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Composition in EL–hyperstructures

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Abstract

The link between ordered sets and hyperstructures is one of the classical areas of research in the hyperstructure theory. In this paper we focus on *EL–hyperstructures*, i.e. a class of hyperstructures constructed from quasi-ordered semigroups. In our paper we link this concept to the concept of a *composition hyperring*, a recent hyperstructure generalization of the classical notion of a composition ring.

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1. Introduction

Since the times of elementary algebra, the scope of this mathematical discipline has widened considerably. Already in 1930s, a step from the study of single-valued structures to the study of multi-valued structures was made. This new creation, the *hyperstructure theory*, has since then grown to a fully established branch of algebra with numerous far-reaching applications in geometry, graph-theory, coding theory, medicine, number theory, physics, chemistry, etc. For basic introduction to the theory and applications see [9, 11].

Two important multi-valued analogues of classical topics of algebra intersect in this paper: *the study of ordered sets* and their connection to hyperstructures and *the study of ring-like hyperstructures*.

The ordered sets have been in the focus of attention of the hyperstructure theory since works of Nieminen, Corsini, Rosenberg, Krasner, Mittas, Davvaz, Leoreanu or Chvalina of 1960s to 1990s. Notice that one of the first chapters of [9], a canonical book of the hyperstructure theory, is dedicated to ordered sets. Selected reading on some aspects of the topic includes also works such as [3,4,8,16]. Furthermore, Heidari and Davvaz [16] have recently introduced the notion of *partially ordered semihypergroups*, i.e. have transferred the concept of partially ordered semigroups to hyperstructures.

Krasner [20] introduced the notion of the *hyperfield* and then *hyperring* in order to approximate a local field of positive characteristic by a system of local fields of characteristic

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zero. The additive part of this hyperring was a special hypergroup while the multiplicative part was a semigroup. Constructions of these structures can be found in [19, 22, 28]. While studying polynomials over Krasner's hyperrings, Mittas [27] introduced *superrings*, in which both parts, additive and multiplicative, were hyperstructures. G. Massouros, approching the theory of languages and automata from the point of view of hypercompositional algebra, was led to the introduction of the concepts of *hyperringoid* and *join hyperring* [23, 24]. Also, Vougiouklis [33] generalizing Mitas' superring introduced *hyperrings in the general sense*. Some recent papers on the topic include [2, 7, 13, 26] and a book [11].

Motivated by the study of properties of the hyperring of polynomials [18], Cristea and Jančić-Rašović in [10] introduced the concept of *composition hyperring* as a multi-valued generalisation of an older concept of the *composition ring* introduced in [1]. Notice that as regards single-valued rings, composition leads to interesting applications in rings of polynomials, power series or in the field of rational functions. In [12], the concept of composition is used to construct *composition* (m, n, k)*–hyperrings*.

In this paper we study composition, suggested by Cristea and Jančić-Rašović, in EL–hyperstructures, i.e. in a class of hyperstructures constructed from quasi-ordered semigroups. The authors of [10] define the *composition hyperoperation* in hyperrings in the general case of [32], i.e. in multivalued systems $(R, +, \cdot)$, where $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup and the multiplication is distributive with respect to the addition. In our paper we partly broaden this environment by suggesting implications also for cases of $(R, +)$ being a semihypergroup (making use of results achieved in [30]).

2. EL–hyperstructures: construction and use

There exist numerous constructions of hyperstructures from given single-valued algebraic structures. The concept of *EL–hyperstructures* was coined by Chvalina in [4] and explored in e.g. [15, 29, 31]. The construction is based on validity of a rather simple and straightforward Lemma 2.1. However, when looking for examples of EL–hyperstructures, the simplicity and straightforwardness disappear. Naturally, there are obvious intuitive face-value examples such as $(\mathbb{N}, +, \leq)$ or $(\mathcal{P}(S), \cap, \subseteq)$. EL-hyperstructures have also been used in papers such as [5, 6, 14] or Sections 8.3 and 8.4 of book [11] in the context of quasi-ordered semigroups such that the nature of their elements and the operation and ordering follow from the application task. In this respect also notice [21], where EL–hyperstructures have been used to construct a class of H_v–matrices. Finally, there is another layer of possible uses: Suppose that we have a set of elements, properties of which can be described by means of numerical values (such as length, cardinality, number of elements of a sequence, etc.). Since number domains with a suitably chosen operation and the natural ordering with respect to size often form quasi-ordered semigroups, Lemma 2.1 presents a natural way of constructing (associative and commutative) hyperstructures out of them. In this paper we intentionally demonstrate our results using the simplest possible examples. For a deeper insight and less obvious and straightforward uses of the construction see the above mentioned references.

Further on we work with *principal ends* (hence EL which stands for "Ends lemma"), i.e. for an arbitrary $a \in (S, \leq)$ we set $[a)_\leq = \{x \in S; a \leq x\}$.

Lemma 2.1. ([4], Theorem 1.3 & Theorem 1.4, pp. 146–147). *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The binary hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ defined by*

$$a * b = [a \cdot b)_\leq \tag{2.1}$$

is associative. The semihypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative. Furthermore, the following conditions are equivalent:

1^0**:** *For any pair* $(a, b) \in S^2$ *there exists a pair* $(c, c') \in S^2$ *such that* $b \cdot c \leq a$ *and* $c' \cdot b \leq a$.

2^0**:** *The associated semihypergroup* $(S, *)$ *is a hypergroup.*

Remark 2.2. If $(S, \cdot, \leq)$ is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S, \cdot, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup. In fact, it is a transposition hypergroup, i.e. our reasoning results in *transposition hyperrings*, which can suggest another line of further research. For the use of transposition axiom in hypercompositional structures see [25]. Cases of $(S, \cdot)$ not being a group yet resulting in a hypergroup $(S, *)$ are discussed in [31]. It can also be easily verified that we can assume *quasi-ordered* structures instead of *partially ordered* ones in Lemma 2.1 (however, beware that in this case commutativity of the hyperoperation does not imply commutativity of the single-valued operation). For details see e.g. [29].

3. Basic notions and concepts, notation

Throughout the paper we work with the following definitions and concepts. By a *hyperring in the general sense* and by a *semihyperring in the general sense* we mean systems $(R, +, \cdot)$ discussed e.g. in [33].

Definition 3.1. ([33], p. 21, included as plain text) $(R, +, \cdot)$ is a hyperring in the general sense if $(R, +)$ is a hypergroup, $(\cdot)$ is associative hyperoperation and the distributive law $x(y + z) \subseteq xy + xz$, $(x + y)z \subseteq xz + yz$ is satisfied for every x, y, z of R. [...] $(R, +, \cdot)$ will be called semihyperring if $(+), (\cdot)$ are associative hyperoperations, where $(\cdot)$ is distributive with respect to $(+)$. The rest of definitions are analogous. If the equality in the distributive law is valid, then the hyperring is called strong or good.

By a *hyperring* and by a *semihyperring* we mean a *good* hyperring, or a *good* semihyperring in the sense of Definition 3.1, respectively. Notice that this means that our concept of *hyperring* is the same as the concept used in [10, 18, 32] yet it permits a generalisation in the sense of inclusions.

Composition hyperrings were introduced in [10] as a special class of hyperrings with one additional property.

Definition 3.2. ([10], Def. 3.1) A composition hyperring is an algebraic structure $(R, +, \cdot, \circ)$, where $(R, +, \cdot)$ is a commutative hyperring and the hyperoperation $\circ$ satisfies the following properties, for any $x, y, z \in R$:

(1) $(x + y) \circ z = x \circ z + y \circ z$
(2) $(x \cdot y) \circ z = (x \circ z) \cdot (y \circ z)$
(3) $x \circ (y \circ z) = (x \circ y) \circ z$.

The binary hyperoperation $\circ$ having the previous properties is called the composition hyperoperation of the hyperring $(R, +, \cdot)$.

To be consistent with the background and reasoning of [1, 10] we further on deal with commutative hyperoperations and composition property only. Notice that in the construction using Lemma 2.1 commutativity of the single-valued operation implies commutativity of the hyperoperation and antisymmetry of $\leq$ turns this implication into equivalence. If $x \circ y$ is a one-element set for all $x, y \in R$, we will speak about an operation rather than a hyperoperation even though it will have to be at certain point applied in an element-wise manner on sets (see below in e.g. (5.7) Theorem 5.10). Throughout the paper we will be interested in the composition (hyper)operation in various types of hyperstructures $(R, +, \cdot)$ – not only in hyperrings but also in hyperrings in the general sense, semihyperrings or semihyperrings in the general sense.

Since we construct hyperoperations from single-valued operations on the same set, we have to alter the standard notation of hyperoperations in ring-like hyperstructures. Thus in our context the symbols $+$ and $\cdot$ will be reserved for single-valued operations and the hyperoperations will be denoted by $\oplus$ and $\bullet$. The hyperoperations will be constructed from *single-valued quasi-ordered semigroups* using Lemma 2.1, i.e. for all $x, y \in R$, where $(R, +, \leq)$ and $(R, \cdot, \leq)$ are quasi-ordered semigroups, we define

$$a \oplus b = [a + b)_{\leq} = \{x \in R; a + b \leq x\} \tag{3.1}$$

and

$$a \bullet b = [a \cdot b)_{\leq} = \{y \in R; a \cdot b \leq y\} \tag{3.2}$$

and get hyperstructures $(R, \oplus, \bullet)$ which we then study. Since $(R, \oplus)$ and $(R, \bullet)$ are EL–hyperstructures, it is possible to apply results achieved in [29–31] and immediately state further properties of both $(R, \oplus)$, $(R, \bullet)$ and $(R, \oplus, \bullet)$.

4. EL–hyperstructures with two hyperoperations

First we show the variety of EL–hyperstructures with two hyperoperations which can be obtained using hyperoperations (3.1) and (3.2). Thus the following lemma, included in [30] as Theorems 5.2, 5.4 and 5.5., bounds the area of our future considerations.

Lemma 4.1. *Let $(R, +, \leq)$ and $(R, \cdot, \leq)$ be quasi-ordered semigroups and $\oplus$, $\bullet$ hyperoperations defined by (3.1) and (3.2) respectively. Furthermore, let $\cdot$ distribute over $+$ from both left and right.*

1. *$(R, \oplus, \bullet)$ is a semihyperring in the general sense.*
2. *If $(R, +)$ is a group or if $(R, \oplus)$ is a hypergroup, then $(R, \oplus, \bullet)$ is a hyperring in the general sense.*
3. *If $(R, \cdot)$ is a group, then $(R, \oplus, \bullet)$ is a semihyperring.*
4. *If $(R, +)$ is a group with neutral element 0 and $(R \setminus \{0\}, \cdot)$ is a group, then $(R, \oplus, \bullet)$ is a hyperring.*

Proof. The proof is included in [30] and is based on use of [30], Lemma 4.1, Lemma 4.4, which discuss distributivity, and Remark 4.8, which discusses the role of the absorbing element of the single-valued ring-like structures. Since Lemma 4.1 is important in the context of this paper and not including at least a sketch of its proof would not be correct, we include the main idea of the proof here.

First we show that, for all $a, b, c \in R$, where $(R, +, \leq)$ and $(R, \cdot, \leq)$ are quasi-ordered semigroups, there is

$$\begin{aligned} a \cdot (b + c) = a \cdot b + a \cdot c \quad &\Rightarrow \quad a \bullet (b \oplus c) \subseteq a \bullet b \oplus a \bullet c \\ (a + b) \cdot c = a \cdot c + b \cdot c \quad &\Rightarrow \quad (a \oplus b) \bullet c \subseteq a \bullet c \oplus b \bullet c \end{aligned} \tag{4.1}$$

This is done in the usual way of rewriting both sides of the inclusions using (3.1) and (3.2) and then proving that an arbitrary element from one side of the inclusion is included in the other one.

If we now suppose that $(R, \cdot, \leq)$ is a quasi-ordered group, then with the help of inverse elements we are able to prove the opposite inclusions, i.e.

$$\begin{aligned} a \cdot (b + c) = a \cdot b + a \cdot c \quad &\Rightarrow \quad a \bullet (b \oplus c) \supseteq a \bullet b \oplus a \bullet c \\ (a + b) \cdot c = a \cdot c + b \cdot c \quad &\Rightarrow \quad (a \oplus b) \bullet c \supseteq a \bullet c \oplus b \bullet c \end{aligned} \tag{4.2}$$

for all $a, b, c \in R$.

To complete the proof we need to discuss the role of the potentially existing absorbing elements. Suppose $a = 0$ (or $c = 0$ in the second inclusion) in (4.1). We get $[0)_{\leq} \subseteq \bigcup\limits_{x,y \in [0)_{\leq}} [x + y)_{\leq}$ for $a = 0$ or $\{0\} \subseteq [0)_{\leq}$ for $c = 0$. Since the relation $\leq$ is reflexive, this obviously holds and does not cause any problems. If we suppose $a = 0$ (or $c = 0$) in (4.2), we get that

$$\bigcup_{x,y\in[0)_\leq} [x+y)_\leq \subseteq \bigcup_{h\in[b+c)_\leq} [0\cdot h)_\leq = [0)_\leq.$$

However $x, y \in [0)_\leq$ means that $0 \leq x$, $0 \leq y$, i.e. $0 = 0 + 0 \leq x + y$, i.e.

$$\bigcup_{x,y\in[0)_\leq} [x+y)_\leq = [0)_\leq,$$

i.e. we get equality $[0)_\leq = [0)_\leq$. If in the second inclusion $c = 0$, then we get the same equality $[0)_\leq = [0)_\leq$.

Thus we have shown the respective parts on distributivity. The rest follows from Lemma 2.1 and definitions of the respective ring-like hyperstructures. □

Remark 4.2. Notice that [31] discusses conditions under which Lemma 2.1 applied on a quasi-ordered semigroup which is not a group constructs a hypergroup. In this respect Lemma 4.1, item 2, could be made stronger – see Example 4.3. The same holds for analogous situations, e.g. below in Theorem 6.2.

Example 4.3. Regard an arbitrary set S and its power set $\mathcal{P}(S)$. The operations $\cap$, $\cup$ of set intersection and set union are associative, thus $(\mathcal{P}(S), \cap)$ and $(\mathcal{P}(S), \cup)$ are semigroups. The relation $\subseteq$ on $\mathcal{P}(S)$ is obviously reflexive and transitive and for arbitrary $A, B, C \in \mathcal{P}(S)$ such that $A \subseteq B$ there is $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Thus if we define hyperoperations $\oplus$, $\bullet$ for arbitrary $A, B \in \mathcal{P}(S)$ by

$$A \oplus B = [A \cup B)_\leq = \{X \in \mathcal{P}(S); A \cup B \subseteq X\} \tag{4.3}$$

and

$$A \bullet B = [A \cap B)_\leq = \{Y \in \mathcal{P}(S); A \cap B \subseteq Y\}, \tag{4.4}$$

we get semihypergroups $(\mathcal{P}(S), \oplus)$ and $(\mathcal{P}(S), \bullet)$. Moreover, as set intersection is distributive with respect to set union, $(\mathcal{P}(S), \oplus, \bullet)$ is a semihyperring in the general sense.

5. The composition hyperoperation in various EL–ring-like hyperstructures

In this section we study the potential and limitations of hyperstructures suggested in Section 4 with respect to the composition hyperoperation (or operation). Since the hyperstructures are constructed from single-valued structures, we concentrate on properties of the hyperstructures which follow from properties of the single-valued structures.

In the text below notice the precise meaning of symbols $\oplus$ and $\bullet$. When applied on single elements, they are used in the meanings (3.1) and (3.2) respectively. However, for all sets $A, B \subseteq R$ there is

$$A \oplus B = \bigcup_{\substack{a\in A\\ b\in B}} [a+b)_\leq = \bigcup_{\substack{a\in A\\ b\in B}} \{x \in R; a+b \leq x\} \tag{5.1}$$

and

$$A \bullet B = \bigcup_{\substack{a\in A\\ b\in B}} [a\cdot b)_\leq = \bigcup_{\substack{a\in A\\ b\in B}} \{y \in R; a\cdot b \leq y\}. \tag{5.2}$$

First of all we discuss a rather trivial case of *constant composition*.

Definition 5.1. If there is $x \circ y = r \circ s$ for an arbitrary quadruple of elements $x, y, r, s \in R$, we call the composition operation (hyperoperation) $\circ$, defined in Definition 3.2, constant composition operation (hyperoperation).

The following theorem holds for all types of hyperstructures discussed in Lemma 4.1.

Theorem 5.2. *Let $(R, \oplus, \bullet)$ be a semihyperring in the general sense constructed in Lemma 4.1 from idempotent quasi-ordered semigroups $(R, +, \leq)$ and $(R, \cdot, \leq)$. Consider $r \in R$ arbitrary. Then $\circ$ defined by*

$$a \circ b = [r)_{\leq} \tag{5.3}$$

for all $a, b \in R$, is a constant composition hyperoperation on $(R, \oplus, \bullet)$. It is a constant operation if $\leq$ is antisymmetric and r is the greatest element of $(R, \leq)$.

Proof. In the $\oplus, \bullet$ notation, the left-hand side of Definition 3.2, property 1, reads $(x \oplus y) \circ z$. This is

$$[x+y)_{\leq} \circ z = \bigcup_{\substack{\text{number of elements} \\ \text{of } [x+y)_{\leq} \text{ –times}}} [r)_{\leq} = [r)_{\leq}.$$

The right-hand side reads $(x \circ z) \oplus (y \circ z)$, which is

$$[r)_{\leq} + [r)_{\leq} = \bigcup_{a,b \in [r)_{\leq}} [a+b)_{\leq} = \bigcup_{\substack{r \leq a \\ r \leq b}} [a+b)_{\leq}.$$

Since $r \leq a$, $r \leq b$ implies $r + r \leq a + b$ and the relation $\leq$ is reflexive, there is $[r)_{\leq} + [r)_{\leq} = [r+r)_{\leq}$. For idempotent $+$ there is $r + r = r$, i.e. $[r)_{\leq} + [r)_{\leq} = [r)_{\leq}$.

The same reasoning can be applied on property 2 of Definition 3.2. Property 3 holds obviously. Finally, if r is the greatest element of $(R, \leq)$, then $[r)_{\leq} = \{r\}$, thus we can speak about an operation instead of a hyperoperation. □

Example 5.3. If we continue with Example 4.3, where the semihyperring in the general sense of the power set $\mathcal{P}(S)$ is discussed, and define

$$A \circ B = [R)_{\subseteq} = \{T \in \mathcal{P}(S); R \subseteq T\}$$

for an arbitrary pair of $A, B \in \mathcal{P}(S)$, we get a constant composition hyperoperation on $\mathcal{P}(S)$. If $R = S$, then $\circ$ becomes a constant composition operation.

Theorem 5.2 obviously does not hold when operations $+$ or $\cdot$ are non-idempotent. Not even one of the inclusions holds because neither $r \in [r+r)_{\leq}$ nor $r + r \in [r)_{\leq}$ in a general case. Yet for all types of hyperstructures discussed in Lemma 4.1 we might prove the following.

Theorem 5.4. *Let $(R, \oplus, \bullet)$ be a semihyperring in the general sense constructed in Lemma 4.1 from partially ordered semigroups $(R, +, \leq)$ and $(R, \cdot, \leq)$. If they exist, denote e_s the neutral element of $(R, +)$ and e_p the neutral element of $(R, \cdot)$.*

(1) *If simultaneously $e_p \leq e_p + e_p$ and $e_s \leq e_s \cdot e_s$, then $\circ_{\min e}$ defined by*

$$a \circ_{\min e} b = [\min\{e_s, e_p\})_{\leq} \tag{5.4}$$

for all $a, b \in R$, is a constant composition hyperoperation on $(R, \oplus, \bullet)$.

(2) *If simultaneously $e_p + e_p \leq e_p$ and $e_s \cdot e_s \leq e_s$, then $\circ_{\max e}$ defined by*

$$a \circ_{\max e} b = [\max\{e_s, e_p\})_{\leq} \tag{5.5}$$

for all $a, b \in R$, is a constant composition hyperoperation on $(R, \oplus, \bullet)$.

Before proving the theorem, agree that, if the elements e_s, e_p are incomparable, then since the minimum does not exist, we set $a \circ_{\min e} b = \emptyset$. Moreover, if only e_s exists, then we set $\min\{e_s, e_p\} = e_s$ (and the same for e_p). And make the similar agreement for the maxima.

Proof. We will prove the theorem for $\circ_{\min e}$ only. The proof for $\circ_{\max e}$ is analogous.

In the $\oplus, \bullet$ notation the left-hand-side of Definition 3.2, property 1, reads $(x \oplus y) \circ z$. This is

$$[x+y)_{\leq} \circ_{\min e} z = \bigcup_{\substack{\text{number of elements} \\ \text{of } [x+y)_{\leq}\text{–times}}} [\min\{e_s, e_p\})_{\leq}$$
$$= [\min\{e_s, e_p\})_{\leq}$$

while the right-hand side, which reads $(x \circ z) \oplus (y \circ z)$, is

$$[\min\{e_s, e_p\})_{\leq} + [\min\{e_s, e_p\})_{\leq} = \bigcup_{\substack{\min\{e_s,e_p\} \leq a \\ \min\{e_s,e_p\} \leq b}} [a+b)_{\leq}.$$

Now the following cases are possible:

$e_s \leq e_p$: : This means that $\min\{e_s, e_p\} = e_s$; the left-hand side is $[e_s)_{\leq}$ while the right-hand side is $\bigcup_{\substack{e_s \leq a \\ e_s \leq b}} [a+b)_{\leq} = [e_s + e_s)_{\leq} = [e_s)_{\leq}$, i.e. the same.

$e_p < e_s$: : This means that $\min\{e_s, e_p\} = e_p$; the left-hand side is $[e_p)_{\leq}$ while the right hand side is $\bigcup_{\substack{e_p \leq a \\ e_p \leq b}} [a+b)_{\leq} = [e_p+e_p)_{\leq}$. Suppose now an arbitrary $x \in [e_p)_{\leq}$, i.e. such $x \in R$ that $e_p \leq x$. Since we assume that $e_p < e_s$, there is also $e_p + e_p < x + e_s = x$, i.e. $x \in [e_p + e_p)_{\leq}$. If on the other hand we suppose an arbitrary $x \in [e_p + e_p)_{\leq}$, i.e. $e_p + e_p \leq x$, then on condition assumed in the theorem, i.e. $e_p \leq e_p + e_p$, there is from transitivity that $e_p \leq x$, which means that $x \in [e_p)_{\leq}$. Altogether $[e_p)_{\leq} = [e_p + e_p)_{\leq}$.

If neither e_s nor e_p exists or if e_s and e_p are incomparable, we end up with $\emptyset = \emptyset$. If only e_s exists, we get the same as when $e_s \leq e_p$. If only e_p exists, we get the same as when $e_p < e_s$.

The proof of Definition 3.2 property 2, is completely analogous. The proof of property 3 is obvious. □

Example 5.5. Since $(\mathbb{Z}, +, \leq)$, where $\leq$ is the natural ordering of integers, is a partially ordered group, $(\mathbb{Z}, \cdot, \leq)$ a partially ordered semigroup and $e_s = 0$, $e_p = 1$, the hyperoperation $\circ$ defined for all $a, b \in \mathbb{Z}$ by $a \circ b = [0)_{\leq}$ is an example of a constant composition hyperoperation on the hyperring in the general sense $(\mathbb{Z}, \oplus, \bullet)$, where $\oplus$ and $\bullet$ are defined by (3.1) and (3.2) respectively, in a context when the single-valued operations $+, \cdot$ are non-idempotent. The conditions of Theorem 5.4 obviously hold since $1 \leq 1+1$ and $0 \leq 0 \cdot 0$.

The constant compositions are rather trivial and degenerated cases yet even there the limits of applying the composition property in the context of the “Ends lemma", i.e. on hyperoperations based on the sets of the $[a)_{\leq}$ type, can be seen. It is rather difficult to achieve equality in properties 1 and 2 since the addition (or multiplication) on the left-hand side is applied on elements while on the right-hand side it is (in a general case) applied on sets – and this is done in a context where neither $a \in [a+a)_{\leq}$ nor $a + a \in [a)_{\leq}$ holds generally.

Let us therefore adjust the composition hyperoperation defined in Definition 3.2 to suit EL–hyperstructures better. In order to keep notation uniform with Definition 3.2 we use symbols $+, \cdot$ for the hyperoperations even though below we are going to use Definition 5.6 only in context of hyperoperations $\oplus$, $\bullet$.

In the following definition we speak of “semihyperrings in the general sense". This is because they are the weakest of hyperstructures discussed in Lemma 4.1. Thus we make sure that the future considerations are valid for all types of relevant hyperstructures.

Definition 5.6. A binary operation (hyperoperation) on a semihyperring in the general sense $(R, +, \cdot)$, where $+$ and $\cdot$ are hyperoperations on R, is called a left weak composition operation (hyperoperation) and denoted $\circ_{lw}$ if, for all $x, y, z \in R$,

(1) $(x + y) \circ_{lw} z \subseteq (x \circ_{lw} z) + (y \circ_{lw} z)$
(2) $(x \cdot y) \circ_{lw} z \subseteq (x \circ_{lw} z) \cdot (y \circ_{lw} z)$
(3) $x \circ_{lw} (y \circ_{lw} z) = (x \circ_{lw} y) \circ_{lw} z$.

or the right weak composition operation (hyperoperation) and denoted $\circ_{rw}$ if, for all $x, y, z \in R$:

(1) $(x \circ_{rw} z) + (y \circ_{rw} z) \subseteq (x + y) \circ_{rw} z$
(2) $(x \circ_{rw} z) \cdot (y \circ_{rw} z) \subseteq (x \cdot y) \circ_{rw} z$
(3) $x \circ_{rw} (y \circ_{rw} z) = (x \circ_{rw} y) \circ_{rw} z$.

The hyperstructure $(R, +, \cdot, \circ_W)$ (regardless of type) is called a weak composition hyperstructure (i.e. weak composition semihyperring / weak composition hyperring / etc.) regardless of whether $\circ_W = \circ_{lw}$ or $\circ_W = \circ_{rw}$ or whether $\circ_W$ is single– or multi–valued.

Chvalina has in [3,4] and subsequent papers introduced and studied the concept of *quasi-order hypergroups*, which has been studied by a number of authors since. In the following theorem we not only give necessary conditions for the existence of a left (right) weak composition hyperoperation but also establish a link between quasi-order hypergroups and EL–hyperstructures by defining the composition hyperoperation by $a \circ b = [a)_\leq \cup [b)_\leq$ for all $a, b \in R$, i.e. by a condition used when testing whether a hypergroupoid $(H, \circ)$ is a quasi-order hypergroup. (For details see e.g.[9], chapter 3, §1). Notice that thanks to reflexivity of relation $\leq$ the set $[a)_\leq \cup [b)_\leq$ has for $a \neq b$ always at least two elements.

Theorem 5.7. *Let $(R, \oplus, \bullet)$ be a semihyperring in the general sense constructed in Lemma 4.1 from quasi-ordered semigroups $(R, +, \leq)$ and $(R, \cdot, \leq)$. If, for all $r \in R$, there is $r + r \leq r$ and $r \cdot r \leq r$, then there exists a left weak composition hyperoperation $\circ_{lw}$ on $(R, \oplus, \bullet)$.*

Proof. Define $a \circ_{lw} b = [a)_\leq \cup [b)_\leq$ for all $a, b \in R$. In this context the left-hand side of property 1 of Definition 5.6 is

$$[x + y)_\leq \circ_{lw} z = \bigcup_{x+y \leq a} [a)_\leq \cup [z)_\leq = [x + y)_\leq \cup [z)_\leq$$

while the right-hand side is

$$\begin{aligned}(x \circ_{lw} z) \oplus (y \circ_{lw} z) &= ([x)_\leq \cup [z)_\leq) \oplus ([y)_\leq \cup [z)_\leq) \\ &= \bigcup_{\substack{a \in [x)_\leq \cup [z)_\leq \\ b \in [y)_\leq \cup [z)_\leq}} [a + b)_\leq,\end{aligned}$$

i.e. $(x \circ_{lw} z) \oplus (y \circ_{lw} z) = \{d \in R; a + b \leq d; (x \leq a \text{ or } z \leq a) \text{ and } (y \leq b \text{ or } z \leq b)\}$. Suppose an arbitrary $c \in [x + y)_\leq \circ_{lw} z$. There are two options: $c \in [x + y)_\leq$ or $c \in [z)_\leq$. If $c \in [x + y)_\leq$, then obviously $c \in (x \circ_{lw} z) \oplus (y \circ_{lw} z)$ because $a \in [x)_\leq$, $b \in [y)_\leq$, i.e. $x \leq a$, $y \leq b$ implies $x + y \leq a + b$ which thanks to transitivity of $\leq$ means that $x + y \leq c$ which is what we suppose. If $c \in [z)_\leq$, i.e. $z \leq c$, then if we suppose that $z + z \leq z$, we get from transitivity of $\leq$ that $z + z \leq c$. Yet this is on the right-hand side the case of $a \in [z)_\leq$, $b \in [z)_\leq$, i.e. $z + z \leq a + b$.

The proof of property 2 is analogous, the proof of property 3 is obvious. □

Corollary 5.8. *If $(R, +, \leq)$ and $(R, \cdot, \leq)$ are idempotent quasi-ordered semigroups, then there always exists a left weak composition hyperoperation $\circ_{lw}$ on $(R, \oplus, \bullet)$. The same holds if $r + r \leq r$ for all $r \in R$ and $(R, \cdot, \leq)$ is an idempotent quasi-ordered semigroup or if $r \cdot r \leq r$ for all $r \in R$ and $(R, +, \leq)$ is an idempotent quasi-ordered semigroup.*

Proof. Conditions $r + r \leq r$, $r \cdot r \leq r$ included in Theorem 5.7 in this case turn into $r \leq r$. However, since the relation $\leq$ is reflexive, they hold trivially. □

Remark 5.9. If both $(R,+,\leq)$ and $(R,\cdot,\leq)$ are quasi-ordered groups, then simultaneous validity of $r+r\leq r$ and $r\cdot r\leq r$ for all $r\in R$ is equivalent to the fact that $r\leq e_s$ and $r\leq e_p$, where e_s and e_p are neutral elements of $(R,+)$ and $(R,\cdot)$ respectively. Thus e_s and e_p are the greatest elements of $(R,\leq)$, which means that for groups $(R,+,\leq)$ and $(R,\cdot,\leq)$ validity of the conditions in Theorem 3 implies that $e_s=e_p$.

Theorem 5.10. *There exists a right weak composition operation $\circ_{rw}$ on all types of hyperstructures $(R,\oplus,\bullet)$ discussed in Lemma 4.1 which are constructed from a quasi-ordered semigroup $(R,+,\leq)$ and a commutative idempotent quasi-ordered semigroup $(R,\cdot,\leq)$.*

Proof. For arbitrary $A,B\subseteq R$ denote

$$A\circ_{rw} B=\{a\cdot b; a\in A, b\in B\}, \tag{5.6}$$

where $\cdot$ is the single-valued product of $(R,\cdot,\leq)$. One-element sets A,B will be represented by the elements themselves, i.e. $\{a\}\circ_{rw}\{b\}=a\cdot b$, which will allow us to write

$$a\circ_{rw} b=a\cdot b \tag{5.7}$$

for all $a,b\in R$.

Now in property 1 of Definition 5.6 we get on the left-hand side, which reads $(x\circ_{rw} z)\oplus(y\circ_{rw} z)$, the set $[x\cdot z+y\cdot z)_{\leq}$ which thanks to distributivity of the single-valued structure $(R,+,\cdot)$ is $[(x+y)\cdot z)_{\leq}$. On the right-hand side, which reads $(x\oplus y)\circ_{rw} z$, we get $[x+y)_{\leq}\circ_{rw} z$, which equals $\bigcup\limits_{x+y\leq s}\{s\cdot z\}$. Yet since the relation $\leq$ is reflexive, there is $x+y\leq x+y$ and $[(x+y)\cdot z)_{\leq}\subseteq\bigcup\limits_{x+y\leq s}\{s\cdot z\}$.

In property 2 of Definition 5.6 we get that (thanks to commutativity and idempotency)

$$\begin{aligned}&(x\circ_{rw} z)\bullet(y\circ_{rw} z)=(x\cdot z)\bullet(y\cdot z)=[x\cdot z\cdot y\cdot z)_{\leq}\\&=[x\cdot y\cdot z\cdot z)_{\leq}=[x\cdot y\cdot z)_{\leq}.\end{aligned}$$

On the left-hand side we get that $[x\cdot y)_{\leq}\circ_{rw} z=\bigcup\limits_{x\cdot y\leq r}\{r\cdot z\}$. Thus thanks to reflexivity of the relation $\leq$ property 2 holds.

In property 3 of Definition 5.6 there is $x\circ_{rw}(y\circ_{rw} z)=x\circ_{rw}(y\cdot z)=x\cdot y\cdot z$ and $(x\circ_{rw} y)\circ_{rw} z=(x\cdot y)\circ_{rw} z=x\cdot y\cdot z$. $\square$

Example 5.11. If we continue with Example 4.3 and define

$$A\circ_{lw} B=[A)_{\subseteq}\cup[B)_{\subseteq}=\{R\in\mathcal{P}(S); A\subseteq R \text{ or } B\subseteq R\}$$

for all $A,B\in\mathcal{P}(S)$, then since both set intersection and set union are idempotent, the above defines a left weak composition hyperoperation on $(\mathcal{P}(S),\oplus,\bullet)$, i.e. $(\mathcal{P}(S),\oplus,\bullet,\circ_{lw})$ is a weak composition semihyperring in the general sense.

Example 5.12. If we continue with Example 4.3 and define $A\circ_{rw} B=A\cap B$ for all $A,B\in\mathcal{P}(S)$, then since the set intersection is both commutative and idempotent (and distributive with respect to set union), this defines a weak composition operation on $(\mathcal{P}(S),\oplus,\bullet)$, i.e. that $(\mathcal{P}(S),\oplus,\bullet,\cap)$ is a weak composition semihyperring in the general sense.

Examples 5.13 and 5.14 are partly motivated by the classical *interval binary hyperoperation* on a linearly ordered group discussed e.g. in [17] and defined by

$$\begin{aligned}a*b&=[\min\{a,b\})_{\leq}\cap(\max\{a,b\}]_{\leq}\\&=\{x\in G;\min\{a,b\}\leq x\leq\max\{a,b\}\}\end{aligned}$$

for all $a,b\in G$.

Example 5.13. Regard the ordered semiring of natural numbers, i.e. a distributive structure $(\mathbb{N},+,\cdot)$, where $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$ are semigroups and $\leq$ is the usual ordering of natural numbers with the smallest element 1. Obviously $(\mathbb{N},+,\leq)$ and $(\mathbb{N},\cdot,\leq)$ are quasi-ordered semigroups, which enables us to construct semihypergroups $(\mathbb{N},\oplus)$ and $(\mathbb{N},\bullet)$ using (3.1) and (3.2) respectively. Thus we get a semihyperring in the general sense $(\mathbb{N},\oplus,\bullet)$.

For arbitrary $a,b\in\mathbb{N}$ define

$$a\circ_{rw} b = [\max\{a,b\})_{\leq}. \tag{5.8}$$

Obviously, the maximum always exists and $a\circ_{rw} b$ is never empty or a one-element set. In the proof of Theorem 6.2 we will show that (5.8) is a weak composition hyperoperation on $(\mathbb{N},\oplus,\bullet)$, or rather on every set where there hold implications used in Theorem 6.2, such that it is different from the hyperoperation considered in the proof of Theorem 5.10.

Example 5.14. One can easily show that when changing in (5.8) $\max\{a,b\}$ to $\min\{a,b\}$, we get another weak composition hyperoperation on $(\mathbb{N},\oplus,\bullet)$.

6. Existence theorems

Using Lemma 4.1, results of section 5 might be summed up as follows. Notice that definitions of composition hyperstructures are analogies of Definition 3.2, only the carrier hyperstructure is different.

Theorem 6.1. *Let $(R,+,\leq)$ and $(R,\cdot,\leq)$ be quasi-ordered semigroups and $(R,\oplus)$, $(R,\bullet)$ their associated EL–hyperstructures constructed using (3.1) and (3.2) respectively. Furthermore, let $\cdot$ distribute over $+$ from both left and right.*

(1) *If operations $+$ and $\cdot$ are idempotent, then there exists a composition hyperoperation $\circ$ on $(R,\oplus,\bullet)$ such that $(R,\oplus,\bullet,\circ)$ is a composition semihyperring in the general sense.*
(2) *The same holds if $(R,+)$ or $(R,\cdot)$ are monoids with neutral elements e_s, e_p respectively, and either $e_p\leq e_p+e_p$, $e_s\leq e_s\cdot e_s$ or $e_p+e_p\leq e_p$, $e_s\cdot e_s\leq e_s$.*
(3) *If $(R,+)$ is a group or $(R,\oplus)$ is a hypergroup, then in 1 and 2 $(R,\oplus,\bullet,\circ)$ is a composition hyperring in the general sense.*
(4) *If $(R,\cdot)$ is a group, then in 1 and 2 $(R,\oplus,\bullet,\circ)$ is a composition semihyperring.*
(5) *If $(R,+)$ is a group with neutral element 0 and $(R\setminus\{0\},\cdot)$ is a group, then in 1 and 2 $(R,\oplus,\bullet,\circ)$ is a composition hyperring.*

Proof. Follows immediately from Lemma 4.1, Theorem 5.2 and Theorem 5.4. □

Analogous theorems can be formulated for weak composition hyperstructures using Theorem 5.7, Corollary 5.8 or Theorem 5.10. Or – which is more important – immediately after finding suitable (weak) composition operations (hyperoperations) in some special contexts. An example of this is the following case of linearly ordered commutative semigroups used in Example 5.13 or Example 5.14.

Theorem 6.2. *Let $(R,\oplus)$ and $(R,\bullet)$ be two semihypergroups constructed from linearly ordered commutative semigroups $(R,+,\leq)$ and $(R,\cdot,\leq)$ by (3.1) and (3.2) respectively. Furthermore, let $\cdot$ distribute over $+$ from both left and right. If implications $a+a\leq b\Rightarrow a\leq b$ and $a\cdot a\leq b\Rightarrow a\leq b$ hold for all $a,b\in R$, then there exists a weak composition hyperoperation $\circ_{rw}$ on R such that $(R,\oplus,\bullet,\circ_{rw})$ becomes a weak composition semihyperring in the general sense.*

Proof. The fact that $(R,\oplus,\bullet)$ is a semihyperring follows from Lemma 4.1. The weak composition hyperoperation in question will be (5.8).

Suppose arbitrary $x,y,z\in R$. First we discuss the meaning of property 1 of Definition 5.6 based on definitions of $\oplus$ and $\circ_{rw}$. In our notation the left-hand side reads

$(x \circ_{rw} z) \oplus (y \circ_{rw} z)$. This is

$$[\max\{x,z\})_\leq \oplus [\max\{y,z\})_\leq = \bigcup_{\substack{a\in[\max\{x,z\})_\leq \\ b\in[\max\{y,z\})_\leq}} [a+b)_\leq = \bigcup_{\substack{\max\{x,z\}\leq a \\ \max\{y,z\}\leq b}} [a+b)_\leq,$$

which results in the following four cases based on the relations between x, y and z. Notice that reasoning in cases C) and D) is analogous to reasoning in case B).

A) $x \leq z, y \leq z$**:** In this case $\max\{x,z\} = z$, $\max\{y,z\} = z$ and moreover $x+y \leq z+z$. Thus

$$\bigcup_{\substack{\max\{x,z\}\leq a \\ \max\{y,z\}\leq b}} [a+b)_\leq = \bigcup_{\substack{z\leq a \\ z\leq b}} [a+b)_\leq = = \{c \in R; a+b \leq c; z \leq a, z \leq b\}.$$

At the same time conditions $z \leq a$, $z \leq b$ result in $z+z \leq a+b$ and from transitivity of $\leq$ we get that $z+z \leq c$. Finally

$$(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in R; x+y \leq c\} = = \{c \in R; z+z \leq c\}. \tag{6.1}$$

B) $x \leq z, z \leq y$**:** In this case $\max\{x,z\} = z$, $\max\{y,z\} = y$ and moreover from transitivity of $\leq$ there is $x \leq y$. Thus

$$\bigcup_{\substack{\max\{x,z\}\leq a \\ \max\{y,z\}\leq b}} [a+b)_\leq = \bigcup_{\substack{z\leq a \\ y\leq b}} [a+b)_\leq = = \{c \in R; a+b \leq c; z \leq a, y \leq b\}.$$

At the same time conditions $z \leq a$, $y \leq b$ result in $z+y \leq a+b$ and from transitivity of $\leq$ we get that $z+y \leq c$. Finally

$$(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in R; z+y \leq c\}. \tag{6.2}$$

C) $z \leq x, y \leq z$**:** This results in $(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in R; x+z \leq c\}$.

D) $z \leq x, z \leq y$**:** This results in

$$(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in R; x+y \leq c\} = \{c \in R; z+z \leq c\}$$

The right-hand side of property 1 of Definition 5.6 reads $(x \oplus y) \circ_{rw} z$. Based on definitions of $\oplus$ and $\circ_{rw}$ this is

$$[x+y)_\leq \circ_{rw} z = \bigcup_{r\in[x+y)_\leq} [\max\{r,z\})_\leq = \bigcup_{x+y\leq r} [\max\{r,z\})_\leq.$$

However, in our case this is the same as $[max\{x+y,z\})_\leq$, which is

$$\{d \in R; \max\{x+y,z\} \leq d\}. \tag{6.3}$$

Now we verify the inclusion in property 1 of Definition 5.6. Suppose an arbitrary $c \in (x \circ_{rw} z) \oplus (y \circ_{rw} z)$ and let us find out whether $c \in (x \oplus y) \circ_{rw} z$. We have to test each of the cases **A** – **D**.

ad A: The element c is such that $z+z \leq c$, $x+y \leq c$ and at the same time $x \leq z$, $y \leq z$. Thus

(1) if $\max\{x+y,z\} = x+y$, then (6.3) turns into $\{d \in R; x+y \leq d\}$. Thus $c \in (x \oplus y) \circ_{rw} z$ obviously holds.

(2) if $\max\{x+y,z\} = z$, then (6.3) turns into $\{d \in R; z \leq d\}$ and we have to show that $z \leq c$. Yet since $z+z \leq c$, there is – thanks to the assumption of the theorem – also $z \leq c$ and $c \in (x \oplus y) \circ_{rw} z$.

ad B: The element c is such that $z+y \le c$ and at the same time $x \le z$, $z \le y$. Thus

(1) if $\max\{x+y,z\} = x+y$, then (6.3) turns into $\{d \in R; x+y \le d\}$. Since $x \le z$, there is $x+y \le z+y$ and from transitivity we get that $x+y \le c$. Thus $c \in (x \oplus y) \circ_{rw} z$.

(2) if $\max\{x+y,z\} = z$, then (6.3) turns into $\{d \in R; z \le d\}$ and we have to show that $z \le c$. Since $z \le y$, there is $z+z \le z+y$ and from transitivity of $\le$, there is $z+z \le c$. Yet this means – thanks to the assumption of the theorem – that $z \le c$ and $c \in (x \oplus y) \circ_{rw} z$.

ad C: The element c is such that $x+z \le c$ and at the same time $z \le x$, $y \le z$. Thus

(1) if $\max\{x+y,z\} = x+y$, then (6.3) turns into $\{d \in R; x+y \le d\}$ and we have to show that $x+y \le c$. Suppose on contrary that $c < x+y$. Since $y \le z$, there is $c < x+z$. Yet since simultaneously $x+z \le c$, we get from transitivity that $c < c$ which is impossible. Thus $x+y \le c$ and $c \in (x \oplus y) \circ_{rw} z$.

(2) if $\max\{x+y,z\} = z$, then (6.3) turns into $\{d \in R; z \le d\}$ and we have to show that $z \le c$. Since $z \le x$, there is $z+z \le x+z$ and from transitivity of $\le$, there is $z+z \le c$. Yet this – thanks to the assumption of the theorem – means that $z \le c$ and $c \in (x \oplus y) \circ_{rw} z$.

ad D: The element c is such that $x+y \le c$, $z+z \le c$ and at the same time $z \le x$, $z \le y$. Thus

(1) if $\max\{x+y,z\} = x+y$, then (6.3) turns into $\{d \in R; x+y \le d\}$ and we have to show that $x+y \le c$. Yet this is one of our assumptions. Thus $c \in (x \oplus y) \circ_{rw} z$ holds trivially.

(2) if $\max\{x+y,z\} = z$, then (6.3) turns into $\{d \in R; z \le d\}$ and we have to show that $z \le c$. Yet since $z+z \le c$, there is also – thanks to the assumption of the theorem – that $z \le c$ and $c \in (x \oplus y) \circ_{rw} z$.

Thus we have verified validity of property 1 of Definition 5.6. The proof of property 2 is completely analogous.

Verifying property 3 is rather straightforward. The left-hand side $x \circ_{rw} (y \circ_{rw} z)$ is

$$\begin{aligned} x \circ_{rw} [\max\{y,z\})_{\le} &= \bigcup_{r \in [\max\{y,z\})_{\le}} [\max\{x,r\})_{\le} \\ &= \bigcup_{\max\{y,z\} \le r} [\max\{x,r\})_{\le} \end{aligned}$$

while the right-hand side $(x \circ_{rw} y) \circ_{rw} z$ is

$$\begin{aligned} [\max\{x,y\})_{\le} \circ_{rw} z &= \bigcup_{s \in [\max\{x,y\})_{\le}} [\max\{s,z\})_{\le} \\ &= \bigcup_{\max\{x,y\} \le s} [\max\{s,z\})_{\le}. \end{aligned}$$

Yet since the relation $\le$ is reflexive, i.e. $\max\{y,z\} \le \max\{y,z\}$, $\max\{x,y\} \le \max\{x,y\}$, both sides equal $[\max\{x,y,z\})_{\le}$.

Thus finally (5.8) is a weak composition hyperoperation on $(R, \oplus, \bullet)$ with the assumed properties. □

Remark 6.3. Notice that as regards number domains, the implications used in Theorem 6.2 which obviously hold in $\mathbb{N}$ or $\mathbb{Z}$, do not hold for other number domains. The transition to $\mathbb{Q}$ or $\mathbb{R}$ is not possible as e.g. $0.1 \cdot 0.1 \le 0.02$ yet $0.1 \not\le 0.02$. Notice that if we expanded Example 5.13 to $R = \mathbb{R}^+$ or considered this in the theorem, then e.g. in case **C2** of the proof the conditions would not hold for multiplication and $x = 0.1$, $y = 0.02$, $z = 0.1$. Naturally, we could expand Theorem 6.2 by including analogies of parts 4 and 5 of Theorem 6.1.

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