, y, z \in L$;

(2) A hom-Lie algebra is called a multiplicative hom-Lie algebra if $\alpha$ is an algebraic morphism, i.e., for any $x, y \in L$, we have $\alpha([x,y]_L) = [\alpha(x), \alpha(y)]_L$;

(3) A sub-vector space $\eta \subset L$ is called a hom-Lie subalgebra of $(L, [\cdot,\cdot]_L, \alpha)$ if $\alpha(\eta) \subset \eta$ and $\eta$ is closed under the bracket operation $[\cdot,\cdot]_L$, i.e., $[x,y]_L \in \eta$ for all $x, y \in \eta$;

(4) A sub-vector space $\eta \subset L$ is called a hom-Lie ideal of $(L, [\cdot,\cdot]_L, \alpha)$ if $\alpha(\eta) \subset \eta$ and $[x,y]_L \in \eta$ for all $x \in \eta, y \in L$.

## 2. The equivalent definition of restricted hom-Lie algebras

Let $(L, [\cdot,\cdot]_L, \alpha)$ be a multiplicative hom-Lie algebra over $\mathbb{F}$. For $c \in L$ satisfying $\alpha(c) = c$, we define $\mathrm{ad}c(a) := [\alpha(a), c]$. Put $L_0 := \{x | \alpha(x) \neq x\} \cup \{0\}$ and $L_1 := \{x | \alpha(x) = x\}$. Then $L = L_0 \cup L_1$ and $L_1$ is a hom-Lie subalgebra of $L$.

**2.1. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a multiplicative hom-Lie algebra over $\mathbb{F}$. A mapping $[p] : L_1 \to L_1, a \mapsto a^{[p]}$ is called a $p$-mapping, if

(1) $[\alpha(y), x^{[p]}] = (\mathrm{ad}x)^p(y),\ \forall x \in L_1, y \in L,$

(2) $(kx)^{[p]} = k^p x^{[p]},\ \forall x \in L_1, k \in \mathbb{F},$

(3) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y),$

where $(\mathrm{ad}(x \otimes X + y \otimes 1))^{p-1}(x \otimes 1) = \sum\limits_{i=1}^{p-1} i s_i(x,y) \otimes X^{i-1}$ in $L \otimes_{\mathbb{F}} \mathbb{F}[X], \forall x, y \in L_1, \alpha(x \otimes X) = \alpha(x) \otimes X$. The pair $(L, [,]_L, \alpha, [p])$ is referred to as a restricted hom-Lie algebra.

From the above definition, we may see that (i) $\alpha(x^{[p]}) = (\alpha(x))^{[p]}$ for all $x \in L_1$, i.e., $\alpha \circ [p] = [p] \circ \alpha$; (ii) By (1) of the definition, one gets $\mathrm{ad}x^{[p]} = (\mathrm{ad}x)^p$ for all $x \in L_1$.

Let $(L, \alpha)$ be a hom-Lie algebra over $\mathbb{F}$ and $f : L \to L$ be a mapping. $f$ is called a $p$-semilinear mapping, if $f(kx+y) = k^p f(x) + f(y)$, $\forall x, y \in L$, $\forall k \in \mathbb{F}$. Let $S$ be a subset of a hom-Lie algebra $(L, \alpha)$. We put $C_L(S) := \{x \in L| \ [\alpha(y), x] = 0, \forall y \in S\}$. $C_L(S)$ is called the centralizer of $S$ in $L$. Put $C(L) := \{x \in L| \ [\alpha(y), x] = 0, \forall y \in L\}$. $C(L)$ is called the center of $L$.

**2.2. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha)$ be a restricted hom-Lie algebra over $\mathbb{F}$. A hom-Lie subalgebra $H$ of $L$ is called a $p$-subalgebra, if $x^{[p]} \in H_1$ for all $x \in H_1$, where $H_1 = \{x \in H|\alpha(x) = x\}$.

**2.3. Proposition.** *Let $L$ be a hom-Lie subalgebra of a restricted hom-Lie algebra $(G, [\cdot,\cdot]_G, \alpha, [p])$ and $[p]_1 : L_1 \to L_1$ a mapping. Then the following statements are equivalent:*

(1) *$[p]_1$ is a $p$-mapping on $L_1$.*

(2) *There exists a $p$-semilinear mapping $f : L_1 \to C_G(L)$ such that $[p]_1 = [p] + f$.*

*Proof.* (1)⇒(2). Consider $f : L_1 \to G$, $f(x) = x^{[p]_1} - x^{[p]}$. Since $\mathrm{ad}f(x)(y) = [\alpha(y), f(x)] = 0, \forall x \in L_1, y \in L$, $f$ actually maps $L_1$ into $C_G(L)$. For $x, y \in L_1, k \in \mathbb{F}$, we obtain

$$
\begin{aligned}
&f(kx+y)\\
&= k^p x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(kx, y) - k^p x^{[p]} - y^{[p]} - \sum_{i=1}^{p-1} s_i(kx, y)\\
&= k^p f(x) + f(y),
\end{aligned}
$$

which proves that $f$ is $p$-semilinear.

(2)⇒(1). We next will check three conditions of the definition step and step. For $x, y \in L_1$, we have

$$
\begin{aligned}
&(x+y)^{[p]_1} = (x+y)^{[p]} + f(x+y)\\
&= x^{[p]} + f(x) + y^{[p]} + f(y) + \sum_{i=1}^{p-1} s_i(x, y)\\
&= x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(x, y)
\end{aligned}
$$

and

$$
\begin{aligned}
&(kx)^{[p]_1} = (kx)^{[p]} + f(kx)\\
&= k^p x^{[p]} + k^p f(x)\\
&= k^p (x^{[p]} + f(x))\\
&= k^p x^{[p]_1}.
\end{aligned}
$$

For $x \in L_1, z \in L$, one gets

$$
\begin{aligned}
&\mathrm{ad}x^{[p]_1}(z) = \mathrm{ad}(x^{[p]} + f(x))(z)\\
&= \mathrm{ad}x^{[p]}(z) + \mathrm{ad}f(x)(z)\\
&= \mathrm{ad}x^{[p]}(z)\\
&= (\mathrm{ad}x)^p(z).
\end{aligned}
$$

The proof is complete. □

**2.4. Corollary.** *The following statements hold.*

(1) *If* $C(L) = 0$, *then* $L$ *admits at most one* $p$*-mapping.*

(2) *If two* $p$*-mappings coincide on a basis, then they are equal.*

(3) *If* $(L, [\cdot,\cdot]_L, \alpha, [p])$ *is restricted, then there exists a* $p$*-mapping* $[p]^{'}$ *of* $L$ *such that* $x^{[p]^{'}} = 0, \ \forall x \in C(L_1)$.

*Proof.* (1) We set $G = L$. Then $C_G(L) = C(L)$, the only $p$-semilinear mapping occurring in Proposition 2.3 is the zero mapping.

(2) If two $p$-mappings coincide on a basis, their difference vanishes since it is $p$-semilinear.

(3) $[p]|_{C(L_1)}$ defines a $p$-mapping on $C(L_1)$. Since $C(L_1)$ is abelian, it is $p$-semilinear. Extend this to a $p$-semilinear mapping $f : L_1 \to C(L_1)$. Then $[p]^{'} := [p] - f$ is a $p$-mapping of $L$, vanishing on $C(L_1)$. □

From the proof of Theorem 2 in [18], we see the following definition:

**2.5. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha_L)$ be a hom-Lie algebra, and let $j : L \to U_{HLie}(L)$ be the composition of the maps $L \hookrightarrow F_{HNAs}(L) \twoheadrightarrow U_{HLie}(L)$. The pair $(U_{HLie}(L), j)$ is called a universal enveloping algebra of $L$ if for every hom-associative algebra $(A, \mu_A, \alpha_A)$ and every morphism $f : L \to HLie(A)$ of hom-Lie algebras, there exists a unique morphism $h : U_{HLie}(L) \to A$ of hom-associative algebras such that $f = h \circ j$ (as morphisms of $\mathbb{F}$-modules).

In the special case of $G = U_{HLie}(L)^{-} \supset L$, where $U_{HLie}(L)$ is the universal enveloping algebra of hom-Lie algebra $L$ (see [18]) and $U_{HLie}(L)^{-}$ denotes a hom-Lie algebra given by hom-associative algebra $U_{HLie}(L)$ via the commutator bracket. We have the following theorem:

**2.6. Theorem.** *Let* $(e_j)_{j\in J}$ *be a basis of* $L_1$ *such that there are* $y_j \in L_1$ *with* $(\mathrm{ad}e_j)^p = \mathrm{ad}y_j$. *Then there exists exactly one* $p$*-mapping* $[p] : L_1 \to L$ *such that* $e_j^{[p]} = y_j, \forall j \in J$.

*Proof.* For $z \in L_1$, we have $0 = ((\mathrm{ad}e_j)^p - \mathrm{ad}y_j)(z) = [\alpha(z), e_j^p - y_j]$. Then $e_j^p - y_j \in C_{U_{HLie}(L_1)}(L_1), \forall j \in J$. We define a $p$-semilinear mapping $f : L_1 \to C_{U_{HLie}(L_1)}(L_1)$ by means of

$$f(\sum \alpha_j e_j) := \sum \alpha_j^p (y_j - e_j^p).$$

Consider $V := \{x \in L_1 | x^p + f(x) \in L_1\}$. The equation

$$(kx+y)^p + f(kx+y) = k^p x^p + y^p + \sum_{i=1}^{p-1} s_i(kx, y) + k^p f(x) + f(y)$$

ensures that $V$ is a subspace of $L_1$. Since it contains the basis $(e_j)_{j\in J}$, we conclude that $x^p + f(x) \in L_1, \ \forall x \in L_1$. By virtue of Proposition 2.3, $[p] : L_1 \to L, x^{[p]} := x^p + f(x)$ is a $p$-mapping on $L_1$. In addition, we obtain $e_j^{[p]} = e_j^p + f(e_j) = y_j$, as asserted. The uniqueness of $[p]$ follows from Corollary 2.4. □

**2.7. Definition.** A multiplicative hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha_L)$ is called restrictable, if $(\mathrm{ad}x)^p \in \mathrm{ad}L_1$ for all $x \in L_1$, where $\mathrm{ad}L_1 = \{\mathrm{ad}x | x \in L_1\}$.

**2.8. Theorem.** $L$ *is a restrictable hom-Lie algebra if and only if there is a* $p$*-mapping* $[p] : L_1 \to L_1$ *which makes* $L$ *a restricted hom-Lie algebra.*

*Proof.* ($\Leftarrow$) By the definition of $p$-mapping $[p]$, for $x \in L_1$, there exists $x^{[p]} \in L_1$ such that $(\mathrm{ad}x)^p = \mathrm{ad}x^{[p]} \in \mathrm{ad}L_1$. Hence $L$ is restrictable.

($\Rightarrow$) Let $L$ be restrictable. Then for $x \in L_1$, we have $(\mathrm{ad}x)^p \in \mathrm{ad}L_1$, that is, there exists $y \in L_1$ such that $(\mathrm{ad}x)^p = \mathrm{ad}y$. Let $(e_j)_{j\in J}$ be a basis of $L_1$. Then there exists

$y_j \in L_1$ such that $(\mathrm{ad}e_j)^p = \mathrm{ad}y_j (j \in J)$. By Theorem 2.6, there exists exactly one $p$-mapping $[p] : L_1 \to L_1$ such that $e_j^{[p]} = y_j, \forall j \in J$, which makes $L$ a restricted hom-Lie algebra. □

## 3. Properties of $p$-mappings and restrictable hom-Lie algebras

In the section, we will discuss some properties of $p$-mappings and restrictable hom-Lie algebras.

**3.1. Definition.** [13] Let $(L, [\cdot,\cdot]_L, \alpha)$ and $(\Gamma, [\cdot,\cdot]_\Gamma, \beta)$ be two hom-Lie algebras. A linear map $\phi : L \to \Gamma$ is said to be a morphism of hom-Lie algebras if

$$\phi[u, v]_L = [\phi(u), \phi(v)]_\Gamma, \qquad \forall u, v \in L, \tag{3.1}$$

$$\phi \circ \alpha = \beta \circ \phi. \tag{3.2}$$

Denote by $\mathfrak{G}_\phi = \{(x, \phi(x)) | x \in L\} \subseteq L \oplus \Gamma$ the graph of a linear map $\phi : L \to \Gamma$.

**3.2. Definition.** A morphism of hom-Lie algebras $\phi : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$ is said to be restricted if $\phi(x^{[p]_1}) = (\phi(x))^{[p]_2}$ for all $x \in L$.

**3.3. Proposition.** *Given two restricted hom-Lie algebras $(L, [\cdot,\cdot]_L, \alpha, [p]_1)$ and $(\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$, there is a restricted hom-Lie algebra $(L \oplus \Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha + \beta, [p])$, where the bilinear map $[\cdot,\cdot]_{L\oplus\Gamma} : \wedge^2(L \oplus \Gamma) \to L \oplus \Gamma$ is given by*

$$[u_1 + v_1, u_2 + v_2]_{L\oplus\Gamma} = [u_1, u_2]_L + [v_1, v_2]_\Gamma, \qquad \forall u_1, u_2 \in L, v_1, v_2 \in \Gamma,$$

*and the linear map $(\alpha + \beta) : L \oplus \Gamma \to L \oplus \Gamma$ is given by*

$$(\alpha + \beta)(u + v) = \alpha(u) + \beta(v), \qquad \forall u \in L, v \in \Gamma,$$

*the $p$-mapping $[p] : L \oplus \Gamma \to L \oplus \Gamma$ is given by*

$$(u + v)^{[p]} = u^{[p]_1} + v^{[p]_2}, \qquad \forall u \in L, v \in \Gamma.$$

*Proof.* Recall that $L_1 = \{x \in L | \alpha(x) = x\}$ and $\Gamma_1 = \{x \in \Gamma | \beta(x) = x\}$. For any $u_1, u_2 \in L, v_1, v_2 \in \Gamma$, we have

$$\begin{aligned} [u_2 + v_2, u_1 + v_1]_{L\oplus\Gamma} &= [u_2, u_1]_L + [v_2, v_1]_\Gamma \\ &= -[u_1, u_2]_L - [v_1, v_2]_\Gamma \\ &= -[u_1 + v_1, u_2 + v_2]_{L\oplus\Gamma}. \end{aligned}$$

The bracket is obviously skew-symmetric. By a direct computation we have

$$\begin{aligned} &[(\alpha + \beta)(u_1 + v_1), [u_2 + v_2, u_3 + v_3]_{L\oplus\Gamma}]_{L\oplus\Gamma} \\ &+c.p.((u_1 + v_1), (u_2 + v_2), (u_3 + v_3)) \\ = \ &[\alpha(u_1) + \beta(v_1), [u_2, u_3]_L + [v_2, v_3]_\Gamma]_{L\oplus\Gamma} + c.p. \\ = \ &[\alpha(u_1), [u_2, u_3]_L]_L + c.p.(u_1, u_2, u_3) + [\beta(v_1), [v_2, v_3]_\Gamma]_\Gamma \\ &+c.p.(v_1, v_2, v_3) \\ = \ &0, \end{aligned}$$

where $c.p.(a, b, c)$ means the cyclic permutations of $a, b, c$. For any $u_1 \in L_1, v_1 \in \Gamma_1, u_2 \in L, v_2 \in \Gamma$, we obtain

$$\begin{aligned} &\mathrm{ad}(u_1 + v_1)^{[p]}(u_2 + v_2) = [(\alpha + \beta)(u_2 + v_2), (u_1 + v_1)^{[p]}]_{L\oplus\Gamma} \\ &= [\alpha(u_2) + \beta(v_2), u_1^{[p]_1} + v_1^{[p]_2}]_{L\oplus\Gamma} \\ &= [\alpha(u_2), u_1^{[p]_1}]_L + [\beta(v_2), v_1^{[p]_2}]_\Gamma \\ &= \mathrm{ad}u_1^{[p]_1}(u_2) + \mathrm{ad}v_1^{[p]_2}(v_2) \end{aligned}$$

$$= (\mathrm{adu}_1)^p(u_2) + (\mathrm{adv}_1)^p(v_2)$$

and

$$\begin{aligned}
&(\mathrm{ad}(u_1+v_1))^p(u_2+v_2)\\
&= [[[\alpha^p(u_2)+\beta^p(v_2), \overbrace{u_1+v_1], u_1+v_1], \cdots, u_1+v_1}^{p}]_{L\oplus\Gamma}\\
&= [[[\alpha^p(u_2), \overbrace{u_1], u_1], \cdots, u_1}^{p}]_L + [[[\beta^p(v_2), \overbrace{v_1], v_1], \cdots, v_1}^{p}]_\Gamma\\
&= (\mathrm{adu}_1)^p(u_2) + (\mathrm{adv}_1)^p(v_2).
\end{aligned}$$

Hence $\mathrm{ad}(u_1+v_1)^{[p]}(u_2+v_2) = (\mathrm{ad}(u_1+v_1))^p(u_2+v_2)$, thus $\mathrm{ad}(u_1+v_1)^{[p]} = (\mathrm{ad}(u_1+v_1))^p$. Moreover, for any $u_1, u_2 \in L_1, v_1, v_2 \in \Gamma_1$, one gets

$$\begin{aligned}
&((u_1+v_1)+(u_2+v_2))^{[p]} = ((u_1+u_2)+(v_1+v_2))^{[p]} = (u_1+u_2)^{[p]_1} + (v_1+v_2)^{[p]_2}\\
&= u_1^{[p]} + u_2^{[p]} + \sum_{i=1}^{p-1} s_i(u_1,u_2) + v_1^{[p]} + v_2^{[p]} + \sum_{i=1}^{p-1} s_i(v_1,v_2)\\
&= (u_1^{[p]} + v_1^{[p]}) + (u_2^{[p]} + v_2^{[p]}) + (\sum_{i=1}^{p-1} s_i(u_1,u_2) + \sum_{i=1}^{p-1} s_i(v_1,v_2))\\
&= (u_1+v_1)^{[p]} + (u_2+v_2)^{[p]} + \sum_{i=1}^{p-1}(s_i(u_1,u_2)) + s_i(v_1,v_2))\\
&= (u_1+v_1)^{[p]} + (u_2+v_2)^{[p]} + \sum_{i=1}^{p-1} s_i((u_1,v_1)+(u_2,v_2))
\end{aligned}$$

and

$$\begin{aligned}
&(k(u_1+v_1))^{[p]} = (ku_1+kv_1)^{[p]} = (ku_1)^{[p]_1} + (kv_1)^{[p]_2}\\
&= k^p u_1^{[p]_1} + k^p v_1^{[p]_2} = k^p(u_1^{[p]_1} + v_1^{[p]_2})\\
&= k^p(u_1+v_1)^{[p]}.
\end{aligned}$$

Therefore, $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$ is a restricted hom-Lie algebra. □

**3.4. Proposition.** *A linear map* $\phi : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta, [p]_2)$ *is a restricted morphism of restricted hom-Lie algebras if and only if the graph* $\mathfrak{G}_\phi \subseteq L\oplus\Gamma$ *is a restricted hom-Lie subalgebra of* $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$.

*Proof.* Let $\phi : (L, [\cdot,\cdot]_L, \alpha) \to (\Gamma, [\cdot,\cdot]_\Gamma, \beta)$ be a restricted morphism of restricted hom-Lie algebras. By (3.1), we have

$$[u+\phi(u), v+\phi(v)]_{L\oplus\Gamma} = [u,v]_L + [\phi(u),\phi(v)]_\Gamma = [u,v]_L + \phi[u,v]_L.$$

Then the graph $\mathfrak{G}_\phi$ is closed under the bracket operation $[\cdot,\cdot]_{L\oplus\Gamma}$. Furthermore, by (3.2), we have

$$(\alpha+\beta)(u+\phi(u)) = \alpha(u) + \beta\circ\phi(u) = \alpha(u) + \phi\circ\alpha(u),$$

which implies that $(\alpha+\beta)(\mathfrak{G}_\phi) \subseteq \mathfrak{G}_\phi$. Thus, $\mathfrak{G}_\phi$ is a hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta)$. Moreover, for $u+\phi(u) \in \mathfrak{G}_\phi$, one gets

$$(u+\phi(u))^{[p]} = u^{[p]_1} + (\phi(u))^{[p]_2} = u^{[p]_1} + \phi(u^{[p]_1}) \in \mathfrak{G}_\phi.$$

Thereby, the graph $\mathfrak{G}_\phi \subseteq L\oplus\Gamma$ is a restricted hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$.

Conversely, if the graph $\mathfrak{G}_\phi \subseteq L \oplus \Gamma$ is a restricted hom-Lie subalgebra of $(L \oplus \Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$, then we have

$$[u+\phi(u), v+\phi(v)]_{L\oplus\Gamma} = [u,v]_L + [\phi(u),\phi(v)]_\Gamma \in \mathfrak{G}_\phi,$$

which implies that

$$[\phi(u),\phi(v)]_\Gamma = \phi[u,v]_L.$$

Furthermore, $(\alpha+\beta)(\mathfrak{G}_\phi) \subset \mathfrak{G}_\phi$ yields that

$$(\alpha+\beta)(u+\phi(u)) = \alpha(u) + \beta\circ\phi(u) \in \mathfrak{G}_\phi,$$

which is equivalent to the condition $\beta\circ\phi(u) = \phi\circ\alpha(u)$, i.e. $\beta\circ\phi = \phi\circ\alpha$. Therefore, $\phi$ is a morphism of restricted hom-Lie algebras. Since $\mathfrak{G}_\phi$ is a restricted hom-Lie subalgebra of $(L\oplus\Gamma, [\cdot,\cdot]_{L\oplus\Gamma}, \alpha+\beta, [p])$, we have

$$(u+\phi(u))^{[p]} = u^{[p]_1} + (\phi(u))^{[p]_2} \in \mathfrak{G}_\phi.$$

Thus, $(\phi(u))^{[p]_2} = \phi(u^{[p]_1})$ for $u \in L$, i.e., $\phi$ is a restricted morphism. □

One advantage in considering restrictable hom-Lie algebras instead of restricted ones rests on the following theorem.

**3.5. Theorem.** *Let $f : (L, [\cdot,\cdot]_L, \alpha, [p]_1) \to (L^{'}, [\cdot,\cdot]_{L^{'}}, \beta, [p]_2)$ be a surjective restricted morphism of hom-Lie algebras. If $L$ is restrictable, so is $L^{'}$.*

*Proof.* It follows from $f$ is a surjective mapping that $L^{'} = f(L)$. Then for $x \in L_1$, we have $\beta(f(x)) = f(\alpha(x)) = f(x)$ and $f(x) \in L_1^{'}$, where $L_1 = \{x \in L | \alpha(x) = x\}$ and $L_1^{'} = \{x \in L^{'} | \beta(x) = x\}$. For $y \in L$, one gets

$$\begin{aligned}
&(\mathrm{ad}f(x))^p(f(y)) = (\mathrm{ad}f(x))^{p-1}[\beta(f(y)), f(x)]\\
&= (\mathrm{ad}f(x))^{p-2}[[\beta^2(f(y)), \beta(f(x))], f(x)]\\
&= [[[\beta^p f(y), \underbrace{f(x)], f(x)], \cdots, f(x)]}_{p}\\
&= \beta^p[[[f(y), \underbrace{f(x)], f(x)], \cdots, f(x)]}_{p}\\
&= \beta^p \circ f[[[y, \underbrace{x], x], \cdots, x]}_{p} = f[[[\alpha^p(y), \underbrace{x], x], \cdots, x]}_{p}\\
&= f((\mathrm{ad}x)^p(y)) = f((\mathrm{ad}x^{[p]_1})(y)) = f[\alpha(y), x^{[p]_1}]\\
&= f[\alpha(y), \alpha(x^{[p]_1})] = f\circ\alpha[y, x^{[p]_1}] = \beta\circ f[y, x^{[p]_1}]\\
&= \beta[f(y), f(x^{[p]_1})] = [\beta(f(y)), \beta(f(x^{[p]_1}))]\\
&= [\beta(f(y)), f(x^{[p]_1})] = \mathrm{ad}f(x^{[p]_1})(f(y))\\
&= \mathrm{ad}(f(x))^{[p]_2}(f(y)).
\end{aligned}$$

We have $(\mathrm{ad}f(x))^p = \mathrm{ad}(f(x))^{[p]_2} \in \mathrm{ad}{L_1}^{'}$. Hence $L^{'}$ is restrictable. □

**3.6. Theorem.** *Let $A$ and $B$ be hom-Lie ideals of hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha)$ such that $L = A \oplus B$. Then $L$ is restrictable if and only if $A, B$ are restrictable.*

*Proof.* $(\Leftarrow)$ If $A, B$ are restrictable, for $x \in L_1$ with $\alpha(x) = x$, we may suppose that $x = x_1 + x_2$, where $x_1 \in A, x_2 \in B$. Then $\alpha(x_1+x_2) = \alpha(x_1) + \alpha(x_2) = x_1 + x_2$. Since $A$ and $B$ are hom-Lie ideals, one gets $\alpha(x_1) \in A, \alpha(x_2) \in B$. we obtain $\alpha(x_1) = x_1$ and

$\alpha(x_2) = x_2$. As $A, B$ are restrictable, then there exists $y_1 \in A_1, y_2 \in B_1$ with $\alpha(y_1) = y_1$ and $\alpha(y_2) = y_2$, such that $(\mathrm{ad}x_1)^p = \mathrm{ad}y_1$ and $(\mathrm{ad}x_2)^p = \mathrm{ad}y_2$. Thus,

$$\begin{aligned}&(\mathrm{ad}(x_1 + x_2))^p = (\mathrm{ad}x_1 + \mathrm{ad}x_2)^p\\&= (\mathrm{ad}x_1)^p + (\mathrm{ad}x_2)^p = \mathrm{ad}y_1 + \mathrm{ad}y_2\\&= \mathrm{ad}(y_1 + y_2).\end{aligned}$$

Therefore, $L$ is restrictable.

($\Rightarrow$) If $L$ is restrictable , so are $A \cong L/B$, $B \cong L/A$ by Theorem 3.5. □

**3.7. Corollary.** *Let $A, B$ be restrictable hom-Lie ideals of a restricted hom-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, [p])$ such that $L = A + B$ and $[A, B] = 0$. Then $L$ is restrictable.*

*Proof.* Define a mapping $f : A \oplus B \to L, (x, y) \mapsto x + y$. Clearly, $f$ is a surjection. For $(x_1, y_1), (x_2, y_2) \in A \oplus B$, by $[A, B] = 0$, one gets $[x_1, y_2] = [y_1, x_2] = 0$. We have

$$\begin{aligned}&f[(x_1, y_1), (x_2, y_2)] = f([x_1, x_2], [y_1, y_2])\\&= [x_1, x_2] + [y_1, y_2] = [x_1, x_2] + [x_1, y_2] + [y_1, x_2] + [y_1, y_2]\\&= [x_1 + y_1, x_2 + y_2] = [f(x_1, y_1), f(x_2, y_2)].\end{aligned}$$

Moreover, one gets

$$\begin{aligned}&\alpha \circ f(x, y) = \alpha(x + y)\\&= \alpha(x) + \alpha(y) = f((\alpha(x), \alpha(y)))\\&= f \circ \alpha(x, y).\end{aligned}$$

Therefore, $\alpha \circ f = f \circ \alpha$. For $x \in A, y \in B$, $\alpha(x, y) = (x, y)$, we have

$$\begin{aligned}&f((x, y)^{[p]}) = f((x^{[p]_1}, y^{[p]_2}))\\&= x^{[p]_1} + y^{[p]_2} = (x + y)^{[p]}\\&= (f(x, y))^{[p]}.\end{aligned}$$

Thus, $f$ is a restricted morphism. By Theorem 3.6, we have $A \oplus B$ is restrictable. By Theorem 3.5, one gets $L$ is restrictable. □

**3.8. Definition.** Let $(L, [\cdot, \cdot]_L, \alpha)$ be a hom-Lie algebra and $\psi$ be a symmetric bilinear form on $L$. $\psi$ is called associative, if $\psi(x, [z, y]) = \psi([\alpha(z), x], y)$.

**3.9. Definition.** Let $(L, [\cdot, \cdot]_L, \alpha)$ be a hom-Lie algebra and $\psi$ a symmetric bilinear form on $L$. Set $L^{\perp} = \{x \in L | \psi(x, y) = 0, \forall\, y \in L\}$. $L$ is called nondegenerate, if $L^{\perp} = 0$.

**3.10. Theorem.** *Let $L$ be a subalgebra of the restricted hom-Lie algebra $(G, [\cdot, \cdot]_G, \alpha, [p])$ with $C(L) = \{0\}$. Assume $\lambda : G \times G \to \mathbb{F}$ to be an associative symmetric bilinear form, which is nondegenerate on $L \times L$. Then $L$ is restrictable.*

*Proof.* Since $\lambda$ is nondegenerate on $L \times L$, every linear form $f$ on $L$ is determined by a suitably chosen element $y \in L : f(z) = \lambda(y, z), \forall z \in L$. Let $x \in L_1$. Then there exists $y \in L$ such that

$$\lambda(x^{[p]}, z) = \lambda(y, z), \forall z \in L.$$

This implies that $0 = \lambda(x^{[p]} - y, [L, L]) = \lambda([\alpha(L), x^{[p]} - y], L)$ and $[\alpha(L), x^{[p]} - y] = 0$. Therefore, $x^{[p]} - y \in C(L) = \{0\}$ and $y = x^{[p]} \in L_1$. Moreover, we obtain

$$(\mathrm{ad}x|_L)^p = \mathrm{ad}x^{[p]}|_L = \mathrm{ad}y|_L,$$

which proves that $L$ is restrictable. □

**3.11. Proposition.** *Let $(L, [\cdot,\cdot]_L, \alpha)$ be a restrictable hom-Lie algebra and $H$ a subalgebra of $L$. Then $H$ is a $p$-subalgebra for some mapping $[p]$ on $L$ if and only if $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$.*

*Proof.* ($\Rightarrow$) If $H$ is a $p$-subalgebra, then for $x \in H_1$, $x^{[p]} \in H_1$, and $(\mathrm{ad}x)^p = \mathrm{ad}x^{[p]} \subseteq \mathrm{ad}H_1|_L$. Hence, $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$.

($\Leftarrow$) If $(\mathrm{ad}H_1|_L)^p \subseteq \mathrm{ad}H_1|_L$, then $H$ is restrictable. By Theorem 2.8, $H$ is restricted. Thereby, $H$ is a $p$-subalgebra of $L$. □

## 4. Cohomology of restricted hom-Lie algebras

In this section, we will discuss the cohomology of restricted hom-Lie algebras in the abelian case, which is similar to the reference [5].

**4.1. Definition.** [12] A hom-associative algebra is a triple $(V, \mu, \alpha)$ consisting of a linear space $V$, a bilinear map $\mu : V \times V \longrightarrow V$ and a linear space homomorphism $\alpha : V \longrightarrow V$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

There is a functor from the category of hom-associative algebras in the category of hom-Lie algebras.

**4.2. Proposition.** [12] *Let $(A, \mu, \alpha)$ be a hom-associative algebra defined on the linear space $A$ by the multiplication $\mu$ and a homomorphism $\alpha$. Then the triple $(A, [\cdot,\cdot], \alpha)$ where the bracket is defined for $x, y \in A$ by $[x, y] = \mu(x, y) - \mu(y, x)$, is a hom-Lie algebra. We also denote it by $(A^-, [\cdot,\cdot], \alpha)$.*

The following definition is analogous to that of the restricted universal enveloping algebra in the reference [14].

**4.3. Definition.** Let $(L, [\cdot,\cdot]_L, \alpha, [p])$ be a restricted hom-Lie algebra. The $(u(L), \mu^{'}, \alpha^{'}, i)$ consisting of a hom-associative algebra $(u(L), \mu^{'}, \alpha^{'})$ with unity and a restricted hom-morphism $i : (L, [\cdot,\cdot]_L, \alpha, [p]) \to (u(L)^-, \mu^{'}, \alpha^{'})$ is called a restricted hom-universal enveloping algebra of $L$ if given any hom-associative algebra $(A, \mu^{''}, \alpha^{''})$ with unity and any restricted hom-morphism $f : (L, [\cdot,\cdot]_L, \alpha, [p]) \to (A^-, \mu^{''}, \alpha^{''})$, there exists a unique morphism $\bar{f} : (u(L), \mu^{'}, \alpha^{'}) \to (A, \mu^{''}, \alpha^{''})$ of hom-associative algebras such that $\bar{f} \circ i = f$.

**4.4. Definition.** [11] Let $A = (V, \mu, \alpha)$ be a hom-associative $\mathbb{F}$-algebra. An $A$-module is a triple $(M, f, \gamma)$ where $M$ is $\mathbb{F}$-vector space and $f, \gamma$ are $\mathbb{F}$-linear maps, $f : M \longrightarrow M$ and $\gamma : V \otimes M \longrightarrow M$, such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes M & \xrightarrow{\gamma} & M \\ \uparrow{\scriptstyle \alpha\otimes\gamma} & & \uparrow{\scriptstyle \gamma} \\ V \otimes V \otimes M & \xrightarrow{\mu\otimes f} & V \otimes M \end{array}$$

We let $S^*(L)$ and $\Lambda^*(L)$ denote the symmetric and alternating algebras of restricted hom-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, [p])$, respectively. Bases for the homogeneous subspaces of degree $k$ for these spaces consist of monomials $e^\mu = e_1^{\mu_1} \cdots e_n^{\mu_n}$ and $e_{\vec{i}} = e_{i_1} \wedge \cdots \wedge e_{i_k}$, respectively, where

$\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$ satisfies $\mu_j \geq 0, |\mu| = \sum_j \mu_j = k$;

$\vec{i} = (i_1, \cdots, i_k) \in \mathbb{Z}^k$ satisfies $1 \leq i_1 < \cdots < i_k \leq n$.

Let $\gamma : \lambda \mapsto \lambda^p$ denote the Frobenius automorphism of $\mathbb{F}$. If $V$ is an abelian group with an $\mathbb{F}$-vector space structure given by $\mathbb{F} \to \mathrm{End}(V)$, then the composition

$$\mathbb{F} \xrightarrow{\gamma^{-1}} \mathbb{F} \to \mathrm{End}(V)$$

gives another vector space structure on $V$ which we will denote by $\overline{V}$. Of course $\overline{V}$ is isomorphic to $V$ as an $\mathbb{F}$-vector space (they have the same dimension). We note that if $W$ is any other $\mathbb{F}$-vector space, then a $p$-semilinear map $V \to W$ is a linear map $\overline{V} \to W$ and vice versa.

In sequel, $(L, [\cdot,\cdot]_L, \alpha, [p])$ denotes a finite-dimensional restricted hom-Lie algebra over $\mathbb{F}$ such that $[g_i, g_j] = 0$ for all $g_i, g_j \in L$ and $(u(L), \alpha^{'}, i)$ denotes the restricted hom-universal enveloping algebra of $L$. Here we take $\alpha = \alpha^{'}$ and $\alpha$ satisfies $\alpha(u_1u_2) = \alpha(u_1)\alpha(u_2)$ for $u_1, u_2 \in u(L)$. For $s, t \geq 0$, we define

$$C_{s,t} = S^t\overline{L}_1 \otimes \Lambda^s L \otimes u(L)$$

with the $u(L)$-module structure given by

$$u(h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes x) = h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes \alpha(u)x,$$

where $h_i, g_j \in L$ and $u, x \in u(L)$. If either $s < 0$ or $t < 0$, we put $C_{s,t} = 0$ and define

$$C_k = \bigoplus_{2t+s=k} C_{s,t}$$

for all $k \in \mathbb{Z}$. Note that each $C_k$ is a free $u(L)$-module. If not both $t = 0$ and $s = 0$, we then define a map

$$d_{s,t} : C_{s,t} \to C_{t,s-1} \oplus C_{t-1,s+1}$$

by the formulas

$$d_{t,s}(h_1 \cdots h_t \otimes g_1 \wedge \cdots \wedge g_s \otimes x)$$

$$= \sum_{i=1}^{s} (-1)^{i-1} h_1 \cdots h_t \otimes \alpha(g_1) \wedge \cdots \widehat{\alpha(g_i)} \cdots \wedge \alpha(g_s) \otimes \alpha(g_i)x \tag{4.1}$$

$$+ \sum_{j=1}^{t} h_1 \cdots \widehat{h_j} \cdots h_t \otimes h_j^{[p]} \wedge \alpha(g_1) \wedge \cdots \wedge \alpha(g_s) \otimes \alpha(x) \tag{4.2}$$

$$- \sum_{j=1}^{t} h_1 \cdots \widehat{h_j} \cdots h_t \otimes h_j \wedge \alpha(g_1) \wedge \cdots \wedge \alpha(g_s) \otimes h_j^{p-1}x. \tag{4.3}$$

For $k \geq 1$, we define the map $d_k : C_k \to C_{k-1}$ by $d_k = \bigoplus_{2t+s=k} d_{s,t}$. Then we obtain the following theorem.

**4.5. Theorem.** *The maps $d_k$ defined above satisfy $d_{k-1}d_k = 0$ for $k \geq 1$, so that $C = (C_k, d_k)$ is an augmented complex of free $u(g)$-modules.*

*Proof.* The terms in the sum (4.1) are elements of $C_{t,s-1}$ whereas the terms in the sums (4.2) and (4.3) lie in $C_{t-1,s+1}$. Therefore, in order to compute $d_{k-1}d_k = 0$, we must apply $d_{t,s-1}$ to (4.1) and $d_{t-1,s+1}$ to (4.2) and (4.3). Applying $d_{t,s-1}$ to (5), we have

$$\begin{aligned}
& d_{t,s}\big(\sum_{i=1}^{s}(-1)^{i-1} h_1 \cdots h_t \otimes \alpha(g_1) \wedge \cdots \widehat{\alpha(g_i)} \cdots \wedge \alpha(g_s) \otimes \alpha(g_i)x\big) \\
= \quad & \sum_{i=1}^{s}(-1)^{i-1}\big(\sum_{\sigma<i}(-1)^{\sigma-1} h_1 \cdots h_t \otimes \alpha^2(g_1) \wedge \cdots \widehat{\alpha^2(g_\sigma)} \cdots \widehat{\alpha^2(g_i)} \cdots \wedge \alpha^2(g_s) \\
& \otimes \alpha^2(g_\sigma)(\alpha(g_i)x) \\
& + \sum_{\sigma>i}(-1)^{\sigma} h_1 \cdots h_t \otimes \alpha^2(g_1) \wedge \cdots \widehat{\alpha^2(g_i)} \cdots \widehat{\alpha^2(g_\sigma)} \cdots \wedge \alpha^2(g_s) \otimes \alpha^2(g_\sigma)(\alpha(g_i)x)
\end{aligned}$$

$$+\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes\alpha(\alpha(g_i)x)$$

$$-\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes h_j^{p-1}(\alpha(g_i)x)\big)$$

$$= \quad \sum_{i=1}^{s}(-1)^{i-1}\big(\sum_{\sigma<i}(-1)^{\sigma-1}h_1\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_\sigma)}\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)$$

$$\otimes(\alpha(g_\sigma)\alpha(g_i))\alpha(x)$$

$$+\sum_{\sigma>i}(-1)^{\sigma}h_1\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\widehat{\alpha^2(g_\sigma)}\cdots\wedge\alpha^2(g_s)\otimes(\alpha(g_\sigma)\alpha(g_i))\alpha(x)$$

$$(4.4) + \quad \sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes\alpha(\alpha(g_i)x)$$

$$(4.5) - \quad \sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_i)}\cdots\wedge\alpha^2(g_s)\otimes(h_j^{p-1}\alpha(g_i))\alpha(x)\big).$$

Since $\alpha(g_i)\alpha(g_j)=\alpha(g_j)\alpha(g_i)$ in $u(g)$, the terms in the first two sums in the parentheses cancel in pairs when summed over all $i$. This leaves the sum over $i$ of (4.4) and (4.5). Now we apply $d_{t-1,s+1}$ to (4.2).

$$d_{t-1,s+1}\big(\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha(g_1)\wedge\cdots\wedge\alpha(g_s)\otimes\alpha(x)\big) = \sum_{j=1}^{t}$$

$$(4.6) \quad \big(\sum_{\sigma=1}^{s}(-1)^{\sigma}h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\widehat{\alpha^2(g_\sigma)}\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(g_\sigma)\alpha(x)$$

$$(4.7) \quad +h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{[p]})\alpha(x)$$

$$(4.8) \quad +\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(x)$$

$$(4.9) \quad -\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j^{[p]}\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_\tau^{p-1}\alpha(x)\big).$$

We note that the terms in (4.8) cancel in pairs since interchanging the first two terms in the alternating product multiplies the term by $-1$. Finally, we apply $d_{t-1,s+1}$ to (4.3) to get

$$d_{t-1,s+1}\big(-\sum_{j=1}^{t} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha(g_s)\otimes h_j^{p-1}x\big)$$

$$= -\sum_{j=1}^{t}\big(\sum_{\sigma=1}^{s}(-1)^{\sigma}h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(g_\sigma)(h_j^{p-1}x)$$

$$+h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha^2(h_j)(h_j^{p-1}x)$$

$$+\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{p-1}x)$$

$$-\sum_{\tau\neq j}h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_\tau^{p-1}(h_j^{p-1}x)\big)$$

$$(4.10)\quad = -\sum_{j=1}^{t}\Big(\sum_{\sigma=1}^{s}(-1)^{\sigma} h_1\cdots\widehat{h_j}\cdots h_t\otimes h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(g_\sigma h_j^{p-1})\alpha(x)$$

$$(4.11)\quad +h_1\cdots\widehat{h_j}\cdots h_t\otimes\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes h_j^p\alpha(x)$$

$$(4.12)\quad +\sum_{\tau\neq j} h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau^{[p]}\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes\alpha(h_j^{p-1}x)$$

$$(4.13)\quad -\sum_{\tau\neq j} h_1\cdots\widehat{h_\tau}\cdots\widehat{h_j}\cdots h_t\otimes h_\tau\wedge h_j\wedge\alpha^2(g_1)\wedge\cdots\wedge\alpha^2(g_s)\otimes(h_\tau^{p-1}h_j^{p-1})\alpha(x)\Big).$$

This time the terms in (4.13) cancel in pairs. Moreover, the terms in (4.4) and (4.6) are identical (with $\sigma = i$) except for sign and hence they cancel. The terms in (4.5) and (4.10) cancel in pairs since $\alpha(h_i^{p-1})\alpha(g_j) = \alpha(g_j)\alpha(h_i^{p-1})$. The terms in (4.9) and (4.12) have the same sign but are equal apart from interchanging the first two terms in the alternating part. Finally the terms in (4.7) and (4.11) match except for sign since $h_j^{[p]} = h_j^p$ in $u(g)$ and hence the entire sum is zero as claimed. This completes the proof. □

We next will consider the cohomology of restricted hom-Lie algebras in the case of simpleness. A basis for the space $C_{t,s}$ consists of the monomials

$$e^\mu\otimes e_I\otimes e^r = e_1^{\mu_1}\cdots e_n^{\mu_n}\otimes e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes e_1^{r_1}\cdots e_n^{r_n},$$

where $\mu = (\mu_1,\cdots,\mu_n), I = (i_1,\cdots,i_s), r = (r_1,\cdots,r_n)$ and

$$\mu_j\geq 0, |\mu| = \sum_j \mu_j = t, 1\leq i_1<\cdots<i_s\leq n, 0\leq r_j\leq p-1.$$

For each $i = 1,\cdots,n$ and $e_i\in L_1$, we let

$$c_i = 1\otimes e_i^{[p]}\otimes 1 - 1\otimes e_i\otimes e_i^{p-1}$$

and we easily note that $c_i\in C_{0,1}$ is a cycle for all $i$. Now we define

$$(\partial/\partial e_i\otimes c_i): C_{t,s}\longrightarrow C_{t-1,s+1}$$

by the formula

$$(\frac{\partial}{\partial e_i}\otimes c_i)(e^\mu\otimes e_I\otimes e^r) = \frac{\partial e^\mu}{\partial e_i}\otimes e_i^{[p]}\wedge\alpha(e_I)\otimes\alpha(e^r) - \frac{\partial e^\mu}{\partial e_i}\otimes e_i\wedge\alpha(e_I)\otimes e_i^{p-1}\alpha(e^r).$$

If $\mu = (\mu_1,\cdots,\mu_n)$ satisfies $|\mu| = t$ and $I = (i_1,\cdots,i_s)$ is increasing, then by the definition we write

$$e^\mu\otimes c_I = \sum_{J\subset\{1,\cdots,s\}}(-1)^{|J|}e^\mu\otimes f_{i_1}\wedge\cdots\wedge f_{i_s}\otimes e_{i_1}^{q_{i_1}}\cdots e_{i_s}^{q_{i_s}}$$

and

$$e^\mu\otimes\alpha(c_I) = \sum_{J\subset\{1,\cdots,s\}}(-1)^{|J|}e^\mu\otimes\alpha(f_{i_1})\wedge\cdots\wedge\alpha(f_{i_s})\otimes\alpha(e_{i_1}^{q_{i_1}})\cdots\alpha(e_{i_s}^{q_{i_s}}),$$

where

$$f_{i_j} = \begin{cases} e_{i_j}, & j\in J\\ e_{i_j}^{[p]}, & j\notin J;\end{cases}\qquad q_{i_j} = \begin{cases} p-1, & j\in J\\ 0, & j\notin J.\end{cases}$$

We then define $\mathfrak{C}_{t,s}$ to be the $\mathbb{F}$-subspace of $C_{t,s}$ spanned by the elements $\{e^\mu\otimes\alpha(c_I) : |\mu| = t$ and $I$ is increasing $\}$ and

$$\mathfrak{C}_k = \bigoplus_{2t+s=k}\mathfrak{C}_{t,s}.$$

The boundary operator $\partial_k = \partial : \mathfrak{C}_k \longrightarrow \mathfrak{C}_{k-1}$ is defined by

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial e_j} \otimes c_j.$$

Then we may show that $\partial^2 = 0$. In fact,

$$\begin{aligned}
&\partial^2(e^\mu \otimes c_I) = \partial(\partial(e^\mu \otimes c_I)) \\
&= \partial(\sum_{j=1}^{n} \frac{\partial}{\partial e_j} \otimes c_j( \sum_{J\subset\{1,\cdots,s\}} (-1)^{|J|} e^\mu \otimes f_{i_1} \wedge \cdots \wedge f_{i_s} \otimes e_{i_1}^{q_{i_1}} \cdots e_{i_s}^{q_{i_s}})) \\
&= \partial(\sum_{j=1}^{n} \sum_{J\subset\{1,\cdots,s\}} (-1)^{|J|} (\frac{\partial e^\mu}{\partial e_j} \otimes e_j^{[p]} \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}) \\
&\quad - \frac{\partial e^\mu}{\partial e_j} \otimes e_j \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes e_j^{p-1} \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}))) \\
&= \sum_{l=1}^{n}\sum_{j=1}^{n} \sum_{J\subset\{1,\cdots,s\}} (-1)^{|J|} \{ \frac{\partial}{\partial e_l} \otimes c_l( \frac{\partial e^\mu}{\partial e_j} \otimes e_j^{[p]} \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes \alpha(e_{i_1}^{q_{i_1}}) \cdots \\
&\quad \alpha(e_{i_s}^{q_{i_s}})) - \frac{\partial}{\partial e_l} \otimes c_l( \frac{\partial e^\mu}{\partial e_j} \otimes e_j \wedge \alpha(f_{i_1}) \wedge \cdots \wedge \alpha(f_{i_s}) \otimes e_j^{p-1} \alpha(e_{i_1}^{q_{i_1}}) \cdots \alpha(e_{i_s}^{q_{i_s}}))\} \\
&= \sum_{l=1}^{n}\sum_{j=1}^{n} \sum_{J\subset\{1,\cdots,s\}} (-1)^{|J|} \{ \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l^{[p]} \wedge \alpha(e_j^{[p]}) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s})
\end{aligned}$$

$$\otimes \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.14}$$

$$- \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l \wedge \alpha(e_j^{[p]}) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes e_l^{p-1} \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.15}$$

$$- \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l^{[p]} \wedge \alpha(e_j) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes \alpha(e_j^{p-1}) \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \tag{4.16}$$

$$+ \frac{\partial(\frac{\partial e^\mu}{\partial e_j})}{\partial e_l} \otimes e_l \wedge \alpha(e_j) \wedge \alpha^2(f_{i_1}) \wedge \cdots \wedge \alpha^2(f_{i_s}) \otimes e_l^{p-1} \alpha(e_j^{p-1}) \alpha^2(e_{i_1}^{q_{i_1}}) \cdots \alpha^2(e_{i_s}^{q_{i_s}}) \}. \tag{4.17}$$

This time the terms in (4.14) cancel in pairs, and the terms in (4.17) cancel in pairs since $e_l^{p-1}\alpha(e_j^{p-1}) = \alpha(e_l^{p-1}) e_j^{p-1}$. Moreover, the terms in (4.15) and (4.16) are identical except for sign and hence they cancel, so that $\mathfrak{C} = \{\mathfrak{C}_k, \partial_k\}_{k\geq 0}$ is a complex.

**4.6. Theorem.** *If $\mathfrak{C}$ is the complex defined above, we define $H_k(\mathfrak{C}) := \mathrm{Ker}\partial_k / \mathrm{Im}\partial_k$. Then*

$$H_k(\mathfrak{C}) = \begin{cases} U_{res.}(g), & k = 0 \\ 0, & 0 < k < p. \end{cases}$$

*Proof.* Define a map $D : \mathfrak{C}_k \to \mathfrak{C}_{k+1}$ by the formula

$$D(e^\mu \otimes \alpha(c_I)) = \sum_{a=1}^{s} (-1)^{a-1} e^\mu e_{i_a} \otimes c_{i_1} \cdots \widehat{c_{i_a}} \cdots c_{i_s}$$

and compute for any monomial $e^\mu \otimes \alpha(c_I)$ :

$$D\partial(e^{\mu}\otimes\alpha(c_I)) = D(\sum_{j=1}^{n}(\frac{\partial}{\partial e_j}\otimes c_j)(e^{\mu}\otimes\alpha(c_I)))$$

$$= \sum_{j=1,j\neq i_1,\cdots,i_s}^{n} D(\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_n^{\mu_n}\otimes c_j\alpha^2(c_I))$$

$$= \sum_{j=1,j\neq i_1,\cdots,i_s}^{n} D(\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_n^{\mu_n}\otimes \alpha(c_j)\alpha^2(c_I))$$

$$= \big(\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j\big)e^{\mu}\otimes\alpha(c_I)$$

$$+ \sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\sum_{a=1}^{s}(-1)^a\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}$$

$$\otimes\alpha(c_j)\alpha(c_{i_1})\cdots\widehat{\alpha(c_{i_a})}\cdots\alpha(c_{i_s}) \tag{4.18}$$

and $\partial D(e^{\mu}\otimes\alpha(c_I)) = \partial(\sum_{a=1}^{s}(-1)^{a-1}e^{\mu}e_{i_a}\otimes c_{i_1}\cdots\widehat{c_{i_a}}\cdots c_{i_s})$

$$= \sum_{a=1}^{s}(-1)^{a-1}\partial(e_1^{\mu_1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}\otimes c_{i_1}\cdots\widehat{c_{i_a}}\cdots c_{i_s})$$

$$= \big(\sum_{a=1}^{s}\mu_{i_a}+1\big)e^{\mu}\otimes\alpha(c_I)$$

$$-\sum_{a=1}^{s}(-1)^a\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j e_1^{\mu_1}\cdots e_j^{\mu_j-1}\cdots e_{i_a}^{\mu_{i_a}+1}\cdots e_n^{\mu_n}$$

$$\otimes\alpha(c_j)\alpha(c_{i_1})\cdots\widehat{\alpha(c_{i_a})}\cdots\alpha(c_{i_s}). \tag{4.19}$$

Clearly the terms (4.18) and (4.19) are identical apart from sign so that we have

$$(D\partial+\partial D)(e^{\mu}\otimes\alpha(c_I)) = \big(\sum_{j=1,j\neq i_1,\cdots,i_s}^{n}\mu_j+\sum_{a=1}^{s}\mu_{i_a}+s\big)(e^{\mu}\otimes\alpha(c_I)) = (t+s)(e^{\mu}\otimes\alpha(c_I)).$$

Therefore we see that every cycle in $\mathfrak{C}_k(k=2t+s)$ is a boundary provided that $t+s\neq 0(\mathrm{mod}p)$. In particular, if $0<k<p$, then $0<t+s<p$ so that $H_k(\mathfrak{C})=0$. Moreover, $\mathfrak{C}_1=\mathfrak{C}_{0,1}$ is spanned by the $c_i$ and $\partial c_i=0$ for all $i$. Therefore $H_0(\mathfrak{C})=\mathfrak{C}_0=U_{res.}(g)$, the proof of the theorem is complete. □

**Acknowledgement** The authors would like to thank the referee for carefully reading and checking the paper both for its English as well as its mathematics.

## References

[1] Ammar, F., Ejbehi, Z. and Makhlouf, A., Cohomology and Deformations of Hom-algebras. *J. Lie Theory,* 2011, 21: 813-836.

[2] Bahturin, Y., Mikhalev, A., Petrogradski, V. M. and Zaicev, M., Infinite-dimensional Lie superalgebras, Walter de Gruyter, Berlin, New York, 1992.

[3] Benayadi, S., Makhlouf, A., Hom-Lie Algebras with Symmetric Invariant NonDegenerate Bilinear Forms. *J. Geom. Phys.,* 2014, 76: 38-60.

[4] Dokas, I., Loday, J. L., On restricted Leibniz algebras. *Comm. Algebra,* 2006, 34: 4467-4478.

[5] Evans, T. J., Fuchs, D., A complex for the cohomology of restricted Lie algebras. *J. fixed point theory appl.,* 2008, 3: 159-179.

[6] Farnsteiner, R., Note on Frobenius extensions and restricted Lie superalgebras. *J. Pure Appl. Algebra*, 1996, 108: 241-256.
[7] Hartwig, J. T., Larsson D. and Silvestrov, S. D., Deformations of Lie algebras using $\sigma$-derivations. *J. Algebra,* 2006, 295: 314-361.
[8] Hodge, T. L., Lie triple system, restricted Lie triple system and algebraic groups. *J. Algebra,* 2001, 244: 533-580.
[9] Larsson, D., Silvestrov, S., Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities. *J. Algebra,* 2005, 288: 321-344.
[10] Larsson, D., Silvestrov, S., Quasi-Lie algebras. *Contemp. Math.,* 2005, 391: 241-248.
[11] Makhlouf, A., Silvestrov, S., Notes on formal deformations of hom-associative and hom-Lie algebras. *Forum Math.,* 2010, 22(4): 715-739.
[12] Makhlouf, A., Silvestrov, S., Hom-algebra structures. *J. Gen. Lie Theory Appl.,* 2008, 2(2), 51-64.
[13] Sheng, Y. H., Representations of hom-Lie algebras. *Algebra Representation Theory,* 2012, 15: 1081–1098.
[14] Strade, H., Farnsteiner, R., Modular Lie algebras and their representations. New York: Dekker. 1988.
[15] Strade, H., The classification of the simple modular Lie algebras. *Ann. Math.*, 1991, 133: 577-604.
[16] Yau, D., Hom-Yang-Baxter equation, hom-Lie algebras, and quasi-triangular bialgebras. *J. Phys. A: Math. Theory,* 2009, 42, 165202.
[17] Yau, D., Hom-algebras and homology. *J. Lie Theory,* 2009, 19: 409-421.
[18] Yau, D., Enveloping algebras of Hom-Lie algebras. *Journal of Generalized Lie Theory and Applications,* 2008, 2: 95-108.