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Lyapunov-type inequalities for third order linear differential equations with two points boundary conditions

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Abstract

In this paper, by using Green's functions for second order differential equations, we establish new Lyapunov-type inequalities for third order linear differential equations with two points boundary conditions. By using such inequalities, we obtain sharp lower bounds for the eigenvalues of corresponding equations.

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1. Introduction

In [15], Lyapunov obtained the following remarkable result: If $q \in C\left([0,\infty),\mathbb{R}^{+}\right)$ and $y(t)$ is a nontrivial solution of

$$y'' + q\left(t\right)y = 0 \tag{1.1}$$

with Dirichlet boundary condition

$$y\left(a\right) = y\left(b\right) = 0 \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $y\left(t\right) \not\equiv 0$ for $t \in (a,b)$, then the following inequality

$$\frac{4}{b-a} \leq \int_a^b q\left(s\right)ds \tag{1.3}$$

holds. The inequality (1.3) is the best possible in the sense that if the constant 4 in the left hand side of (1.3) is replaced by any larger constant, then there exists an example of (1.1) for which (1.3) no longer holds (see [12, p. 345], [14, p. 267]). The inequality (1.3) provides a lower bound for the distance between two consecutive zeros of y. Furthermore, this result has found many applications in areas like eigenvalue problems, stability, oscillation theory, disconjugacy, etc. Since then, there have been several results to generalize the above linear equation in many directions [1–19]. Before stating many efforts, it is worth to the mention following work.

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By using Green's function, Hartman [12] obtained the generalized inequality as follows: If $q \in C([0,\infty),\mathbb{R})$ and $y(t)$ is a nontrivial solution on (a,b) for problem (1.1)-(1.2), then

$$1 \leq \int_a^b \frac{(s-a)(b-s)}{b-a} q^+(s)\, ds \tag{1.4}$$

holds, where $q^+(t) = \max\{q(t), 0\}$. It is easy to see that the function $M(t) = (t-a)(b-t)$ takes the maximum value at $\frac{a+b}{2}$, i.e.

$$M(t) \leq \max_{a \leq t \leq b} M(t) = M\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^2. \tag{1.5}$$

Thus, from (1.5), the inequality (1.4) is a natural generalization of the inequality (1.3).

In this paper, we prove new Lyapunov-type inequalities for third order linear differential equation of the form

$$y''' + q(t)\, y = 0, \tag{1.6}$$

where $q \in C(\mathbb{R},\mathbb{R})$ and $y(t)$ is a real solution of (1.6) satisfying the following linearly independent two-point boundary conditions

$$\begin{cases} Y_1(y) := \gamma_{11} y(a) + \gamma_{12} y'(a) + \gamma_{13} y(b) + \gamma_{14} y'(b) = 0 \\ Y_2(y) := \gamma_{21} y(a) + \gamma_{22} y'(a) + \gamma_{23} y(b) + \gamma_{24} y'(b) = 0 \\ Y_3(y) := y''(a) + y''(b) = 0 \end{cases} \tag{1.7}$$

or

$$\begin{cases} Y_4(y) := \gamma_{11} y'(a) + \gamma_{12} y''(a) + \gamma_{13} y'(b) + \gamma_{14} y''(b) = 0 \\ Y_5(y) := \gamma_{21} y'(a) + \gamma_{22} y''(a) + \gamma_{23} y'(b) + \gamma_{24} y''(b) = 0 \\ Y_6(y) := y(a) + y(b) = 0 \end{cases} \tag{1.8}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $y(t) \not\equiv 0$ for $t \in (a,b)$.

Now, we present Green's functions to be used in the proofs of our main results. Assume that $y(t)$ is a nontrivial solution of (1.1) satisfying the linearly independent two-point boundary conditions $Y_1(y) = Y_2(y) = 0$. Thus, this condition implies that, of six determinants contained in the matrix

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \end{bmatrix}, \tag{1.9}$$

not all are zero. Therefore, either

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} \gamma_{13} & \gamma_{14} \\ \gamma_{23} & \gamma_{24} \end{vmatrix} \neq 0 \tag{1.10}$$

or else [13, p. 216]. We know that the solution of (1.1) satisfying $Y_1(y) = Y_2(y) = 0$ is given by

$$y(t) = \int_a^b G_1(t,s)\, y''(s)\, ds \tag{1.11}$$

with Green's function

$$G_1(t,s) = \begin{cases} \dfrac{A_1(s) B_2(t) - A_2(s) B_1(t)}{C} + t - s & ; \quad a \leq s \leq t \\ \dfrac{A_1(s) B_2(t) - A_2(s) B_1(t)}{C} & ; \quad t \leq s \leq b \end{cases} \tag{1.12}$$

where

$$\gamma_{11} + \gamma_{13} \neq 0, \tag{1.13}$$

$$A_i(t) = (b-t)\gamma_{i3} + \gamma_{i4}, \tag{1.14}$$

$$B_i(t) = (t-a)(\gamma_{i1} + \gamma_{i3}) - (\gamma_{i2} + (b-a)\gamma_{i3} + \gamma_{i4}) \tag{1.15}$$

for $i = 1, 2$, and

$$C = \begin{vmatrix} \gamma_{11} + \gamma_{13} & \gamma_{12} + \gamma_{13}(b-a) + \gamma_{14} \\ \gamma_{21} + \gamma_{23} & \gamma_{22} + \gamma_{23}(b-a) + \gamma_{24} \end{vmatrix} \tag{1.16}$$

(See the proof of the following Lemma 2.1 for the construction of the Green's function (1.12)). We also know that non-homogeneous linear boundary value problem $y''(t) = g(t)$ satisfying $Y_1(y) = Y_2(y) = 0$ has only the trivial solution under the condition

$$D(Y) = \begin{vmatrix} Y_1(y_1) & Y_1(y_2) \\ Y_2(y_1) & Y_2(y_2) \end{vmatrix} \neq 0, \tag{1.17}$$

where $y_1(t) = 1$ and $y_2(t) = t$ are the solutions of the corresponding homogeneous linear equation. Thus, we have the following condition

$$D(Y) = \begin{vmatrix} \gamma_{11} + \gamma_{13} & \gamma_{11}a + \gamma_{12} + \gamma_{13}b + \gamma_{14} \\ \gamma_{21} + \gamma_{23} & \gamma_{21}a + \gamma_{22} + \gamma_{23}b + \gamma_{24} \end{vmatrix} \neq 0 \tag{1.18}$$

instead of (1.17). It is clear that $D(Y) = C$. Here we note that the condition (1.18) is also valid for the problem (1.6) with the two-point boundary conditions (1.7) or (1.8). We also know that if the problem (1.1) satisfying $Y_1(y) = Y_2(y) = 0$ is well posed (if, in other words, the problem (1.1) satisfying $Y_1(y) = Y_2(y) = 0$ has only the trivial solution $y(t) \equiv 0$), then it has a unique Green's function.

It is easy to see that under the condition

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} = \begin{vmatrix} \gamma_{13} & \gamma_{14} \\ \gamma_{23} & \gamma_{24} \end{vmatrix}, \tag{1.19}$$

the Green's function $G_1(t,s)$ is symmetric, that is, $G_1(t,s) = G_1(s,t)$ for $t, s \in [a,b]$. Moreover, we know that this symmetry is a result of self-adjoint of the equation (1.1) satisfying $Y_1(y) = Y_2(y) = 0$ [13, p. 215]. Thus, if the condition (1.19) holds, then we have

$$y(t) = \int_a^b G(t,s)\, y''(s)\, ds, \tag{1.20}$$

where

$$G(t,s) = \begin{cases} \dfrac{A_1(t)B_2(s) - A_2(t)B_1(s)}{C}; & a \le s \le t \\ \dfrac{A_1(s)B_2(t) - A_2(s)B_1(t)}{C}; & t \le s \le b \end{cases} \tag{1.21}$$

is a symmetrized Green's function instead of (1.12). Therefore, in this paper, by using the symmetrized Green's function (1.21) for the equation (1.1) satisfying $Y_1(y) = Y_2(y) = 0$ under the condition (1.19), we prove new Lyapunov-type inequalities for third order linear differential equation (1.6) with the two-point boundary conditions (1.7) or (1.8). By using such inequalities, we obtain sharp lower bounds for the eigenvalues of corresponding equations.

2. Main results

We state some important lemmas which we will be used in the proofs of our main results. In the following first lemma, we construct Green's function for the second order nonhomogeneous differential equation

$$y'' = g(t) \tag{2.1}$$

with two-point boundary conditions $Y_1(y) = Y_2(y) = 0$.

Lemma 2.1. *If $y(t)$ is a solution of (2.1) satisfying $Y_1(y) = Y_2(y) = 0$, then the integral equation (1.11) holds.*

Proof. Integrating Eq. (2.1) from a to t to find y, we get

$$y'(t) = d_1 + \int_a^t g(s)\,ds \tag{2.2}$$

and

$$y(t) = d_0 + d_1(t-a) + \int_a^t (t-s)\,g(s)\,ds, \tag{2.3}$$

where d_0 and d_1 are arbitrary constants. Thus, the general solution of (2.1) is (2.3). Now, by using the boundary conditions $Y_1(y) = Y_2(y) = 0$, we can find the constants d_0 and d_1. Thus, we have

$$d_1 = \int_a^b \frac{(\gamma_{11}+\gamma_{13})\,A_2(s) - (\gamma_{21}+\gamma_{23})\,A_1(s)}{C} g(s)\,ds \tag{2.4}$$

and

$$d_0 = -\int_a^b \left[\frac{(\gamma_{12}+(b-a)\gamma_{13}+\gamma_{14})[(\gamma_{11}+\gamma_{13})A_2(s)-(\gamma_{21}+\gamma_{23})A_1(s)]+CA_1(s)}{C(\gamma_{11}+\gamma_{13})}\right] g(s)\,ds, \tag{2.5}$$

where $A_i(t)$, $i = 1, 2$, and C are given in (1.14) and (1.16), respectively. Substituting the constants d_0 and d_1 in the general solution (2.3), we get

$$\begin{aligned} y(t) &= \int_a^t \left[\frac{A_1(s)\,B_2(t) - A_2(s)\,B_1(t)}{C} + t - s\right] g(s)\,ds + \\ &\quad \int_t^b \frac{A_1(s)\,B_2(t) - A_2(s)\,B_1(t)}{C} g(s)\,ds. \end{aligned} \tag{2.6}$$

This completes the proof. □

Lemma 2.2. *Let (1.19) hold. If $y(t)$ is a solution of (1.6) satisfying the two-point boundary conditions (1.8), then the following inequality*

$$|y(t)| \le \int_a^b G(s)\,|y'''(s)|\,ds \tag{2.7}$$

holds, where

$$G(t) = \frac{1}{2}\int_a^b |G(u,t)|\,du \tag{2.8}$$

and $G(t,s)$ is given in (1.21).

Proof. Assume that $y(t)$ is a solution of (1.6) satisfying $Y_4(y) = Y_5(y) = Y_6(y) = 0$. It is easy to see that, by using $Y_4(y) = Y_5(y) = 0$ and proceeding as in the proof of Lemma 2.1, we have

$$y'(t) = \int_a^b G(t,s)\,y'''(s)\,ds, \tag{2.9}$$

where $G(t,s)$ is the Green's function (1.21). Integrating (2.9) from a to t, we get

$$y(t) = y(a) + \int_a^t \left(\int_a^b G(u,s)\,y'''(s)\,ds\right) du. \tag{2.10}$$

Similarly, integrating (2.9) from t to b, we get

$$y(t) = y(b) + \int_t^b \left(-\int_a^b G(u,s)\,y'''(s)\,ds\right) du. \tag{2.11}$$

Adding (2.10) and (2.11), and by using $Y_6(y) = 0$, we have

$$y(t) = \frac{1}{2}\left\{\int_a^t \left(\int_a^b G(u,s)\, y'''(s)\, ds\right) du + \int_t^b \left(-\int_a^b G(u,s)\, y'''(s)\, ds\right) du\right\}. \tag{2.12}$$

By taking the absolute value of (2.12), we obtain

$$|y(t)| \le \frac{1}{2}\int_a^b \left(\int_a^b |G(u,s)|\, |y'''(s)|\, ds\right) du \tag{2.13}$$

and hence

$$|y(t)| \le \frac{1}{2}\int_a^b |y'''(s)| \left(\int_a^b |G(u,s)|\, du\right) ds, \tag{2.14}$$

where

$$G(u,s) = \begin{cases} \dfrac{A_1(u)\, B_2(s) - A_2(u)\, B_1(s)}{C}; & s \le u \le b \\ \dfrac{A_1(s)\, B_2(u) - A_2(s)\, B_1(u)}{C}; & a \le u \le s. \end{cases} \tag{2.15}$$

Therefore, we have the inequality (2.7). This completes the proof. □

By using the inequality (2.7), we have the following result which is an useful tool to determine a lower bound of distance between a and b points of solution of the equation (1.6) under the boundary conditions (1.8).

Theorem 2.3. *Let (1.19) hold. If $y(t)$ is a nontrivial solution of (1.6) satisfying the two-point boundary conditions (1.8), then the following Lyapunov-type inequality*

$$1 \le \int_a^b G(s)\, |q(s)|\, ds \tag{2.16}$$

holds, where $G(t)$ is given in (2.8).

Proof. Assume that $y(t)$ is a solution of (1.6) satisfying $Y_4(y) = Y_5(y) = Y_6(y) = 0$ and y is not identically zero on (a,b). From (1.6) and (2.7), we get

$$|y'''(t)| = |q(t)|\, |y(t)| \le |q(t)| \int_a^b G(s)\, |y'''(s)|\, ds. \tag{2.17}$$

Multiplying both sides of (2.17) by $G(t)$ and integrating from a to b, we get

$$\int_a^b G(s)\, |y'''(s)|\, ds \le \int_a^b G(s)\, |y'''(s)|\, ds \int_a^b G(s)\, |q(s)|\, ds. \tag{2.18}$$

Next, we prove that

$$0 < \int_a^b G(s)\, |y'''(s)|\, ds. \tag{2.19}$$

If (2.19) is not true, then we have

$$\int_a^b G(s)\, |y'''(s)|\, ds = 0. \tag{2.20}$$

From (2.7), we get

$$|y(t)| \le \int_a^b G(s)\, |y'''(s)|\, ds = 0. \tag{2.21}$$

It follows from (2.21) that $y(t) \equiv 0$ for $t \in (a,b)$, which contradicts with (1.8) since $y(t) \neq 0$ for all $t \in (a,b)$. Thus, by using (2.19) in (2.18), we get the inequality (2.16). □

Now, we give another main result for the equation (1.6) under the boundary conditions (1.7).

Theorem 2.4. *Let (1.19) hold. If $y(t)$ is a nontrivial solution of (1.6) satisfying the two-point boundary conditions (1.7), then the following Lyapunov-type inequality*

$$1 \leq G_0 \int_a^b |q(s)|\, ds \tag{2.22}$$

holds, where $G_0 = \frac{1}{2} \int_a^b |G(t_0, s)|\, ds$ *and* $|y(t_0)| = \max\{|y(t)| : a \leq t \leq b\}$.

Proof. Assume that $y(t)$ is a solution of (1.6) satisfying $Y_1(y) = Y_2(y) = Y_3(y) = 0$ and y is not identically zero on (a, b). By integrating $y'''(t)$ from a to t, we get

$$y''(t) = y''(a) + \int_a^t y'''(s)\, ds. \tag{2.23}$$

Similarly, by integrating $y'''(t)$ from t to b, we have

$$y''(t) = y''(b) - \int_t^b y'''(s)\, ds. \tag{2.24}$$

Adding the inequalities (2.23) and (2.24), and by using $Y_3(y) = 0$, we have

$$2y''(t) = \int_a^t y'''(s)\, ds - \int_t^b y'''(s)\, ds. \tag{2.25}$$

By taking the absolute value of (2.25), we obtain

$$|y''(t)| \leq \frac{1}{2} \int_a^b |y'''(s)|\, ds. \tag{2.26}$$

Next, pick $t_0 \in (a, b)$ so that $|y(t_0)| = \max\{|y(t)| : a \leq t \leq b\}$. From (1.20), (2.26), and (1.6), we get

$$\begin{aligned} |y(t_0)| &\leq \int_a^b |G(t_0, s)|\, |y''(s)|\, ds \\ &\leq \frac{1}{2} \int_a^b |G(t_0, s)|\, ds \int_a^b |y'''(s)|\, ds \qquad (2.27) \\ &= G_0 \int_a^b |q(s)|\, |y(s)|\, ds \\ &\leq G_0\, |y(t_0)| \int_a^b |q(s)|\, ds. \qquad (2.28) \end{aligned}$$

Dividing both sides by $|y(t_0)|$, we get the inequality (2.22). □

Remark 2.5. To the best of our knowledge, the inequality (2.16) (or (2.22)) is new Lyapunov-type inequality for third order linear differential equation (1.6) under the two-point boundary conditions (1.8) (or (1.7)).

It is easy to see that since

$$G(t) = \frac{1}{2} \int_a^b |G(u, t)|\, du \leq \frac{\widetilde{C}(t)}{2\,|C|}, \tag{2.29}$$

where

$$\begin{aligned} \widetilde{C}(t) &= \widetilde{A}_1(t) \int_a^t \widetilde{B}_2(u)\, du + \widetilde{A}_2(t) \int_a^t \widetilde{B}_1(u)\, du \\ &\quad + \widetilde{B}_2(t) \int_t^b \widetilde{A}_1(u)\, du + \widetilde{B}_1(t) \int_t^b \widetilde{A}_2(u)\, du, \end{aligned} \tag{2.30}$$

$$\widetilde{A}_i(t) = (b-t)|\gamma_{i3}| + |\gamma_{i4}|, \tag{2.31}$$

and

$$\widetilde{B}_i(t) = (|\gamma_{i1}| + |\gamma_{i3}|)(t-a) + |\gamma_{i2}| + |\gamma_{i3}|(b-a) + |\gamma_{i4}| \tag{2.32}$$

for $i = 1, 2$, we have the following result from Theorem 2.3 and hence the proof is omitted.

Corollary 2.6. *Let (1.19) hold. If $y(t)$ is a nontrivial solution of (1.6) satisfying the two-point boundary conditions (1.8), then the following Lyapunov-type inequality*

$$1 \leq \int_a^b \frac{\widetilde{C}(s)}{2|C|}|q(s)|\,ds \tag{2.33}$$

holds, where C and $\widetilde{C}(t)$ are given in (1.16) and (2.30), respectively.

Remark 2.7. Note that if we take $\gamma_{11} = \gamma_{13} = \gamma_{23} = 1$, $\gamma_{21} = -1$, and $\gamma_{i2} = \gamma_{i4} = 0$ for $i = 1, 2$ in (1.8), we have

$$8 \leq \int_a^b (b-s)(b-4a+3s)|q(s)|\,ds \tag{2.34}$$

from (2.33), and hence

$$\frac{6}{(b-a)^2} \leq \int_a^b |q(s)|\,ds. \tag{2.35}$$

Now, we give another result for the equation (1.6) by using the following inequality

$$|G(t,s)| \leq \frac{\widetilde{A}_1(s)\widetilde{B}_2(s) + \widetilde{A}_2(s)\widetilde{B}_1(s)}{|C|} \tag{2.36}$$

obtained by taking the absolute value of (1.21). Thus, we have the following result from Theorem 2.4 and hence the proof is omitted.

Corollary 2.8. *Let (1.19) hold. If $y(t)$ is a nontrivial solution of (1.6) satisfying the two-point boundary conditions (1.7), then the following Lyapunov-type inequality*

$$1 \leq \int_a^b \frac{\widetilde{A}_1(s)\widetilde{B}_2(s) + \widetilde{A}_2(s)\widetilde{B}_1(s)}{2|C|}ds \int_a^b |q(s)|\,ds \tag{2.37}$$

holds, where C, $\widetilde{A}_i(t)$, $\widetilde{B}_i(t)$, $i = 1, 2$, are given in (1.16), (2.31), (2.32), respectively.

Now, we give an application of the obtained Lyapunov-type inequalities for the following eigenvalue problem

$$y''' + \lambda k(t)\,y = 0 \tag{2.38}$$

under the boundary conditions (1.7). Thus, if there exists a nontrivial solution $y(t)$ of linear homogeneous problem (2.38), then we have

$$\frac{2|C|}{\int_a^b \left(\widetilde{A}_1(s)\widetilde{B}_2(s) + \widetilde{A}_2(s)\widetilde{B}_1(s)\right)ds \int_a^b |k(s)|\,ds} \leq |\lambda|, \tag{2.39}$$

where C, $\widetilde{A}_i(t)$, $\widetilde{B}_i(t)$, $i = 1, 2$, are given in (1.16), (2.31), (2.32), respectively.

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