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Simple groups with m-regular first prime graph component

M. Foroudi Ghasemabadi* and A. Iranmanesh†‡

Abstract

Let G be a finite simple group and $\mathrm{GK}(G)$ be the prime graph of G. The connected component of $\mathrm{GK}(G)$ whose vertex set contains 2 is denoted by $\pi_1(G)$. In this paper, our purpose is to classify the finite simple groups G such that $\pi_1(G)$ is regular. We prove that $\pi_1(G)$ is regular if and only if all the connected components of $\mathrm{GK}(G)$ are cliques.

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1. Introduction

For a positive integer n, let $\pi(n)$ denote the set of all prime divisors of n. Given a finite group G, we set $\pi(G) = \pi(|G|)$. The *prime graph* (or *Gruenberg-Kegel graph*) $\mathrm{GK}(G)$ of G is a simple graph which is defined as follows. The vertex set of $\mathrm{GK}(G)$ is the set $\pi(G)$ and two distinct vertices p and q are adjacent (we write $(p, q) \in \mathrm{GK}(G)$) if G contains an element of order pq. If $2 \in \pi(G)$, then the connected component of $\mathrm{GK}(G)$ whose vertex set contains 2 is denoted by $\pi_1(G)$.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. And after that, prime graphs have received some attention in the theory of finite groups. For instance, it has been proved that some of finite simple groups can be characterized by their prime graphs (see [2, 3, 5, 17]). Moreover, some graph properties of this graph have been studied. It has been showed that for every finite group G, the number of connected components of $\mathrm{GK}(G)$ is at most 6 (see [6, 16]) and the diameter of $\mathrm{GK}(G)$ is at most 5 (see [8]). Also, in [9] the groups G such that $\mathrm{GK}(G)$ is a tree, have been investigated. Moreover, according to [12, 16], we know that if Δ is a connected component of $\mathrm{GK}(G)$ whose vertex set does

*Department of Pure Mathematics, Faculty of Mathematical Sciences
Email : foroudi@modares.ac.ir
†Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran.
Email : iranmanesh@modares.ac.ir
‡Corresponding Author.

not contain 2, then Δ is a clique. Note that a clique in a graph is a subset of its vertices such that every two vertices in the subset are connected by an edge. Motivated by this result, Lucido and Moghaddamfar in [10], described the finite nonabelian simple groups G such that $\pi_1(G)$ is a clique. Finally, in [4, 11, 18], the finite simple groups G such that $\pi_1(G)$ is m-regular, where $m \in \{0,1,2\}$, have been obtained. For a nonnegative integer m, a graph is called m-regular, when the degree of each vertex is m. Also, a graph is regular if the degrees of all vertices are the same. The aim of this paper is to extend the m-regularity results, for an arbitrary m. In fact, we prove the following main theorem:

Main theorem. *Let G be a finite nonabelian simple group and let m be a nonnegative integer. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following statements holds:*

- $G = A_5, A_6, A_2(4), A_1(2^k)$, *where* $k > 1$, ${}^2B_2(2^{2k+1})$, *where* $k \geq 1$ *and* $m = 0$;
- $G = M_{11}, M_{22}, A_7$ *and* $m = 1$;
- $G = J_1, J_2, J_3, HiS, A_9, {}^3D_4(2), {}^2A_3(3),\ {}^2A_5(2),\ C_3(2), D_4(2)$ *and* $m = 2$;
- $G = A_{12}, A_{13}$ *and* $m = 3$;
- $G = A_1(q)$, *where* $q \equiv 1 \pmod 4$ *and* $m = |R_1(q)| - 1$;
- $G = A_1(q)$, *where* $q \equiv 3 \pmod 4$, $q > 3$, *and* $m = |R_2(q)| - 1$;
- $G = A_2(q)$, *where* $(q-1)_3 \neq 3$, $q + 1 = 2^k$, *and* $m = |R_1(q)| + 1$;
- $G = {}^2A_2(q)$, *where* $(q+1)_3 \neq 3$, $q - 1 = 2^k$, $C_2(q)$, *where* $q > 2$ *or* $G_2(3^k)$, *where* $k \geq 1$ *and* $m = |R_1(q)| + |R_2(q)|$.

It is worth remarking that by $R_k(q)$ we mean the set of all primitive prime divisors of $q^k - 1$.
As an immediate consequence of the main theorem, we have the following corollary:

Corollary. *Let G be a finite nonabelian simple group. Then $\pi_1(G)$ is regular if and only if all the connected components of the prime graph* $\mathrm{GK}(G)$ *are cliques.*

2. Notation and preliminary results

Throughout this paper, we use the following notation and definitions: By $\gcd(k,l)$ we denote the greatest common divisor of k and l. Let G be a finite group. For $p \in \pi(G)$, put $\deg(p) := |\{q \in \pi(G) |\ (p,q) \in \mathrm{GK}(G)\}|$.
The notation for groups of Lie type is according to [1] and sometimes for abbreviation, we write $A_n^{\varepsilon}(q)$ and $D_n^{\varepsilon}(q)$, where $\varepsilon \in \{+,-\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$. Also, for an integer n, by $\eta(n)$, $\nu(n)$ and $\nu_{\varepsilon}(n)$ we denote the following functions:

$$\eta(n) = \begin{cases} n & \text{if } n \text{ is odd;} \\ n/2 & \text{otherwise.} \end{cases}$$

$$\nu(n) = \begin{cases} n & \text{if } \ n \equiv 0 \pmod 4; \\ n/2 & \text{if } \ n \equiv 2 \pmod 4; \\ 2n & \text{if } \ n \equiv 1 \pmod 2. \end{cases} \qquad \nu_{\varepsilon}(n) = \begin{cases} n & \text{if } \varepsilon = +; \\ \nu(n) & \text{if } \varepsilon = -. \end{cases}$$

All further unexplained group theory notation is standard and can be found in [1].
The following lemma describes the finite nonabelian simple groups G such that $\pi_1(G)$ is m-regular, where $m \in \{0,1,2\}$:

2.1. Lemma. [4, 11, 18] Let G be a finite nonabelian simple group.

1. If $\pi_1(G)$ is 0-regular, then $G = A_5, A_6; A_2(4), A_1(q)$, where q is a Fermat prime, a Mersenne prime or a prime power of 2; ${}^2B_2(q)$, where q is an odd prime power of 2.

2. If $\pi_1(G)$ is 1-regular, then $G = A_7; M_{11}, M_{22}; A_2(3), {}^2A_2(3), {}^2A_3(2), G_2(3); A_1(q)$, where q is a prime power such that $3 < q \equiv \varepsilon 1 \pmod 4$ and $|\pi(q - \varepsilon 1)| = 2$, for $\varepsilon \in \{+, -\}$.
3. If $\pi_1(G)$ is 2-regular, then $G = A_9; J_1, J_2, J_3, HiS; C_3(2), {}^2A_2(9), {}^2A_3(3), {}^3D_4(2), G_2(9), D_4(2); C_2(q)$, where $q = 4, 5, 7, 8, 9, 17$; $A_1(q)$, where q is a prime power such that $3 < q \equiv \varepsilon 1 \pmod 4$ and $|\pi(q - \varepsilon 1)| = 3$, for $\varepsilon \in \{+, -\}$.

The finite nonabelian simple groups G such that all the connected components of $\mathrm{GK}(G)$ are cliques, have been determined in [10]. Since this result plays a role in the proof of the main theorem, in the following, we state its revised version from [14]:

2.2. Lemma. Let G be a finite nonabelian simple group. Then all the connected components of $\mathrm{GK}(G)$ are cliques if and only if G is one of the following:

1. Sporadic groups $M_{11}, M_{22}, J_1, J_2, J_3, HiS$;
2. Alternating groups A_n, where $n = 5, 6, 7, 9, 12, 13$;
3. Groups of Lie type $A_1(q)$, where $q > 3$; $A_2(4)$; $A_2(q)$, where $(q - 1)_3 \neq 3$, $q+1 = 2^k$; ${}^2A_3(3)$; ${}^2A_5(2)$; ${}^2A_2(q)$, where $(q+1)_3 \neq 3$, $q-1 = 2^k$; $C_3(2)$, $C_2(q)$, where $q > 2$; $D_4(2)$; ${}^3D_4(2)$; ${}^2B_2(q)$, where $q = 2^{2k+1}$; $G_2(q)$, where $q = 3^k$.

2.3. Remark. According to Table 1 in [7], we have $\pi_1({}^2A_5(2)) = \{2, 3, 5\}$ and $\pi_1({}^2A_2(17)) = \{2, 3, 17\}$. Moreover, by Lemma 2.2, the prime graph components of the groups ${}^2A_5(2)$ and ${}^2A_2(17)$ are cliques. Thus these mentioned groups should be added to the list of groups in Lemma 2.1(3).

3. Proof of the main theorem

If G is a finite nonabelian simple group, then by the classification of the finite simple groups, it follows that G is a sporadic simple group, an alternating group or a simple group of Lie type. We will consider each case separately.

According to [1], we can easily conclude the next statement for the sporadic simple groups:

3.1. Lemma. Let G be a sporadic simple group. If $\pi_1(G)$ is m-regular, then one of the following cases holds:

(1) $G = M_{11}, M_{22}$ and $m = 1$;
(2) $G = J_1, J_2, J_3, HiS$ and $m = 2$.

For considering the alternating groups, we need the following lemma:

3.2. Lemma. [7, Lemma 1] If $n \geq 19$ is a natural number, then there are at least three prime numbers q_i such that $(n + 1)/2 < q_i < n$.

3.3. Lemma. Let $G = A_n$ be an alternating group of degree n. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G = A_5, A_6$ and $m = 0$;
(2) $G = A_7$ and $m = 1$;
(3) $G = A_9$ and $m = 2$;
(4) $G = A_{12}, A_{13}$ and $m = 3$.

Proof. According to Lemma 2.1, we can assume that $m \geq 3$. Note that for odd primes $r, s \in \pi(A_n)$, $(r, s) \notin \mathrm{GK}(A_n)$ if and only if $r + s > n$. Also, $(r, 2) \notin \mathrm{GK}(A_n)$ if and only if $r + 4 > n$ (see [14]). So, it easy to see that if $(s, r) \in \mathrm{GK}(A_n)$ and $(p, s) \neq (2, 3)$, where $2 \leq p < s < r$, then $(p, r), (p, s) \in \mathrm{GK}(A_n)$. Moreover, if $(p, r) \in \mathrm{GK}(A_n)$, then $(p, s) \in \mathrm{GK}(A_n)$.

If we denote the i-th prime number, by p_i, then since $\deg(2) = m$, according to $\pi(G)$, we see that $\{2, p_2, p_3, \cdots, p_m, p_{m+1}\} \subseteq \pi_1(G)$ and hence, $p_{m+1} \leq n$. We know that $p_{m+2} \leq n$, otherwise,

$$\pi(G) = \{2, p_2, p_3, \cdots, p_m, p_{m+1}\}$$

which implies that $\mathrm{GK}(G)$ is complete and this is impossible according to Lemma 2.2. Since $m \geq 3$, we have $p_{m+2} \geq 11$. Also, since $\pi_1(G)$ is m-regular, we conclude that $(3, p_{m+2}) \notin \mathrm{GK}(G)$, otherwise, $\deg(3) = m + 1$ which is a contradiction. Therefore, $n \leq 2 + p_{m+2}$. On the other hand, since $p_{m+1} \in \pi_1(G)$ and $\deg(p_{m+1}) = m$, we deduce that $(p_m, p_{m+1}) \in \mathrm{GK}(G)$ and hence, $p_m + p_{m+1} \leq n$. Thus $p_m + p_{m+1} \leq n \leq 2 + p_{m+2}$ which implies that

$$p_{m+2} - (p_m + p_{m+1}) \geq -2 \tag{3.1}$$

Now, if $p_{m+2} \geq 19$, then by Lemma 3.2 there exist at least three distinct primes q_i such that $(p_{m+2} - 1)/2 < q_i < p_{m+2}$. Thus we conclude that $(p_{m+2} - 1)/2 < p_{m-1} < p_m < p_{m+1} < p_{m+2}$ and hence,

$$1 + (p_{m+2} - 1)/2 \leq p_{m-1},$$

$$2 + (p_{m+2} - 1)/2 \leq p_m, \tag{3.2}$$

$$3 + (p_{m+2} - 1)/2 \leq p_{m+1}. \tag{3.3}$$

Summing 3.2 and 3.3, implies that $5 + 2 \times (p_{m+2} - 1)/2 \leq p_m + p_{m+1}$ and hence, $p_{m+2} - (p_m + p_{m+1}) \leq -4$, which contradicts 3.1. Thus $p_{m+2} \in \{11, 13, 17\}$. If $p_{m+2} = 11$, 13 or 17, then $m = 3$, 4 or 5 respectively. But according to 3.1, the last two cases cannot happen. So, $m = 3$ and by 3.1, we see that $n \in \{12, 13\}$, as desired. $\square$

The rest of the paper will be devoted to the proof of the main theorem for the simple groups of Lie type. We will consider the classical and the exceptional groups of Lie type separately. For the classical simple groups, our method is based on the results of [14], concerning the arithmetic criterion of adjacency in their prime graphs.

Let s be a prime and let k be a natural number. The s-part of k which is denoted by k_s is equal to s^t if $s^t \mid k$ and $s^{t+1} \nmid k$. If q is a natural number, r is an odd prime and $\gcd(r, q) = 1$, then by $e(r, q)$ we denote the smallest natural number k such that $q^k \equiv 1 \pmod r$. If q is odd, we put $e(2, q) = 1$ whenever $q \equiv 1 \pmod 4$, and $e(2, q) = 2$ otherwise. The following lemma is considered as a corollary to Zsigmondy's theorem:

3.4. Lemma. [14, Lemma 1.4] Let q be a natural number greater than 1. For every natural number k, there exists a prime r with $e(r, q) = k$, but for the cases $q = 2$ and $k = 1$, $q = 3$ and $k = 1$, and $q = 2$ and $k = 6$.

A prime r with $e(r, q) = k$ is called a *primitive prime divisor* of $q^k - 1$. It is obvious that $q^k - 1$ can have more than one primitive prime divisor. We denote by $R_k(q)$ the set of all primitive prime divisors of $q^k - 1$ and by $r_k(q)$ any element of $R_k(q)$. When no confusion can arise, we will write r_k instead of $r_k(q)$ and R_k instead of $R_k(q)$.

3.5. Lemma. [14, Propositions 2.1-2.2],[15, Propositions 2.4-2.5] Let G be a finite simple group of Lie type over a field of order $q = p^\alpha$, for some prime p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$.

(1) If $G = A^\varepsilon_{n-1}(q)$ and $2 \leq \nu_\varepsilon(k) \leq \nu_\varepsilon(l)$, then r and s are nonadjacent if and only if $\nu_\varepsilon(k) + \nu_\varepsilon(l) > n$ and $\nu_\varepsilon(k)$ does not divide $\nu_\varepsilon(l)$.

(2) If $G = B_n(q)$ or $C_n(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and l/k is not an odd natural number.

(3) If $G = D_n^{\varepsilon}(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ and l/k is not an odd natural number. Moreover, if $\varepsilon = +$, then the chain of equalities $n = l = 2\eta(l) = 2\eta(k) = 2k$, is not true as well.

3.6. Lemma. [14, Proposition 3.1] Let G be a finite simple classical group of Lie type over a field of characteristic p, and let $r \in \pi(G)$ and $r \neq p$. Then r and p are nonadjacent if and only if one of the following holds:

(1) $G = A_{n-1}^{\varepsilon}(q)$, r is odd, and $\nu_{\varepsilon}(e(r, q)) > n - 2$;
(2) $G = C_n(q)$ or $G = B_n(q)$, $\eta(e(r, q)) > n - 1$;
(3) $G = D_n^{\varepsilon}(q)$, $\eta(e(r, q)) > n - 2$;
(4) $G = A_1(q), r = 2$;
(5) $G = A_2^{\varepsilon}(q), r = 3$ and $(q - \varepsilon 1)_3 = 3$.

3.7. Lemma. [14, Proposition 4.1-4.2] Let $G = A_{n-1}^{\varepsilon}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q - \varepsilon 1$, and s be an odd prime distinct from the characteristic. Put $k = e(s, q)$. Then s and r are nonadjacent if and only if one of the following holds:

(1) $\nu_{\varepsilon}(k) = n$, $n_r \leq (q - \varepsilon 1)_r$, and if $n_r = (q - \varepsilon 1)_r$, then $2 < (q - \varepsilon 1)_r$;
(2) $\nu_{\varepsilon}(k) = n - 1$ and $(q - \varepsilon 1)_r \leq n_r$.

3.8. Lemma. [14, Propositions 4.3-4.4] Let G be a finite simple group of Lie type over a field of order $q = p^{\alpha}$, for some prime p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r, q)$.

(1) If $G = B_n(q)$ or $C_n(q)$, then r and 2 are nonadjacent if and only if $\eta(k) = n$ and one of the following holds:
  (a) n is odd and $k = (3 - e(2, q))n$;
  (b) n is even and $k = 2n$.
(2) If $G = D_n^{\varepsilon}(q)$, then r and 2 are nonadjacent if and only if one of the following holds:
  (a) $\eta(k) = n$ and $\gcd(4, q^n - \varepsilon 1) = (q^n - \varepsilon 1)_2$;
  (b) $\eta(k) = k = n - 1$, n is even, $\varepsilon = +$, and $e(2, q) = 2$;
  (c) $\eta(k) = k/2 = n - 1$, $\varepsilon = +$, and $e(2, q) = 1$;
  (d) $\eta(k) = k/2 = n - 1$, n is odd, $\varepsilon = -$, and $e(2, q) = 2$.

3.9. Remark. Let G be a finite simple group over a field of order q, where $q = p^{\alpha}$ for an odd prime p. According to the above lemmas, it is evident that 2 and p are adjacent in all classical simple groups except $A_1(q)$. Moreover, for a fixed k, every two elements in $R_k(q)$ are adjacent in $\mathrm{GK}(G)$.

From now on, we assume that $q = p^{\alpha}$, where p is a prime number.

3.10. Lemma. Let G be a finite simple classical group of Lie type. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G = C_2(q)$, where $q > 2$, and $m = |R_1(q)| + |R_2(q)|$.
(2) $G = A_1(q)$, where $q > 3$. In this case, if $q \equiv 1 \pmod 4$, then $m = |R_1(q)| - 1$; and if $q \equiv 3 \pmod 4$, then $m = |R_2(q)| - 1$; also if q is even, then $m = 1$.
(3) $G = A_2(q)$, where $(q - 1)_3 \neq 3$, $q + 1 = 2^k$, and $m = 1 + |R_1(q)|$;
(4) $G = A_2(4)$, and $m = 0$;
(5) $G = {}^2A_3(3)$, ${}^2A_5(2)$, $C_3(2)$ or $D_4(2)$, and $m = 2$;
(6) $G = {}^2A_2(q)$, where $(q + 1)_3 \neq 3$, $q - 1 = 2^k$, and $m = |R_1(q)| + |R_2(q)|$;

Proof. According to the types of the classical groups, the proof will be divided into five parts.

Part A. $G = B_n(q)$ or $G = C_n(q)$, where $n \geq 2$ and $(n,q) \neq (2,2)$:
If $(n,q) = (3,2)$, then $B_n(q) \cong C_n(q)$ and according to Lemma 2.1(3), the result is obvious. Also, if $n = 2$, then $q > 2$ and $B_n(q) \cong C_n(q)$ and hence, Lemma 2.2 implies that $\pi_1(G)$ is a clique. Thus according to Remark 3.9, it is enough to calculate $\deg(p)$. Since $\pi(G) = \{p\} \cup R_1(q) \cup R_2(q) \cup R_4(q)$, Lemma 3.6(2) implies that $m = |R_1| + |R_2|$ as desired. So we may assume that $n \geq 3$ and $(n,q) \neq (3,2)$.
Case 1. Let n be an odd number.

• If $2 \notin R_2(q)$, then $q \equiv 1 \pmod 4$ or $p = 2$. Since $n \geq 3$, we can see that $R_{2n} \cap R_2 = \emptyset$. Also, Lemma 3.4 and the fact that $(n,q) \neq (3,2)$ imply that $R_{2n}(q)$ is nonempty. Moreover, since $n \geq 3$ is odd, according to Lemmas 3.8(1), 3.6(2) and 3.5(2), we have $(2, r_2), (r_2, r_{2n}) \in \mathrm{GK}(G)$. Thus $\{r_{2n}, r_2\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_2) \in \mathrm{GK}(G)$:

Since $n \geq 3$ is odd, by Lemmas 3.8(1) and 3.6(2), we see that $(r_2, 2), (r_2, p) \in \mathrm{GK}(G)$. Thus, we may assume that $r \in R_l(q) \setminus \{2\}$, where $l \in \mathbb{N}$. Considering Lemma 3.5(2) implies that $2n/l$ is an odd number, so is $l/2$. Lemma 3.5(2) now yields r and r_2 are adjacent in $\mathrm{GK}(G)$.

Moreover, since $n \geq 3$ and $2 \notin R_2(q)$, considering Lemma 3.5(2) implies that $(r_2, r_{2(n-1)}) \in \mathrm{GK}(G)$ but $(r_{2n}, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Thus, $\deg(r_2) > \deg(r_{2n})$ and hence, in this case $\pi_1(G)$ cannot be m-regular.

• If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, q is odd. In this case, according to Lemma 3.8(1), we have $(2, r) \in \mathrm{GK}(G)$ if and only if $r \notin R_n(q)$. Also, Lemma 3.5(2) implies that $(r_n, r_{2n}) \notin \mathrm{GK}(G)$. Thus if the vertex r is adjacent to r_{2n}, then r and 2 are adjacent as well. On the other hand, according to Lemma 3.5(2), we have r_{2n} and $r_{2(n-1)}$ are nonadjacent and hence $\deg(2) > \deg(r_{2n})$, which implies that $\pi_1(G)$ cannot be m-regular.
Case 2. If n is even, then $n \geq 4$. In this case, by Lemmas 3.6(2) and 3.8(1), we conclude that r and 2 are adjacent if and only if $r \notin R_{2n}$. Thus if $r_{2n} \notin \pi_1(G)$, then $\pi_1(G)$ is a clique, which is impossible according to Lemma 2.2. Thus $r_{2n} \in \pi_1(G)$ and $\mathrm{GK}(G)$ is connected. In this case, we have $n = 2^k \times m$, where $m \geq 1$ is odd and $k \geq 1$. If $m = 1$, then Lemmas 3.6(2), 3.5(2) and 3.8(1) imply that $R_{2n}(q)$ is an odd connected component of $\mathrm{GK}(G)$, which is a contradiction. Therefore, $m \geq 3$ and $n \neq 2^k$. Now we claim that, if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_{2^{k+1}}) \in \mathrm{GK}(G)$:

Since n is even, by Lemmas 3.6(2) and 3.8(1), we conclude that

$$(2, r_{2n}), (p, r_{2n}) \notin \mathrm{GK}(G).$$

If $(r, r_{2n}) \in \mathrm{GK}(G)$, then $r \in R_l(q) \setminus \{2\}$, where $l \in \mathbb{N}$ and according to Lemma 3.5(2), $l = 2^{k+1} \times j$, where $j \mid m$. Therefore, we can easily infer our assertion by using Lemma 3.5(2).

On the other hand, since $n \neq 2^k$, Lemma 3.8(1) implies that $(2, r_{2^{k+1}}) \in \mathrm{GK}(G)$. Thus $\deg(r_{2n}) < \deg(r_{2^{k+1}})$ and $\pi_1(G)$ cannot be m-regular.

Consequently, if $G = B_n(q)$ or $C_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ is m-regular if and only if $(n,q) = (3,2)$ or $n = 2$ and $q > 2$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.
Part B. $G = D_n(q)$, where $n \geq 4$:
Case 1. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, $p \neq 2$. In this case, if n is even, then according to Lemma 3.6(3) and $|G|$, we conclude that $(r, p) \in \mathrm{GK}(G)$ if and only if $r \notin R_{n-1} \cup R_{2(n-1)}$. Also, considering Lemma 3.8(2) and $|G|$ imply that $(r, 2) \in \mathrm{GK}(G)$ if and only if $r \notin R_{n-1}$. Thus $R_{2(n-1)} \subseteq \pi_1(G)$ and $\deg(2) > \deg(p)$. If n is odd, then by the same procedure, we can conclude that $(r, 2) \in \mathrm{GK}(G)$ if and only if $r \notin R_n$ and $(r, p) \in \mathrm{GK}(G)$ if and only if $r \notin R_n \cup R_{2(n-1)}$. Thus $\deg(2) > \deg(p)$ and $\pi_1(G)$ cannot be m-regular.

Case 2. If $2 \notin R_2(q)$, then $p = 2$ or $q \equiv 1 \pmod 4$. If $(n, q) = (4, 2)$, then according to Lemma 2.1(3), the result is obvious. Thus we may assume that $(n, q) \neq (4, 2)$ and hence, $R_2 \cap R_{2(n-1)} = \emptyset$. Lemma 3.4 now implies $R_{2(n-1)}$ is nonempty. Also, by Lemmas 3.6(3) and 3.8(2), we have $(p, r_2), (2, r_2) \in \mathrm{GK}(G)$. Moreover, by Lemma 3.5(3), we can easily see that if r is an odd number which is adjacent to $r_{2(n-1)}$, then r is adjacent to r_2 as well. On the other hand, if $n \geq 5$, then $(r_2, r_3) \in \mathrm{GK}(G)$, but $(r_3, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Also, if $n = 4$, then $(r_2, r_4) \in \mathrm{GK}(G)$, but $(r_4, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Therefore, $\deg(r_2) > \deg(r_{2(n-1)})$ and $\pi_1(G)$ cannot be m-regular.

Consequently, if $G = D_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ is m-regular if and only if $(n, q) = (4, 2)$ and $m = 2$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.

Part C. $G = {}^2D_n(q)$, where $n \geq 4$:

Case 1. In this case, we assume that $2 \notin R_2(q)$ and hence, $q \equiv 1 \pmod 4$ or $p = 2$.

• If n is odd, then according to Lemmas 3.8(2), 3.6(3) and 3.5(3), we see that $(p, r_2), (2, r_2), (r_2, r_{2n}) \in \mathrm{GK}(G)$ and hence, $\{r_2, r_{2n}\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_2) \in \mathrm{GK}(G)$:

Since $(2, r_2), (p, r_2) \in \mathrm{GK}(G)$, it is sufficient to consider the case $r \in \pi_1(G) \setminus \{2, p\}$. Thus if $(r, r_{2n}) \in \mathrm{GK}(G)$, then there exists a natural number l, such that $r \in R_l(q)$. Applying Lemma 3.5(3) implies that $2n/l$ is an odd number, so is $l/2$. Thus by Lemma 3.5(3), we have $(r, r_2) \in \mathrm{GK}(G)$.

Moreover, Lemma 3.5(3) implies that $(r_2, r_4) \in \mathrm{GK}(G)$, but $(r_{2n}, r_4) \notin \mathrm{GK}(G)$. Thus $\deg(r_2) > \deg(r_{2n})$ and $\pi_1(G)$ cannot be m-regular.

• If $n \geq 4$ is even and $(n, q) \neq (4, 2)$, then $r_{2(n-1)} \in \pi(G)$ and it is enough to replace r_{2n} with $r_{2(n-1)}$ in the previous argument and conclude that $r_{2(n-1)} \in \pi_1(G)$ and $\deg(r_2) > \deg(r_{2(n-1)})$. If $G = {}^2D_4(2)$, then according to Lemma 3.6(3), we see that 2 is just adjacent to 3 and 5. Thus $\pi_1(G)$ should be 2-regular. But according to Lemma 3.5(3), 3 is adjacent to 2,5,7 and hence, $\pi_1(G)$ cannot be m-regular.

Case 2. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, q is odd and $q \equiv \varepsilon_0 \pmod 8$, where $\varepsilon_0 \in \{3, 7\}$.

• If n is even, then according to Lemma 3.8(2), we have $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_{2n}$. Thus $(R_{2(n-1)} \cup R_{n-1}) \subseteq \pi_1(G)$. Also, Lemma 3.6(3) implies that $(r, p) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)} \cup R_{n-1})$. Consequently, $\deg(2) > \deg(p)$ and Remark 3.9 implies that $\pi_1(G)$ cannot be m-regular.

• If n is odd and $q \equiv 7 \pmod 8$, then as in the even case we can see that $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_{2(n-1)}$. Also, $(r, p) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)})$. Thus similarly, we can conclude that $\pi_1(G)$ in not m-regular. Therefore, we may assume that n is odd and $q \equiv 3 \pmod 8$. Now, by Lemma 3.8(2), $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)})$. Thus $(r_3, 2), (2, r_{2(n-2)}) \in \mathrm{GK}(G)$. Also, Lemma 3.5(3) implies that $(r_3, r_{2(n-1)}), (r_3, r_{2n}), (r_3, r_{2(n-2)}) \notin \mathrm{GK}(G)$. Thus $\{r_3, r_{2(n-2)}\} \subseteq \pi_1(G)$ and the same argument in the above discussion conclude that $\deg(2) > \deg(r_3)$.

Consequently, if $G = {}^2D_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ cannot be m-regular.

Part D. $G = A_{n-1}(q)$, where $n \geq 2$ and $(n, q) \neq (2, 2), (2, 3)$:

Case 1. In this case, we consider $n \in \{2, 3\}$. If $n = 2$, then Lemma 2.2 implies that the simple group G has complete prime graph components. Also, we can see that $\pi(G) = \{p\} \cup R_1 \cup R_2$. So, we can easily conclude the result by Lemmas 3.6(1,4) and 3.7. Thus it remains to consider the case $n = 3$. In this case, if $p \neq 2$, then $\pi(G) = \{p\} \cup R_i$, where $1 \leq i \leq 3$, and according to Remark 3.9 and Lemma 3.7, we infer that $(r, 2) \notin \mathrm{GK}(G)$ if and only if $r \in R_3(q)$. Thus since $2 \in R_1 \cup R_2$ and $(2, p) \in \mathrm{GK}(G)$, we conclude that $\deg(2) = |R_1| + |R_2|$. Lemma 3.6(1,5) now yields $(r, p) \in \mathrm{GK}(G)$ if and only if

$$r \in R_1(q) \cup \{2\}, \text{ where } (q-1)_3 \neq 3; \tag{3.4}$$

$$r \in (R_1(q) \cup \{2\}) \setminus \{3\}, \text{ where } (q-1)_3 = 3. \tag{3.5}$$

Now we find the possible cases which $\pi_1(A_2(q))$ is m-regular:

• If $\{2,3\} \cap R_1(q) = \emptyset$, then $2 \in R_2(q)$ and $(q-1)_3 \neq 3$. Thus according to 3.4 we have $\deg(p) = 1 + |R_1|$ and since $\pi_1(G)$ is m-regular and $\deg(2) = |R_1| + |R_2|$, we are supposed to have $|R_2(q)| = 1$. Therefore, $q + 1 = 2^k$ and by Lemma 2.2, the result is obtained.

• If $2 \in R_1$ and $3 \notin R_1$, then according to 3.4, $\deg(p) = |R_1|$ and hence $\deg(2) > \deg(p)$ which implies that $\pi_1(G)$ is not m-regular.

• If $2 \notin R_1$ and $3 \in R_1$, then since $p \neq 2$, we conclude that $2 \in R_2$. Also, by 3.4 and 3.5, we have $\deg(p) = 1 + |R_1|$, where $(q-1)_3 > 3$ and $\deg(p) = |R_1|$, where $(q-1)_3 = 3$. Thus since $\pi_1(G)$ is m-regular, we infer that $\deg(2) = \deg(p)$. Since R_2 is nonempty, we conclude that $q + 1 = 2^k$. Therefore, the result is obvious by Lemma 2.2.

• If $\{2,3\} \subseteq R_1$. As in the previous case, we conclude that $\deg(p) = |R_1| - 1$, where $(q-1)_3 = 3$ and $\deg(p) = |R_1|$, where $(q-1)_3 > 3$. Thus both cases imply that $\deg(2) > \deg(p)$ and hence, in this case $\pi_1(G)$ is not m-regular.

Now it remains to consider the case $n = 3$ and $q = 2^\alpha \geq 4$:

• If $R_1 = \{3\}$, then according to Lemma 3.6(1,5), we conclude that 2 is a vertex with degree zero. On the other hand, since $3 \mid (2^\alpha - 1)$, we conclude that $\alpha = 2k$, where $k \in \mathbb{N}$. Thus $2^\alpha - 1 = (2^k - 1)(2^k + 1)$. But since $R_1 = \{3\}$, we have $2^k - 1 = 1$ and hence, $G \cong A_2(4)$ and $\pi_1(G)$ is 0-regular.

• If $R_1 \neq \{3\}$, then there is $r_1 \in R_1 \setminus \{3\}$. Now, according to Lemmas 3.6(5) and 3.7, we conclude that $(r, r_1) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_3$. Thus $\deg(r_1) = |R_1| + |R_2|$. Lemma 3.6(1,5) now yields $\deg(2) \leq |R_1|$ and hence, $\deg(2) < \deg(r_1)$ and in this case $\pi_1(G)$ cannot be m-regular.

Case 2. Let $n \geq 4$. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, $p \neq 2$. According to Lemma 3.6(1), $(p, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_n \cup R_{n-1})$. On the other hand, since $4 \mid (q+1)$, so $(q-1)_2 = 2$ and Lemma 3.7 implies that $(2, r) \in \mathrm{GK}(G)$ if and only if either $r \in \pi(G) \setminus R_n$ or $r \in \pi(G) \setminus R_{n-1}$. Thus $\deg(2) > \deg(p)$ and in this case $\pi_1(G)$ cannot be m-regular.

If $2 \notin R_2(q)$, then Lemmas 3.5(1), 3.6(1) and 3.7 imply that $(r_2, r_4), (r_2, 2) \in \mathrm{GK}(G)$ and hence, $\{r_2, r_4\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_4) \in \mathrm{GK}(G)$, then r and r_2 are adjacent as well:

Since $2 \notin R_2(q)$ and $n \geq 4$, according to Lemmas 3.6(1) and 3.7, we conclude that $(p, r_2), (2, r_2), (r_1, r_2) \in \mathrm{GK}(G)$. Thus if $(r, r_4) \in \mathrm{GK}(G)$, then it is enough to consider the case $r \in R_l(q)$, where $l \geq 2$. Since $(r_l, r_4) \in \mathrm{GK}(G)$, by Lemma 3.5(1), we have $l + 4 \leq n$ or $4 \mid l$ or $l \mid 4$. In each case, by using Lemma 3.5(1), we conclude that $(r_l, r_2) \in \mathrm{GK}(G)$.

If $n \geq 5$, then we can choose $l \in \{n-2, n-3\}$ as an odd integer greater than 1. Now by Lemma 3.5(1), we can easily check that $(r_l, r_4) \notin \mathrm{GK}(G)$, but $(r_l, r_2) \in \mathrm{GK}(G)$. Therefore, $\deg(r_2) > \deg(r_4)$. Now it remains to consider the case $n = 4$. In this case, we have $\pi(G) = \{p\} \cup R_i$, where $1 \leq i \leq 4$, and since $2 \notin R_2$, we have $p = 2$ or $4 \mid (q-1)$. Lemmas 3.7, 3.6(1) and 3.5(1) now yields $(r, r_4) \in \mathrm{GK}(A_3(q))$ if and only if $r \in R_4 \cup R_2$. Also, $(r, r_2) \in \mathrm{GK}(A_3(q))$ if and only if $r \in \pi(A_3(q)) \setminus R_3$. Thus $\deg(r_2) > \deg(r_4)$ and in this case $\pi_1(G)$ cannot be m-regular.

Consequently, according to Cases 1 and 2, we conclude that $\pi_1(A_{n-1}(q))$ is m-regular if and only if $\pi_1(A_{n-1}(q))$ is a clique.

Part E. $G = {}^2A_{n-1}(q)$, where $n \geq 3$ and $(n, q) \neq (3, 2)$:

Case 1. If $n = 3$, we consider the cases "q is even" and "q is odd", separately:

• If q is even, then $\pi(G) = \{2\} \cup R_1 \cup R_2 \cup R_6$. According to Lemma 3.6(1), we know that 2 is nonadjacent to r_1 and r_6. If $R_2 \neq \{3\}$, then Lemmas 3.7 and 3.6(1,5) imply that $(r_2, r) \in \mathrm{GK}(G)$, where $r_2 \in R_2 \setminus \{3\}$ and $r \in R_1 \cup R_2 \cup \{2\}$. Thus $\deg(2) \in \{|R_2|, |R_2| - 1\}$ and $\deg(r_2) = |R_1| + |R_2|$ which imply that $\deg(2) < \deg(r_2)$. But this is a contradiction to the fact that $\pi_1(G)$ is m-regular and hence, $R_2 = \{3\}$. Now, since $q \neq 2$ is even we deduce that $(q+1) = 3^k$ and $(q+1)_3 \neq 3$. Thus by Lemmas 3.7, 3.5(1) and 3.6(1,5) we can see that $(2, r) \in \mathrm{GK}(G)$ if and only if $r = 3$ and also, $(3, r) \in \mathrm{GK}(G)$ if and only if $r \in \{2\} \cup R_1$. Therefore, $\deg(2) = 1 < \deg(3) = |R_1| + 1$ and in this case $\pi_1(G)$ cannot be m-regular.

• If q is odd, then by Lemma 3.7 and Remark 3.9, we can easily see that $\deg(2) = |R_1| + |R_2|$. Since $\pi_1(G)$ is m-regular, Remark 3.9 implies that $\deg(p) = \deg(2) = |R_1| + |R_2|$. On the other hand, according to Lemma 3.6(1,5) and Remark 3.9, we conclude that $(r, p) \in \mathrm{GK}(G)$ if and only if

$$r \in (\{2\} \cup R_2) \setminus \{3\} \text{ and } (q+1)_3 = 3 \tag{3.6}$$

or

$$r \in \{2\} \cup R_2 \text{ and } (q+1)_3 \neq 3 \tag{3.7}$$

Thus if 3.6 holds, then $\deg(p) = |R_2|$ or $\deg(p) = |R_2| - 1$, where $2 \in R_1$ or $2 \in R_2$ respectively. If 3.7 holds, then $\deg(p) = |R_2| + 1$ or $\deg(p) = |R_2|$, where $2 \in R_1$ or $2 \in R_2$ respectively.

Therefore, according to the above statements, we can easily conclude that $(q+1)_3 \neq 3$, $q - 1 = 2^k$ and $m = |R_1| + |R_2|$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.

Case 2. If $n \geq 4$, then we consider the following two subcases:

Subcase a. If $R_1(q) = \emptyset$, then $q \in \{2, 3\}$. First we deal with the case $q = 2$. Since ${}^2A_3(2) \cong C_2(3)$ is 1-regular by Lemma 2.1(2), we can assume that $n \neq 4$. In this case, according to Lemma 3.6(1), $(r, 2) \notin \mathrm{GK}(G)$ if and only if $r \in R_l$, where $\nu(l) \in \{n, n-1\}$. Since $(n, q) \neq (4, 2)$, Lemma 3.7 implies that $\deg(2) < \deg(3)$. Similarly, if $n_3 > 3$, then we can conclude that $\deg(2) < \deg(3)$. Therefore, it remains to consider the case $n_3 = 3$. If $n = 6$, then by Lemmas 3.6(1), 3.7 and 3.5(1), we have $\pi_1(G) = \{2, 3, 5\}$ is 2-regular. Thus we may assume that $n \geq 12$ and in this case we know that $R_4(2) \cup R_8(2) = \{5, 17\} \subseteq \pi(G)$. According to Lemmas 3.6(1) and 3.7, we have $(2, r_4), (3, r_4) \in \mathrm{GK}(G)$. Now we claim that if $(r, r_8) \in \mathrm{GK}(G)$, then $(r, r_4) \in \mathrm{GK}(G)$:

Since r_4 is adjacent to 2 and 3 and $R_2(2) = \{3\}$, it is enough to consider the case $r \in R_l(2)$, where $\nu(l) \geq 2$. Thus if $(r_l, r_8) \in \mathrm{GK}(G)$, then by Lemma 3.5(1), we can see that $\nu(l) + 8 \leq n$, $\nu(l) \mid 8$ or $8 \mid \nu(l)$ which imply that $\nu(l) + 4 \leq n$, $\nu(l) \in \{2, 4, 8\}$ or $8 \mid \nu(l)$, respectively and hence, $(r_l, r_4) \in \mathrm{GK}(G)$.

Set l be an integer, where $\nu(l) \in \{n-5, n-4\}$ and $\nu(l)$ is odd. Thus by Lemma 3.5(1), we can conclude that $(r_l, r_4) \in \mathrm{GK}(G)$, but $(r_l, r_8) \notin \mathrm{GK}(G)$. Therefore, $\deg(r_4(2)) > \deg(r_8(2))$. Thus if $n \geq 4$, then $\pi_1({}^2A_{n-1}(2))$ is m-regular if and only if $(n, m) = (4, 1)$ or $(n, m) = (6, 2)$. Moreover, according to Lemma 2.2, in both cases $\pi_1({}^2A_{n-1}(2))$ is a clique.

If $q = 3$, then according to Lemma 3.6(1), we have $(r, 3) \notin \mathrm{GK}(G)$ if and only if $\nu(e(r, 3)) \in \{n-1, n\}$. On the other hand, if $n_2 \neq 4$, then by Lemma 3.7 and as in the above discussion, we can see that $\deg(2) = \deg(r_2(3)) > \deg(3)$. Thus it is enough to consider the case $n_2 = 4$. Since according to Lemma 2.1(3), $\pi_1({}^2A_3(3))$ is 2-regular, we may assume that $n \geq 8$. Also, we know that $R_n(3) \subseteq \pi(G)$. Now, we claim that if $(r, r_n) \in \mathrm{GK}(G)$, then $(r, r_4) \in \mathrm{GK}(G)$:

Since $n \geq 8$, according to Lemmas 3.6(1) and 3.7, we can see that $(3, r_4), (2, r_4) \in \mathrm{GK}(G)$ and since $R_2(3) = \{2\}$ we may assume that $r \in R_l(3)$, where $\nu(l) \geq 2$. Now Lemma 3.5(1) implies that $\nu(l) \mid n$. If $\nu(l) = n$, then since $4 \mid n$, by Lemma 3.5(1), we conclude that $(r, r_4) \in \mathrm{GK}(G)$. If $\nu(l) \neq n$, then $\nu(l) \leq n/2$ and since $n \geq 8$, so we have $\nu(l) + 4 \leq /2 + 4 \leq n$. Now, using Lemma 3.5(1) completes the proof of our claim.

Since $(2, r_4) \in \mathrm{GK}(G)$ and $(2, r_n) \notin \mathrm{GK}(G)$, according to the above discussion, we conclude that $\deg(r_4) > \deg(r_n)$. As $n \geq 4$, we can see that $\pi_1({}^2A_{n-1}(3))$ is m-regular if and only if $(n,m)=(4,2)$. Moreover, according to Lemma 2.2, in this case $\pi_1({}^2A_{n-1}(3))$ is a clique.

Subcase b. If $R_1(q)\neq\emptyset$, then we have the following cases:

• If $2\in R_1$, then $q\equiv 1 \pmod 4$ and hence q is odd. According to Lemma 3.6(1) and Remark 3.9, we conclude that $(r,p)\notin \mathrm{GK}(G)$ if and only if $\nu(e(r,q))\in\{n-1,n\}$. Since $\pi_1(G)$ is m-regular, we should have $\deg(2)=\deg(p)$ and hence, Lemma 3.7 implies that $n_2=(q+1)_2>2$, which is impossible according to $q\equiv 1 \pmod 4$.

• If $2\notin R_1$ and $n\geq 5$, then by using Lemmas 3.6(1), 3.7 and 3.5(1), we can easily see that $(p,r_1),(2,r_1),(r_1,r_4)\in \mathrm{GK}(G)$ and each vertex which is adjacent to r_4 is adjacent to r_1, as well. On the other hand, according to Lemma 3.5(1), the vertex r_l, where $\nu(l)\in\{n-3,n-2\}$ is odd, is adjacent to r_1 but is nonadjacent to r_4. Thus $\deg(r_1)>\deg(r_4)$ and $\pi_1(G)$ cannot be m-regular. If $2\notin R_1$ and $n=4$, then $p=2$ or $q\equiv -1 \pmod 4$. Thus Lemmas 3.6(1) and 3.7 imply that $(2,r_4),(p,r_4)\notin \mathrm{GK}(G)$. Now, by Lemma 3.5(1), we conclude that $(r,r_4)\in \mathrm{GK}(G)$ if and only if $r\in R_1\cup R_4$. Also, we can see that $(r,r_1)\in\mathrm{GK}(G)$ if and only if $r\in\{p\}\cup R_1\cup R_2\cup R_4$. Thus $\deg(r_1)>\deg(r_4)$ and hence, $\pi_1(G)$ cannot be m-regular.

Consequently, according to Cases 1 and 2, we conclude that $\pi_1({}^2A_{n-1}(q))$ is m-regular if and only if $\pi_1({}^2A_{n-1}(q))$ is a clique. □

In order to complete the proof of the main theorem, we need the following lemmas for considering the Ree groups ${}^2G_2(3^{2n+1})$ and ${}^2F_4(2^{2n+1})$, where n is a natural number.

3.11. Lemma. [14, Lemma 1.5(2-3)] Let n be a natural number.

(1) Let $m_1(G,n)=3^{2n+1}-1$, $m_2(G,n)=3^{2n+1}+1$, $m_3(G,n)=3^{2n+1}-3^{n+1}+1$, $m_4(G,n)=3^{2n+1}+3^{n+1}+1$. Then $\gcd(m_1(G,n),m_2(G,n))=2$ and $\gcd(m_i(G,n),m_j(G,n))=1$ otherwise.

(2) Let $m_1(F,n)=2^{2n+1}-1$, $m_2(F,n)=2^{2n+1}+1$, $m_3(F,n)=2^{4n+2}+1$, $m_4(F,n)=2^{4n+2}-2^{2n+1}+1$, $m_5(F,n)=2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$, $m_6(F,n)=2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$.
Then $\gcd(m_2(F,n),m_4(F,n))=3$ and $\gcd(m_i(F,n),m_j(F,n))=1$ otherwise.

3.12. Lemma. [14, Propositions 3.3(2-3)] Let G be a finite simple Ree group over a field of characteristic p, let $r\in\pi(G)\setminus\{p\}$. Then r, p are nonadjacent if and only if one of the following holds:

(1) $G={}^2G_2(3^{2n+1})$, r divides $m_k(G,n)$ and $r\neq 2$.
(2) $G={}^2F_4(2^{2n+1})$, r divides $m_k(F,n)$, $r\neq 3$ and $k>2$.

3.13. Lemma. [14, Propositions 4.5(8)] If $G={}^2G_2(3^{2n+1})$ and $r\in\pi(G)\setminus\{2,3\}$, then r and 2 are nonadjecent if and only if r divides $m_3(G,n)$ or $m_4(G,n)$.

If $G={}^2F_4(2^{2n+1})$, then denote by $\mathcal{S}_i(G)$ the set $\pi(m_i(F,n))\setminus\{3\}$. Thus we have the following lemma:

3.14. Lemma. [15, Propositions 2.9(3)] Let $G={}^2F_4(2^{2n+1})$ and $r,s\in\pi(G)\setminus\{2\}$. Then r and s are nonadjecent if and only if either $r\in\mathcal{S}_k(G)$ and $s\in\mathcal{S}_l(G)$, where $l\neq k$, $\{k,l\}\neq\{1,2\},\{1,3\}$; or $r=3$ and $s\in\mathcal{S}_l(G)$, where $l\in\{3,5,6\}$.

3.15. Lemma. Let G be a finite simple exceptional group of Lie type. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G={}^2B_2(2^{2n+1})$ and $m=0$;
(2) $G={}^3D_4(2)$ and $m=2$;
(3) $G=G_2(3^n)$ and $m=|R_1(q)|+|R_2(q)|$.

Proof. According to the compact form of $\mathrm{GK}(G)$ in [15], where

$$G\in\{E_7(q),E_8(q),E_6(q),{}^2E_6(q),F_4(q)\},$$

we can easily find two vertices p and q in $\pi_1(G)$ which have the following properties:

(1) If $(p, r) \in \mathrm{GK}(G)$, then $(q, r) \in \mathrm{GK}(G)$;
(2) There exists a prime s in $\pi(G)$, where $(p, s) \notin \mathrm{GK}(G)$ but $(q, s) \in \mathrm{GK}(G)$.

Thus $\deg(q) > \deg(p)$ which implies that $\pi_1(G)$ cannot be m-regular. In the same manner we can see that $\pi_1(^3D_4(q))$ is m-regular if and only if $q = 2$. We omit the details for convenience. Also, according to the compact form of $\mathrm{GK}(G_2(q))$, we can see that $\pi_1(G_2(q))$ is m-regular if and only if $q = 3^\alpha$. In this case, we have $m = |R_1| + |R_2|$. Moreover, by Lemma 2.1(1), we know that $\pi_1(^2B_2(2^{2n+1}))$, where $n \in \mathbb{N}$, is 0-regular. Additionally, if $G = {}^2F_4(2)'$, then using [1] implies that $\deg(2) = 2$, $\deg(3) = 1$ and $(2, 3) \in \mathrm{GK}(G)$ and hence, $\pi_1(^2F_4(2)')$ is not m-regular. Thus it remain to consider the simple groups, $^2G_2(3^{2n+1})$ and $^2F_4(2^{2n+1})$, where $n \in \mathbb{N}$. If $G = {}^2G_2(3^{2n+1})$, then Lemma 3.12(1) implies that $(3, r) \in \mathrm{GK}(G)$ if and only if $r = 2$. Also, according to Lemma 3.13, we can see that $(2, r) \in \mathrm{GK}(G)$ if and only if $r = 3$ or $r \mid m_1(G, n)$ or $r \mid m_2(G, n)$. Thus $\deg(2) > \deg(3)$ and $\pi_1(^2G_2(3^{2n+1}))$ is not m-regular. Finally, if $G = {}^2F_4(2^{2n+1})$, then Lemma 3.12(2) implies that $(2, r) \in \mathrm{GK}(G)$ if and only if $r = 3$ or $r \mid m_1(F, n)$ or $r \mid m_2(F, n)$. Moreover, according to Lemma 3.14, we can see that $(3, r) \in \mathrm{GK}(G)$ if and only if $r = 2$ or $r \mid m_1(F, n)$ or $r \mid m_2(F, n)$ or $r \mid m_4(F, n)$. Thus $\deg(2) < \deg(3)$ and $\pi_1(^2F_4(2^{2n+1}))$ is not m-regular. □

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