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Affine singular control systems on Lie groups

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Abstract

The purpose of this paper is to show that an affine singular control system S on a connected Lie group G leads to two subsystems: An affine control system on a homogeneous space G/H and an algebraic-differential control system on H of G, where H is some closed subgroup of G.

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1. INTRODUCTION

Let G denote a connected Lie group with Lie algebra $L(G)$ (the set of right invariant vector fields on G). Let us denote by $Af(G)$ the affine group on G. An affine singular control system S on G is a family of differential equations

$$E_{g(t)}\left(\dot{g}(t)\right) = F(g(t)) + \sum_{j=1}^{d} u_j(t) F^j(g(t)), \qquad g(t) \in G, \tag{1.1}$$

where $u \in U$ is the class of unrestricted piecewise constant admissible controls with values on $\mathbb{R}^d$, i.e., the set

$$U = \left\{ u : [0, T_u] \to \mathbb{R}^d \mid u \text{ is a piecewise constant function} \right\}.$$

Here, the vector fields $F, F^1, ..., F^d$ belong to the affine algebra $af(G)$ and E is a non-invertible derivation on $L(G)$. The operator $E_g : T_gG \to T_gG$ is defined by $E_g = (l_g)_* \circ E \circ (l_{g^{-1}})_*$, where

$$(l_{g^{-1}})_* : T_gG \to T_eG, \qquad E : T_eG \to T_eG, \qquad (l_g)_* : T_eG \to T_gG.$$

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The singular control system on Euclidean spaces was introduced by Dai [5]. The system has been well developed on Lie groups, see [3, 4]. Thus, there exist the basic ingredients to start with the study of affine singular control systems on Lie groups.

Throughout this paper H, which is the subgroup of G will be assumed to be closed, because in this case the quotient set G/H is a homogeneous space. We also assume that the vector fields $F, F^1, \ldots, F^d$ are projectable on the homogeneous space G/H leads to a decomposition of (1.1) in two systems, one on G/H and the one other on H. The algebraic-differential subsystem plays a crucial role in the understanding of the trajectories for affine singular control systems on Lie groups. Actually, the solvability of (1.1) depends just on when we are able to solve (3.1). Furthermore, we establish a special solution of (3.1) and hence the solution of (1.1).

This paper is organized as follows. In the next section we introduce the notion of an affine control system on a connected Lie group G. In Section 3, vector fields of the affine singular control system on homogeneous space are introduced, and we obtain the decomposition for the affine singular control system S on G, as well as the solution of the decomposition (3.1).

2. Affine Control Systems

In this section, the definition of affine vector fields are recalled. More details can found in [2, 8, 7].

Let G denote a connected Lie group of dimension n with Lie algebra $L(G)$. The affine group $Af(G)$ of G is the semidirect product of $Aut(G)$ and G, i.e., $Af(G) = Aut(G) \times_s G$. The semidirect product consists of all pairs $(\phi, g) \in Af(G)$, with the group structure given by

$$(\phi, g_1) \cdot (\psi, g_2) = (\phi \circ \psi, g_1 \phi(g_2)),$$

that (Id, e) is the group identity and that $\left(\phi^{-1}, \phi^{-1}\left(g^{-1}\right)\right)$ is the inverse of (ϕ, g). Then, the mapping $g \to (Id, g)$ embeds G into $Af(G)$ and $\phi \to (\phi, e)$ embeds $Aut(G)$ into $Af(G)$. Therefore, G and $Aut(G)$ are subgroups of $Af(G)$. There is a natural action

$$Af(G) \times G \to G$$

defined by

$$(\phi, g_1) \cdot g_2 \to g_1 \phi(g_2),$$

where $(\phi, g_1) \in Af(G)$ and $g_2 \in G$. This action is transitive. Indeed, if it is taken $g_2 = e$, then $(\phi, g_1) \cdot e = g_1$ since $\phi(g_2) = e$.

Denote by $AutL(G)$ the automorphism group of $L(G)$ and whose Lie algebra is $DerL(G)$, the Lie algebra of derivations of $L(G)$. If G is simply connected, then $Aut(G)$ and $AutL(G)$ are isomorphic. In fact, there is an isomorphism Φ which assigns to each automorphism ϕ of G its differential $d\phi \mid_{Id}$ at the identity. Any automorphism ϕ of $L(G)$ extends to an automorphism of G, therefore, Φ is indeed an isomorphism between $Aut(G)$ and $AutL(G)$. Thus, in this case, the Lie algebra of $Aut(G)$ is $DerL(G)$.

The Lie bracket in $af(G)$ is given by

$$\left[\left(\mathcal{D}^1, Y^1\right), \left(\mathcal{D}^2, Y^2\right)\right] = \left(\left[\mathcal{D}^1, \mathcal{D}^2\right], \mathcal{D}^1 Y^2 - \mathcal{D}^2 Y^1 + \left[Y^1, Y^2\right]\right),$$

where the first coordinate in the bracket is that of $DerL(G)$, while the second is that of $L(G)$ and $\mathcal{D}X$ denotes the derived action of $DerL(G)$ on $L(G)$. The Lie algebra $af(G)$ of $Af(G)$ is the semidirect product $DerL(G) \times_s L(G)$. An affine vector field F on G can be exclusively separated decomposed into a sum

$$F = \mathcal{D} + Y,$$

where $\mathcal{D} \in DerL(G)$ and $Y \in L(G)$. Thus, an affine control system on G is determined by the dynamic parametrized by $u \in U$,

$$\dot{g}(t) = (\mathcal{D} + Y)(g(t)) + \sum_{j=1}^{d} u_j(t) \left(\mathcal{D}^j + Y^j\right)(g(t)), \qquad g(t) \in G,$$

where right invariant vector fields $Y, Y^1, ..., Y^d \in L(G)$ and $\mathcal{D}, \mathcal{D}^1, ..., \mathcal{D}^d \in DerL(G)$.

As usual, for any $g \in G$, denote by r_g the right translation on G by g; that is, $r_g(x) = xg$ for all x in G. l_g will denote the left translation by g; that is, $l_g(x) = gx$. We recall that $L(G)$ is isomorphic to the tangent space T_eG of G at the identity element e. Thus, a right invariant vector field Y on G is determined by its value at e. In particular, $Y(g) = (r_g)_*Y(e)$ and its flow is given by $Y(g(t)) = r_g(Y(e(t)))$, where $(r_g)_*$ is derivative of r_g.

Let $\mathcal{X}$ be an infinitesimal automorphism of the Lie group G, that is, the flow $(\mathcal{X}_t)_{t\in\mathbb{R}}$ induced by the vector field $\mathcal{X}$ is a one-parameter subgroup of $Aut(G)$. Then, $\mathcal{X}$ induces a derivation $\mathcal{D} = -ad_{\mathcal{X}}$ on $L(G)$ for $\mathcal{D} \in DerL(G)$. This condition on ad means

$$\mathcal{D}Y = -[\mathcal{X}, Y]$$

for $\forall Y \in L(G)$ and verifies $\mathcal{X}(e) = 0$.

3. Affine Singular Control Systems

Throughout this section, we can always assume that G is simply connected and Π_*Y is one-to-one.

Let G denote a Lie group and let H denote a closed Lie subgroup of G with Lie algebra $L(H)$. For closed subgroup H of G, $G/H = \{gH : g \in G\}$ denotes the homogeneous space of left cosets of H, and we denote by Π the natural projection of G onto G/H. In order to any right invariant vector field $Y \in L(G)$, Y projects to Π_*Y on G/H, will be induced to as a well-defined invariant vector field on G/H. Furthermore, $\Pi_*L(G) = \{\Pi_*Y;\ Y \in L(G)\}$ is a Lie algebra and Π_* is a Lie algebra morphism from $L(G)$ onto $\Pi_*L(G)$. Also the projection Π_*Y of $Y \in L(G)$ vanishes at the point H iff $Y \in L(H)$.

We consider an affine singular control system S with derivation $E \in Der(L(G))$ and vector field $\mathcal{X}$ induced by a derivation $\mathcal{D} \in Der(L(G))$. Now, we wish to show the existence of a vector field Π-related to $\mathcal{X}$ on G/H. There exists a vector field π-related to $\mathcal{X}$ on G/H such that

$$\Pi(\mathcal{X}(g(t)x(t))) = \Pi(\mathcal{X}(g(t)))$$

for $\forall g \in G, \forall x \in H$ and $\forall t \in \mathbb{R}$. On the other hand, the corresponding flows on G/H are related by

$$\Pi\left(\mathcal{X}(g(t)x(t))\right) = \Pi\left(\mathcal{X}(g(t))\mathcal{X}(x(t))\right) = \Pi\left(\mathcal{X}(g(t))\right)\mathcal{X}(x(t))H,$$

where $\mathcal{X}(x(t))$ is the one-parameter subgroup in H. Because of the existence of the projection, the subgroup H is invariant under the flow of $\mathcal{X}$; thus, $\mathcal{X}$ is tangent to H.

Now, let H be connected. Because of the elements of H, which are products of exponentials, the invariance of H under $\mathcal{X}$ writes

$$\forall Y \in L(H),\ \forall t \in \mathbb{R} \qquad \mathcal{X}_t(\exp Y) = \exp(e^{t\mathcal{D}}Y) \in H,$$

or equivalently as

$$\forall Y \in L(H),\ \forall t \in \mathbb{R} \qquad e^{t\mathcal{D}}Y \in L(H).$$

Finally, its Lie algebra $L(H)$ is invariant under $\mathcal{D}$.

Under the above assumptions, the projection of $\mathcal{X}$ onto G/H will be denoted by $\Pi_*\mathcal{X}$.

Now, we take an affine vector field $F = \mathcal{X} + Y$ on G. This decomposition is chosen in order to ensure that the projection $\Pi_* Y$ of Y onto G/H is well defined. If $\Pi_* \mathcal{X}$ exists, then F is Π-related to a vector field on G/H. It follows that $\Pi_* F = \Pi_* \mathcal{X} + \Pi_* Y$ will stand for the projection of F onto G/H. Then, there exists an affine control system on G

$$\dot{g}(t) = \Pi\left(F(g(t))\right) + \sum_{j=1}^{d} u_j(t)\, \Pi\left(F^j(g(t))\right), \quad g(t) \in G,$$

which projects down onto G/H.

Now, it follows that $(\Pi_* E)^{-1} D \in Der(L(G)/L(H))$ since $E, D \in Der(L(G))$ and $L(H)$-invariant. Let us denote by $\Pi_*\left((\Pi_* E)^{-1} D\right) \in Der(L(G))$ such that its restriction to $L(G)/L(H)$ coincide with $(\Pi_* E)^{-1} D$. Thus, $\Pi_* (\Pi_* E)^{-1} \mathcal{X} = \Pi_* \mathcal{X}$ on G/H. On the other hand, we define $\Pi_*\left((\Pi_* E)^{-1} Y\right)$ as the only invariant vector fields determined by $(\Pi_* E)^{-1} Y(e) \in L(G)/L(H)$. Thus, the mapping $E_g : T_g G \to T_g G$ is invertible on the homogeneous space G/H for any $g \in G$. In particular, we can consider the affine control system $\Pi(S)$ on G/H in the following way:

$$\begin{aligned} \dot{y}(t) &= \left(E_{y(t)}\right)^{-1} \circ \Pi\left(\mathcal{X}(y(t))\right) + \left(E_{y(t)}\right)^{-1} \circ \Pi\left(Y(y(t))\right) + \\ &\quad \left(E_{y(t)}\right)^{-1} \circ \sum_{j=1}^{d} u_j(t)\, \Pi\left(\mathcal{X}^j(y(t))\right) + \left(E_{y(t)}\right)^{-1} \circ \sum_{j=1}^{d} u_j(t)\, \Pi\left(Y^j(y(t))\right), \end{aligned}$$

where $y(t) \in G/H$ is an integral curve of the projected affine control system on the homogeneous space G/H. Also $\dot{y}(t)$ has a well-defined solution for each piecewise admissible control u and any initial condition in G.

3.1. Theorem. *Let G be a connected Lie group with Lie algebra $L(G)$ and assume that the connected Lie subgroup H of G with Lie algebra $L(H)$ is closed. The curve $g(t)$ is solution of the affine singular control system S for the initial condition $y(0) = y \in G/H$ associated to the control $u \in U$. Then, there exists a one parameter group $x(t)$ of the closed subgroup H which together satisfies the algebraic-differential equation*

$$\begin{aligned} E_{g(t)}\left(y(t)\,\dot{x}(t)\right) &= \left(l_{y(t)}\right)_* \left(\mathcal{X}_{L(H)}(x(t)) + \sum_{j=1}^{d} u_j(t)\, \mathcal{X}^j_{L(H)}(x(t))\right) \\ &\quad + \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right), \end{aligned} \tag{3.1}$$

where $\mathcal{X}_{L(H)}, \mathcal{X}^1_{L(H)}, \ldots, \mathcal{X}^d_{L(H)}$ are infinitesimal automorphisms of the Lie subgroup H and $Y_{L(H)}, Y^1_{L(H)}, \ldots, Y^d_{L(H)} \in L(H)$.

Proof. Assume there exists a solution $g(t)$ of the affine singular control system S with control u and initial condition $y(0) = y$. Then, for almost every t, there exists a curve $x(t) \in H$, with $x(0) = e$, where e is the identity on G, such that

$$\begin{aligned} g(t) &= y(t)\, x(t) \\ \dot{\overline{y(t)\, x(t)}} &= \left(l_{y(t)}\right)_* \dot{x}(t) + \left(r_{x(t)}\right)_* \dot{y}(t) \end{aligned}$$

Applying $E_{g(t)}$ on both sides, equation takes form,

$$E_{g(t)}\left(\dot{g}(t)\right) = E_{g(t)}\left(y(t)\,\dot{x}(t)\right) + E_{g(t)}\left(\dot{y}(t)x(t)\right).$$

Hence, we get

$$\begin{aligned}&\mathcal{X}(g(t))+Y(g(t))+\sum_{j=1}^{d}u_j(t)\mathcal{X}^j(g(t))+\sum_{j=1}^{d}u_j(t)Y^j(g(t))\\ =\;& E_{g(t)}\left(y(t)\dot{x(t)}\right)+\\ &\left(r_{x(t)}\right)_*\left(\Pi(\mathcal{X})(y(t))+\Pi(Y)(y(t))+\sum_{j=1}^{d}u_j(t)\Pi\left(\mathcal{X}^j\right)(y(t))+\sum_{j=1}^{d}u_j(t)\Pi\left(Y^j\right)(y(t))\right).\end{aligned}$$

Since $Y, Y^1, ..., Y^d$ are elements of the Lie algebra $L(G)$, we can project this dynamic on any homogeneous space of G. In particular,

$$\left(r_{x(t)}\right)_*\left(\Pi(Y)(y(t))+\sum_{j=1}^{d}u_j(t)\Pi\left(Y^j\right)(y(t))\right)=\Pi(Y)(g(t))+\sum_{j=1}^{d}u_j(t)\Pi\left(Y^j\right)(g(t)).$$

Thus, it follows that

$$\begin{aligned}&E_{g(t)}\left(y(t)\dot{x(t)}\right)=\mathcal{X}(g(t))-\left(r_{x(t)}\right)_*\Pi(\mathcal{X})(y(t))+\\ &\sum_{j=1}^{d}u_j(t)\mathcal{X}^j(g(t))-\left(r_{x(t)}\right)_*\sum_{j=1}^{d}u_j(t)\Pi\left(\mathcal{X}^j\right)(y(t))\\ &+Y(g(t))-\Pi(Y)(g(t))+\sum_{j=1}^{d}u_j(t)Y^j(g(t))-\sum_{j=1}^{d}u_j(t)\Pi\left(Y^j\right)(g(t))\end{aligned}$$

On the other hand, $\mathcal{X}_t\in Aut(G)$ for any real number t, and therefore,

$$\mathcal{X}(g(t))=\mathcal{X}(y(t)x(t))=\mathcal{X}(y(t))\mathcal{X}(x(t)).$$

By taking a derivative of the product $\mathcal{X}(g(t))$ at time t, we obtain

$$\mathcal{X}(y(t)x(t))=\left(r_{x(t)}\right)_*\mathcal{X}(y(t))+\left(l_{y(t)}\right)_*\mathcal{X}(x(t)).$$

By construction for each $t\in\mathbb{R}:\ \mathcal{X}(y(t))=\Pi(\mathcal{X}(y(t)))$ and $\mathcal{X}(x(t))=\Pi(\mathcal{X}(x(t)))x(t)=x(t)$. Thus, we conclude that

$$\begin{aligned}E_{g(t)}\left(y(t)\dot{x(t)}\right)\;=\;&\left(l_{y(t)}\right)_*\left(\mathcal{X}_{L(H)}(x(t))+\sum_{j=1}^{d}u_j(t)\mathcal{X}^j_{L(H)}(x(t))\right)\\ &+\left(Y_{L(H)}(g(t))+\sum_{j=1}^{d}u_j(t)Y^j_{L(H)}(g(t))\right),\end{aligned}$$

which completes the proof. □

3.2. Theorem. *Under the conditions of theorem3.1, if the derivation E is nilpotent, then the solution of (3.1) is given by*

$$\begin{aligned}\dot{x(t)}\;=\;&-\sum_{i=0}^{k-1}E^i_{x(t)}\circ\left(l_{y(t)^{-1}}\right)_*\circ\left(Y_{L(H)}(g(t))+\sum_{j=1}^{d}u_j(t)Y^j_{L(H)}(g(t))\right)\\ &-\sum_{i=0}^{k-1}\sum_{j=1}^{d}u_j(t)E^i_{x(t)}\circ\mathcal{X}^j_{L(H)}(x(t)).\end{aligned}$$

Proof. Suppose E is nilpotent whose nilpotent index is denoted by k. Let $x(t)\in H$ be such that $x(0)=e$. Taking the left hand side term of (3.1):

$$E_{g(t)}\left(y(t)\dot{x(t)}\right)=\left(l_{g(t)}\right)_*\circ E\circ\left(l_{g(t)^{-1}}\right)_*\circ\left(l_{y(t)}\right)_*\ \dot{x(t)}=\left(l_{g(t)}\right)_*\circ E\circ\left(l_{x(t)^{-1}}\right)_*\ \dot{x(t)}$$

because $\left(l_{g(t)}\right)_* = \left(l_{y(t)}\right)_* \circ \left(l_{x(t)}\right)_*$. Otherwise, we have $\dot{x}(t) = \mathfrak{X}_{L(H)}(x(t))$ where the vector field $\mathfrak{X}_{L(H)}$ is induced by a derivation $\mathcal{D} \in Der(L(H))$ and applying $\left(l_{g(t)^{-1}}\right)_*$ on both sides of (3.1),

$$\begin{aligned} E \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) &= \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right) \\ &\quad + \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right). \end{aligned}$$

If $k = 1$, the algebraic-differential equation (3.1) becomes

$$\dot{x}(t) = -\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t)) - \left(l_{y(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right).$$

Now, let $k > 1$. Then, left multiplying both sides by E, we obtain the following equations:

$$\begin{aligned} E^2 \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) &= E \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + E \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right) \\ &\quad + E \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right) \\ &\cdots \\ E^k \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) &= E^{k-1} \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + E^{k-1} \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right) \\ &\quad + E^{k-1} \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right). \end{aligned}$$

From the addition of these equations and the fact $E^k = 0$, $E^{k-1} \neq 0$, we have

$$\begin{aligned} \dot{x}(t) &= -\sum_{i=0}^{k-1} \left(l_{x(t)}\right)_* \circ E^i \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right) \\ &\quad - \sum_{i=0}^{k-1} \sum_{j=1}^{d} \left(l_{x(t)}\right)_* \circ E^i \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right), \end{aligned}$$

which proves our claim. □

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