Hacettepe Journal of Mathematics and Statistics
Volume 45 (2) (2016), 429 – 446

On second-order linear recurrent functions with period k and proofs to two conjectures of Sroysang

Julius Fergy Tiongson Rabago*

Abstract

Let w be a real-valued function on $\mathbb{R}$ and k be a positive integer. If for every real number $x, w(x+2k) = rw(x+k) + sw(x)$ for some non-negative real numbers r and s, then we call such function a *second-order linear recurrent function with period k*. Similarly, we call a function $w : \mathbb{R} \to \mathbb{R}$ satisfying $w(x+2k) = -rw(x+k) + sw(x)$ an *odd second-order linear recurrent function with period k*. In this work, we present some elementary properties of these type of functions and develop the concept using the notion of f-even and f-odd functions discussed in [9]. We also investigate the products and quotients of these functions and provide in this work a proof of the conjecture of B. Sroysang which he posed in [19]. In fact, we offer here a proof of a more general case of the problem. Consequently, we present findings that confirm recent results in the theory of Fibonacci functions [9] and contribute new results in the development of this topic.

Keywords: Second-order linear recurrent functions with period k, Horadam numbers, generalized Fibonacci sequences, Pell functions with period k, Jacobsthal functions with period k, Sroysang's conjecture.

2000 AMS Classification: Primary: 39A10, Secondary: 11B37.

Received : 21.09.2014 *Accepted* : 06.03.2015 *Doi* : 10.15672/HJMS.20164512497

*Department of Mathematics and Computer Science, College of Science, University of the Philippines Baguio, Governor Pack Road, Baguio City 2600, Benguet, PHILIPPINES
Email: jtrabago@upd.edu.ph, jfrabago@gmail.com

1. Introduction

The *Fibonacci numbers* F_n are defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2, F_{n+1} = F_n + F_{n-1}$. Since the introduction of these numbers, different generalizations have been formulated and were extensively studied. In fact, several books were written solely to study the properties of Fibonacci numbers (*see* for instance [5] and [21]). In a book by T. Koshy [11], any sequences G_n, where $G_1 = a, G_2 = b$, and $G_n = G_{n-1} + G_{n-2}, n \geq 3$ is called the *generalized Fibonacci sequence* (GFS). In [10], A. F. Horadam defined a second-order linear recurrence sequence $\{W_n\}$ by the recurrence relation

$$W_0 = a, \quad W_1 = b, \quad W_{n+1} = rW_n + sW_{n-1}, \quad (n \geq 2).$$

The sequence $\{W_n\}$ can be viewed easily as a certain generalization of $\{F_n\}$. It is now known in literature as *Horadam's sequence*. For a good survey paper regarding Horadam numbers, we refer the readers to [12] (see also [13] for a survey update and extensions). The n^{th} Horadam number W_n with initial conditions $W_0 = 0$ and $W_1 = 1$ can be represented by the following Binet's formula:

$$W_n(0, 1; r, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (n \geq 2)$$

where α and β are the roots of the quadratic equation $x^2 - rx - s = 0$, i.e. $\alpha = (r + \sqrt{r^2 + 4s})/2$ and $\beta = (r - \sqrt{r^2 + 4s})$.

In [15], the author presented a formula for solving the missing terms of $\{W_n\}$ given its first term and last term. Another generalization of Fibonacci numbers is the so-called Fibonacci polynomials (*see* [1] and [4] and the references therein). Recently, Fibonacci numbers were involved in the study of difference and differential equations. Particularly, in [20], D. T. Tollu, Y. Yazlik, and N. Taskara investigated the solutions of two special types of the Riccati difference equation

$$x_{n+1} = \frac{1}{1 + x_n} \quad \text{and} \quad y_{n+1} = \frac{1}{-1 + y_n}$$

such that their solutions are associated with Fibonacci numbers. Another interesting investigation, which involves the Fibonnaci numbers, is presented in [6] where A. Hakami found an application of Fibonacci numbers in the study of continued fractions. This work of Hakami has been recently generalized, to some extent, by the author in [17]. Meanwhile, the author and J. B. Bacani consider in [2] the system

$$x_{n+1} = \frac{q}{p + x_n^\nu} \quad \text{and} \quad y_{n+1} = \frac{q}{-p + y_n^\nu} \quad (p, q \in \mathbb{R}^+ \text{ and } \nu \in \mathbb{N})$$

as a generalization of Tollu et al.'s work [20]. One particular result established in [2] is the solution form of the above system. In fact, it was shown that every solution of the system, for any arbitrary given set of initial values, is expressible in terms of Horadam numbers. In an earlier paper, the author [16], studied homogeneous differential equations of the form

$$w^{(2k)}(x) = rw^{(k)}(x) + sw(x),$$

where $r, s \in \mathbb{R}^+$ and $w^{(k)}$ is the k^{th} derivative of w with respect to x. Intriguingly, it was found that the differential equation has some sort of connection with Horadam numbers.

Other papers dealing with problems involving Fibonacci numbers are what follows. In [7], J. S. Han, H. S. Kim and J. Neggers studied the Fibonacci norm of positive integers, and in [8], they studied Fibonacci sequences in groupoids. Han, Kim, and Neggers also introduced the concept of Fibonacci functions with Fibonacci numbers in [9] which were later on extended by B. Sroysang [19] to Fibonacci functions with period k. In [19], the following generalization of Fibonacci functions was presented.

1.1. Definition ([19])**.** Let $k \in \mathbb{N}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called a Fibonacci function with period k if it satisfies the equation

$$f(x+2k) = f(x+k) + f(x), \quad \forall x \in \mathbb{R}. \tag{1.1}$$

Properties of Fibonacci functions with period k were also presented in [19]. As further generalization of these functions, we define a second-order linear recurrent function with period k as follows:

1.2. Definition. Let k be a positive integer, and r and s be non-negative real numbers. A function $w\colon \mathbb{R} \to \mathbb{R}$ is said to be a second-order linear recurrent function with period k if it satisfies the equation

$$w(x+2k) = rw(x+k) + sw(x), \quad \forall x \in \mathbb{R}. \tag{1.2}$$

We say that w is a complete second-order linear recurrent function if for any arbitrary r and s, w satisfies (1.2), otherwise it is called conditional.

Now, throughout the rest of this paper, we shall refer to a real-valued function w satisfying equation (1.2) as a recurrent function of order k instead of using the term second-order linear recurrent function with period k for convenience.

Definition (1.2) provides a further generalization of Fibonacci functions with Fibonacci numbers [9] and Fibonacci functions with period k [19]. Our main contribution includes a proof of the following open problems posed by Sroysang [19] (which is in fact a proof of a more general case of the problem):

1.3. Conjecture. *If f is a Fibonacci function with period k, then $f(x+k)/f(x) \to \phi$ as $x \to \infty$.*

1.4. Conjecture. *If f is a Fibonacci function with period k, then $f(x+k)/f(x) \to -\phi$ as $x \to \infty$.*

Here ϕ denotes the well-known golden ratio, i.e. $\phi = (1+\sqrt{5})/2 = 1.6180339\ldots$.

Now the rest of the paper is organized as follows: in Section 2 and Section 3, we give examples and basic properties of recurrent functions with period k and odd recurrent functions with period k, respectively. In Section 4, we develop the notion of these types of recurrent functions using the concept of f-even and f-odd functions discussed in [9]. In Section 5, we study the products of these functions and finally, in Section 6, we investigate the quotients of these functions. The proofs of conjectures (1.3) and (1.4) are also presented in the last section.

2. Recurrent functions with period k

In this section we present some properties of recurrent functions with period k. We begin by defining what we call Pell and Jacobsthal functions. If in equation (1.2), $r = 2$ and $s = 1$ (resp. $r = 1$ and $s = 2$), then we call such function a Pell (resp. Jacobsthal) function. That is, for a given natural number k, a Pell function p with period k satisfies

$$p(x+2k) = 2p(x+k) + p(x), \quad \forall x \in \mathbb{R}, \tag{2.1}$$

and a Jacobsthal function j with period k satisfies

$$j(x+2k) = j(x+k) + 2j(x), \quad \forall x \in \mathbb{R}. \tag{2.2}$$

2.1. Example. Let $\alpha > 0, k \in \mathbb{N}$, and $w(x) = \alpha^{x/k}$ be a recurrent function with period k. Substituting w in (1.2), we have

$$\alpha^{(x/k)+2} = r\alpha^{(x/k)+1} + s\alpha^{x/k}, \quad \forall x \in \mathbb{R}.$$

So $\alpha^2 - r\alpha - s = 0$ whose roots are $\alpha_{1,2} = (r \pm \sqrt{r^2+4s})/2$. Thus, $w(x) = \alpha^{x/k}$.

The following are special cases of the previous example.

(1) If $(r, s) = (1, 1)$, then the function $f(x) := \phi^{x/k}$ is an example of a Fibonacci function with period k.
(2) If $(r, s) = (2, 1)$, then $p(x) := \sigma^{x/k}$, where $\sigma = 1 + \sqrt{2}$ is the well known *silver ratio*, is an example of a Pell function with period k.
(3) If $(r, s) = (1, 2)$, then the function $j(x) := 2^{x/k}$ is an example of a Jacobsthal function with period k.

2.2. Remark. Clearly, the functions f, p, and j are conditional. Also, any non-zero constant function is conditional but only for positive real numbers r and s such that $r + s = 1$. On the other hand, the function $w(x) \equiv 0$ is an example of a complete type. It can be verified directly that any scalar multiple of a recurrent function with period k is again a recurrent function with period k. Furthermore, if a differentiable function w is a recurrent function with period k then so is its derivative w'.

2.3. Proposition. *Let $k \in \mathbb{N}$ and $w \colon \mathbb{R} \to \mathbb{R}$ be a recurrent function with period k. Define $g_t(x) = w(x + t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then g_t is also a recurrent function with period k.*

2.4. Corollary ([19])**.** *Let $k \in \mathbb{N}$ and $f \colon \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period k. Define $g_t(x) = f(x+t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, g_t is also a Fibonacci function with period k.*

2.5. Example. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) = \alpha^{(x+t)/k}, \quad \forall x \in \mathbb{R}, \tag{2.3}$$

then g_t is a recurrent function with period k.

As special cases of the previous example, we have the following:

(1) if $(r, s) = (1, 1)$, then we have $g_t(x) := f(x+t) = \phi^{(x+t)/k}$, a Fibonacci function with period k,
(2) if $(r, s) = (2, 1)$, then we have $g_t(x) := p(x + t) = \sigma^{(x+t)/k}$, a Pell function with period k,
(3) if $(r, s) = (1, 2)$, then we have $g_t(x) := j(x+t) = 2^{(x+t)/k}$, a Jacobsthal function with period k.

2.6. Theorem. *Let w be a recurrent function with period k and $\{W_n(0, 1; r, s)\}$, or simply $\{W_n\}$, be a Horadam sequence with initial conditions $W_0 = 0$ and $W_1 = 1$. Then,*

$$w(x + nk) = W_n w(x + k) + sW_{n-1}w(x), \quad \forall x \in \mathbb{R}, n \in \mathbb{N}. \tag{2.4}$$

Proof. The proof is by induction on n. First, we note that $\alpha + \beta = r, \alpha - \beta = \sqrt{r^2 + 4s}$, and $\alpha\beta = -s$. We see that the formula obviously holds for $n = 1, 2$. So we assume (2.4) holds for n and $n + 1$ for some natural number $n \geq 2$. Then,

$$\begin{aligned}
w(x + (n+2)k) &= rw(x + (n+1)k) + sw(x + nk) \\
&= r\left(W_{n+1}w(x + k) + sW_n w(x)\right) \\
&\qquad + s\left(W_n w(x + k) + sW_{n-1}w(x)\right) \\
&= \left(rW_{n+1} + sW_n\right) w(x + k) + s\left(rW_n + sW_{n-1}\right) w(x) \\
&= W_{n+2}w(x + k) + sW_{n+1}w(x), \quad \forall x \in \mathbb{R}, n \in \mathbb{N}.
\end{aligned}$$

By induction principle, conclusion follows. $\square$

2.7. Corollary. *Let w be a recurrent function with period k and let $\{W_n\}$ be the sequence of Horadam numbers. Then, $\alpha^n = \alpha W_n + sW_{n-1}$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In particular, for $r = s = 1$, we have $\phi^n = \phi F_n + F_{n-1}$, where F_n is the n^{th} Fibonacci number, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$.*

Proof. From Example (2.1) we saw that $w(x) = \alpha^{x/k}$ is a recurrent function with period k, so it satisfies equation (2.4), i.e.

$$\begin{aligned}\alpha^{(x+nk)/k} &= w(x+nk) = W_n w(x+k) + sW_{n-1}w(x)\\ &= \alpha^{(x+k)/k}W_n + sW_{n-1}\alpha^{x/k}, \quad \forall x \in \mathbb{R}, n \in \mathbb{N}.\end{aligned}$$

Upon simplifying, we get

$$\alpha^n = \alpha W_n + sW_{n-1}, \quad \forall n \in \mathbb{N},$$

as desired. Letting $r = s = 1$ in $\alpha = (r + \sqrt{r^2+4s})/2$ we get $\phi^n = \phi F_n + F_{n-1}$. $\square$

2.8. Corollary ([19]). *Let f be a Fibonacci function with period k and let $\{F_n\}$ be the sequence of Fibonacci numbers. Then, $f(x+nk) = F_n f(x+k) + F_{n-1}f(x)$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$.*

Similar results can be obtained easily for Pell and Jacobsthal functions.

3. Odd recurrent functions with period k

Here we discuss the notion of odd recurrent functions with period k formally defined as follows:

3.1. Definition. Let $k \in \mathbb{N}$. A function $w : \mathbb{R} \to \mathbb{R}$ is said to be an odd recurrent functions with period k if

$$w(x+2k) = -rw(x+k) + sw(x), \quad \forall x \in \mathbb{R}. \tag{3.1}$$

In particular, a function is an odd Fibonacci, odd Pell, and odd Jacobsthal function if it satisfies equation (3.1) with $(r, s) = (1,1), (2,1)$, and $(1,2)$, respectively.

3.2. Example. Let $k \in \mathbb{N}, \tilde{\alpha} > 0$, and $w(x) = \tilde{\alpha}^{x/k}$ be an odd recurrent function with period k. Then, $\tilde{\alpha} = (-p + \sqrt{p^2+4q})/2 = q\alpha^{-1}$. So the function $w(x) = \left(q\alpha^{-1}\right)^{x/k}$ is an odd recurrent function.

Similar to what we remarked for recurrent functions with period k, if a differentiable function w is an odd recurrent function with period k then so is its derivative w'. Furthermore, any function defined by $g_t = w(x+t)$, where w satisfies (3.1) and $t \in \mathbb{R}$, is an odd recurrent function with period k. For example, the function defined by $g_t(x) = \tilde{\alpha}^{(x+t)/k}$ is an odd recurrent function with period k. Of course, the functions $g_t(x) := f(x+t) = \phi^{-(x+t)/k}$ and $g_t(x) := p(x+t) = \sigma^{-(x+t)/k}$ are also an odd Fibonacci and odd Pell function, respectively.

3.3. Theorem. *Let $w\colon \mathbb{R} \to \mathbb{R}$ be an odd recurrent function with period k, and let $\{W_{-n}\} = \{(-1)^{n+1}W_n(0,1;r,s)\}$, i.e.*

$$W_{-(n+1)} = -rW_{-n} + sW_{-n+1}, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Then, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$w(x+nk) = W_{-n}w(x+k) + sW_{-n+1}w(x). \tag{3.3}$$

The above theorem can be proven similarly as in Theorem 4.9 and we leave this to the reader.

3.4. Corollary. *Let w be an odd recurrent function with period k and let $\{W_{-n}\} = \{(-1)^{n+1}W_n\}$, where W_n is the n^{th} Horadam number. Then, $\tilde{\alpha}^n = \tilde{\alpha}W_{-n} + sW_{-n+1}$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In particular, for $r = 2$ and $s = 1$, we have $\sigma^{-n} = \sigma^{-1}P_n + P_{n-1}$, where P_n is the n^{th} Pell number, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$.*

3.5. Corollary ([19]). *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd Fibonacci function with period k, and let $\{F_{-n}\} = \{(-1)^{n+1}F_n\}$ be a sequence of numbers where F_n is the n^{th} Fibonacci number, i.e.*

$$F_{-(n+1)} = -F_{-n} + F_{-n+1}, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Then, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$f(x + nk) = F_{-n}f(x + k) + F_{-n+1}f(x). \tag{3.5}$$

Similar results can be formulated easily for Pell and Jacobsthal functions. Now we develop the concept of recurrent functions with period k using f-even and f-odd functions with period k.

4. f-even and f-odd functions with period k

We start-off this section with the following definition.

4.1. Definition (cf. [19]). Let $k \in \mathbb{N}$ and $\varphi \colon \mathbb{R} \to \mathbb{R}$ be such that if $\varphi h \equiv 0$, where $h \colon \mathbb{R} \to \mathbb{R}$ is continuous, then $h \equiv 0$. The map φ is said to be an f-even function with period k (resp. f-odd) function with period k if $\varphi(x+k) = \varphi(x)$ (resp. $\varphi(x+k) = -\varphi(x)$) for any $x \in \mathbb{R}$.

By the above definition, we can see immediately that there is no f-even and f-odd Fibonacci function except possibly when the function is the zero function. In fact, in general, a function $w \colon \mathbb{R} \to \mathbb{R}$ satisfying equation (1.2) is an f-even recurrent function with period k if and only if $r + s = 1$ or $w(x) \equiv 0$ for all $x \in \mathbb{R}$. Similarly, w is an f-odd recurrent function with period k if and only if $B - A = 1$ or $w \equiv 0$ for all $x \in \mathbb{R}$.

We first discuss f-even functions.

4.2. Example. Let $\varphi(x) = \cos(\pi x)$ for all $x \in \mathbb{R}$ and let $h \colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $(\varphi h)(x) = 0$. For any $x \notin \frac{\pi}{2}\mathbb{Z}$, we have $\varphi(x) \neq 0$, so $h(x) = 0$. Since $\mathbb{R} \backslash \frac{\pi}{2}\mathbb{Z}$ is dense in $\mathbb{R}$ and h is a continuous function, $h(x) = 0$. Now, let k be an even natural number and $x \in \mathbb{R}$. Then,

$$\varphi(x + k) = \cos(\pi(x + k)) = \cos(\pi x)\cos(k\pi) - \sin(\pi x)\sin(k\pi) = \cos(\pi x) = \varphi(x).$$

Hence, $\varphi(x) = \cos(\pi x)$ is an f-even function.

In [19], we have seen that $\varphi(x) = x - \lfloor x \rfloor$ is also an example of f-even functions.

4.3. Theorem. *Let $k \in \mathbb{N}$ and $\varphi \colon \mathbb{R} \to \mathbb{R}$ be an f-even function with period k and let $w \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, w is a recurrent function with period k if and only if φw is a recurrent function with period k.*

Proof. For the necessity part, we assume that w is a recurrent function with period k satisfying equation (1.2) with $r, s \in \mathbb{R}^+$. Then, for any $x \in \mathbb{R}$ we have

$$\begin{aligned} \varphi(x + 2k)w(x + 2k) &= \varphi(x + k)[rw(x + k) + sw(x)] \\ &= r\varphi(x + k)w(x + k) + s\varphi(x + k)w(x) \\ &= r\varphi(x + k)w(x + k) + s(\varphi w)(x). \end{aligned}$$

Hence, the product φw is a recurrent function with period k.

Now, for the sufficiency part, we assume that φw is a recurrent function with period k satisfying equation (1.2) with $r, s \in \mathbb{R}^+$. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned}\varphi(x+k)w(x+2k) &= \varphi(x+2k)w(x+2k) = (\varphi w)(x+2k)\\ &= p(\varphi w)(x+k) + q(\varphi w)(x)\\ &= r\varphi(x+k)w(x+k) + s(\varphi w)(x)\\ &= r\varphi(x+k)w(x+k) + s\varphi(x+k)w(x)\\ &= \varphi(x+k)[rw(x+k) + sw(x)].\end{aligned}$$

Thus, $w(x+2k) = rw(x+k) + sw(x)$, and this shows that w is a recurrent function with period k. This completes the proof of the theorem. □

4.4. Corollary. *Let $k \in \mathbb{N}$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an f-even function with period k and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, g is a Fibonacci (resp. Pell and Jacobsthal) function with period k if and only if φg is a Fibonacci (resp. Pell and Jacobsthal) function with period k.*

4.5. Example. Let $k \in \mathbb{N}$ and define $\varphi(x) = \cos(\pi x)$ and $\gamma(x) = x - \lfloor x \rfloor$. Note that φ and γ are examples of f-even functions. Furthermore, recall that the function $w(x) = \alpha^{x/k}$ is a recurrent function with period k. Then, for all $x \in \mathbb{R}$, the products

$$(\varphi w)(x) = \cos(\pi x)\alpha^{x/k} \tag{4.1}$$

and

$$(\gamma w)(x) = (x - \lfloor x \rfloor)\alpha^{x/k} \tag{4.2}$$

are both examples of recurrent functions with period k. Specifically, if $(r, s) = (2, 1)$, then $(\varphi p)(x) = \cos(\pi x)\sigma^{x/k}$ and $(\gamma p)(x) = (x - \lfloor x \rfloor)\sigma^{x/k}$ are both Pell functions with period k.

4.6. Example. Let $k \in \mathbb{N}$ and define

$$\varphi(x) = \begin{cases} 1, & \text{for } x \in \mathbb{Q};\\ -1, & \text{otherwise.}\end{cases}$$

Hence, $\varphi(x+k) = \varphi(x)$ for any $x \in \mathbb{R}$. Also, if $\varphi h \equiv 0$, then $h \equiv 0$ whether or not h is continuous. Thus, φ is an f-even function with period k. We know that $w(x) = (x - \lfloor x \rfloor)\,\alpha^{x/k}$ is a recurrent function with period k. So, by Theorem 4.3, the mapping defined by

$$(\varphi w)(x) = \begin{cases} (x - \lfloor x \rfloor)\,\alpha^{x/k}, & \text{for } x \in \mathbb{Q};\\ (\lfloor x \rfloor - x)\,\alpha^{x/k}, & \text{otherwise,}\end{cases}$$

is also a recurrent function. Specifically, if $(r, s) = (1, 2)$, then $w(x) = 2^{x/k}\,(x - \lfloor x \rfloor)$. So we have,

$$(\varphi w)(x) = \begin{cases} 2^{x/k}\,(x - \lfloor x \rfloor), & \text{for } x \in \mathbb{Q};\\ 2^{x/k}\,(\lfloor x \rfloor - x), & \text{otherwise,}\end{cases}$$

a Jacobsthal function with period k.

4.7. Theorem. *Let $k \in \mathbb{N}, \varphi\colon \mathbb{R} \to \mathbb{R}$ be an f-even function with period k, and let $w\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, w is an odd recurrent function with period k if and only if φw is an odd recurrent function with period k.*

We omit the proof since it is similar on how we prove Theorem 4.3.

4.8. Example. Let k be an even natural number and consider the f-even functions φ and γ. In Example (3.2), we saw that $w(x) = \tilde{\alpha}^{x/k}$ is an odd recurrent function with period k. Then, for all $x \in \mathbb{R}$, the functions $(\varphi w)(x) = \cos(\pi x)\tilde{\alpha}^{x/k}$ and $(\gamma w)(x) = (x - \lfloor x \rfloor)\tilde{\alpha}^{x/k}$ are both odd recurrent functions with period k.

We now discuss f-odd functions. Recall that if $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that for $\varphi h \equiv 0$ and h is continuous, we have $h \equiv 0$. The map φ is said to be an f-odd function with period k if $\varphi(x + k) = -\varphi(x)$ for all $x \in \mathbb{R}$. We have seen in [19] that $\varphi(x) = \sin(\pi x)$ is an example of f-odd function.

4.9. Theorem. *Let $k \in \mathbb{N}, \varphi \colon \mathbb{R} \to \mathbb{R}$ be an f-odd function with period k, and let $w \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, w is a recurrent function with period k if and only if φw is an odd recurrent function with period k.*

Again, we leave the proof to the reader.

4.10. Example. Let k be any odd natural number. Define $\varphi(x) = \sin(\pi x)$ and $w(x) = \tilde{\alpha}^{x/k}$ for all $x \in \mathbb{R}$. Hence, $(\varphi w)(x) = \tilde{\alpha}^{x/k} \sin(\pi x)$. We have seen in our discussion that φ is an f-odd function with period k and w is an odd recurrent function with period k. Hence, by Theorem 4.9, the product φw is a recurrent function with period k. We have the following examples for specific values of r and s.

(1) If $(r, s) = (1, 1)$, then $(\varphi f)(x) = \sin(\pi x)\,(\phi - 1)^{x/k}$ is a Fibonacci function with period k.
(2) If $(r, s) = (2, 1)$, then $(\varphi p)(x) = \sin(\pi x)\,(\sigma - 2)^{x/k}$ is a Pell function with period k.
(3) If $(r, s) = (1, 2)$, then $(\varphi j)(x) = \sin(\pi x)$ is a Jacobsthal function with period k.

5. Products of recurrent functions with period k

In this section, we give conditions so that whenever g and h are any two recurrent functions with period k in $\mathbb{R}$, their product forms another recurrent function with period k.

5.1. Theorem. *Let $k \in \mathbb{N}$ and g and h be two recurrent functions with period k satisfying*

$$g(x + 2k) = Ag(x + k) + Bg(x), \quad \forall x \in \mathbb{R}, \tag{5.1}$$

$$h(x + 2k) = Uh(x + k) + Vh(x), \quad \forall x \in \mathbb{R}, \tag{5.2}$$

respectively, where $A, B, U,$ and V are non-negative real numbers. Suppose the following conditions are satisfied:

(C1) $\mathcal{A} = AU$, $\mathcal{B} = BV$, $AV = BU$,
(C2) *g is an f-even function and h is an f-odd function.*

Then, $w(x) := (gh)(x)$ forms another recurrent function with period k satisfying the equation

$$w(x + 2k) = \mathcal{A}w(x + k) + \mathcal{B}w(x), \quad \forall x \in \mathbb{R}. \tag{5.3}$$

Proof. The proof is straightforward. Suppose g and h are two recurrent functions with period k ($k \in \mathbb{N}$) satisfying equations (5.1) and (5.2), respectively. Furthermore, we suppose that conditions (C1) and (C2) are satisfied. Then,

$$\begin{aligned} w(x+2k) &= (gh)(x+2k) = [Ag(x+k)+Bg(x)][Uh(x+k)+Vh(x)] \\ &= \mathcal{A}w(x+k)+\mathcal{B}w(x)+[AVg(x+k)h(x)+BUg(x)h(x+k)] \\ &= \mathcal{A}w(x+k)+\mathcal{B}w(x)+AVg(x)[h(x)+h(x+k)] \\ &= \mathcal{A}w(x+k)+\mathcal{B}w(x), \quad \forall x\in\mathbb{R}, \end{aligned}$$

proving the theorem. □

5.2. Example. Let k be an odd natural number and $t \in \mathbb{R}^+$. Define $g(x) = x - \lfloor x \rfloor$ and let $A = t/(2t+1)$ and $B = (t+1)/(2t+1)$. Furthermore, define $h(x) = \sin(\pi x)$ and $U = V - 1 = t$. We claim that $w(x) := (gh)(x)$ is a recurrent function with period k satisfying the following equation:

$$w(x+2k) = \mathcal{A}w(x+1)+\mathcal{B}w(x), \quad \forall x\in\mathbb{R}, \tag{5.4}$$

where $\mathcal{A} = AU$ and $\mathcal{B} = BV$. We know that $g(x) = x - \lfloor x \rfloor$ and $h(x) = \sin(\pi x)$ are examples of f-even and f-odd functions with period k, respectively. We first show that g satisfies the equation

$$g(x+2k) = \frac{t}{2t+1}g(x+k)+\frac{t+1}{2t+1}g(x) \quad \forall x\in\mathbb{R}, \tag{5.5}$$

and then show that h satisfies

$$h(x+2k) = th(x+k)+(t+1)h(x), \quad \forall x\in\mathbb{R}. \tag{5.6}$$

We have,

$$\begin{aligned} g(x+2k) &= x+2k-\lfloor x+2k\rfloor = x-\lfloor x\rfloor = \left(\frac{t}{2t+1}+\frac{t+1}{2t+1}\right)(x-\lfloor x\rfloor) \\ &= \frac{t}{2t+1}\left(x+1-\lfloor x+1\rfloor\right)+\frac{t+1}{2t+1}\left(x+1-\lfloor x+1\rfloor\right) \\ &= \frac{t}{2t+1}g(x+1)+\frac{t+1}{2t+1}g(x), \quad \forall x\in\mathbb{R}. \end{aligned}$$

and

$$\begin{aligned} h(x+2k) &= \sin(\pi(x+2k)) = \sin(\pi x)\cos(2k\pi) \\ &= -t\sin(\pi x)+(t+1)\sin(\pi x) \\ &= t\sin(\pi x)\cos(k\pi)+(t+1)\sin(\pi x) \\ &= t\sin(\pi(x+k))+(t+1)\sin(\pi x) \\ &= th(x+k)+(t+1)h(x), \quad \forall x\in\mathbb{R}. \end{aligned}$$

Obviously, $AV = BU$. By Theorem 5.1, it follows that $w(x) = (gh)(x)$ is a recurrent function with period k satisfying equation (5.4).

5.3. Corollary. *Let g and h be two recurrent functions with period k satisfying equations* (5.1) *and* (5.2)*, respectively. Suppose conditions* (C1) *and* (C2) *are satisfied, then $w(x) := (gh)(x)$ is never a Fibonacci function with period k except possibly when $g \equiv 0$ or $h \equiv 0$ for all $x \in \mathbb{R}$.*

Proof. Let g and h be two functions satisfying equations (5.1) and (5.2), respectively and suppose that conditions (C1) and (C2) are satisfied. Hence,

$$\begin{aligned} g(x+k) &= g(x+2k) = Ag(x+k)+Bg(x) \\ &= (A+B)g(x+k), \quad \forall x\in\mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} h(x+k) &= h(x+2k) = Uh(x+k) + Vh(x) \\ &= (U-V)h(x+k), \quad \forall x \in \mathbb{R}. \end{aligned}$$

It follows that $A+B=1$ and $U-V=1$. Since $AV=BU$, we have $AV=(1-B)V=(1+V)B=BU$ which implies that $B=V/(2V+1)$. Letting $V=t\in\mathbb{R}^+$ we get the following equations:

$$g(x+2k) = (t+1)(2t+1)^{-1}g(x+k) + t(2t+1)^{-1}g(x), \tag{5.7}$$

$$h(x+2k) = (t+1)h(x+k) + th(x). \tag{5.8}$$

Hence,

$$\begin{aligned} w(x+2k) &= g(x+2k)h(x+2k) \\ &= \left(\frac{t+1}{2t+1}g(x+k) + \frac{t}{2t+1}g(x)\right)((t+1)h(x+k)+th(x)) \\ &= \frac{(t+1)^2}{2t+1}w(x+k) + \frac{t^2}{2t+1}w(x) + \frac{t(t+1)}{2t+1}g(x)\left(h(x+k)+h(x)\right) \\ &= \frac{(t+1)^2}{2t+1}w(x+k) + \frac{t^2}{2t+1}w(x) \\ &= \mathcal{A}w(x+k) + \mathcal{B}w(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Suppose w is a Fibonacci function. Then, $\mathcal{A}=\mathcal{B}=1$. This is impossible since

$$\frac{(t+1)^2}{2t+1} = \frac{t^2}{2t+1} + 1 = 2 > 1 = \frac{t^2}{2t+1}.$$

This proves the theorem. □

5.4. Corollary. *Let g and h be two recurrent functions satisfying equation* (5.1) *and* (5.2)*, respectively. Suppose that $AU=2, BV=1, AV=BU$ and condition* (C2) *is satisfied. Then, $w(x) := (gh)(x)$ is a Pell function.*

Proof. The proof follows a similar argument used in the proof of Corollary 5.3 so we omit it. □

5.5. Example. In the proof of Corollary 5.3 we have seen that the product of equations (5.7) and (5.8) forms a recurrent function provided they satisfy conditions (C1) and (C2). If we set $\mathcal{A}=2$ and $\mathcal{B}=1$, then we obtain a Pell function provided we could find a positive real number t such that $t^2-2t-1=0$. The solution to this equation is given by $t=1\pm\sqrt{2}$, so we choose $t=1+\sqrt{2}\in\mathbb{R}^+$. Hence, equations (5.7) and (5.8) become

$$g(x+2k) = \left(\frac{\sigma+1}{2\sigma+1}\right)g(x+k) + \left(\frac{\sigma}{2\sigma+1}\right)g(x) \tag{5.9}$$

and

$$h(x+2k) = (\sigma+1)h(x+k) + \sigma h(x), \tag{5.10}$$

respectively. Now, our goal is to find functions g and h satisfying condition (C2). We can choose $g(x)=x-\lfloor x\rfloor$ and $h(x)=\sin(\pi x)$, and one can check that these equations satisfy equations (5.9) and (5.10). Thus, by Corollary 5.8, $w(x)=(x-\lfloor x\rfloor)\sin(\pi x)$ is a Pell function.

5.6. Corollary. *Let g and h be any two recurrent functions with period k satisfying equation* (5.1) *and* (5.2)*, respectively, such that $AU=1, BV=2, AV=BU$ and condition* (C2) *is satisfied. Then, $w(x) := (gh)(x)$ is a Jacobsthal function.*

5.7. Example. In Example (5.2), we have seen that the function defined by $w(x) := (gh)(x) = (x - \lfloor x \rfloor)\sin(\pi x)$ is a recurrent function with period k satisfying equation (5.4). If we set $\mathcal{A}$ and $\mathcal{B}$ to be in the set of natural numbers such that $\mathcal{B} = \mathcal{A} + 1$, then we have the following

$$\begin{aligned} g(x+2k) &= \theta(2\theta+1)^{-1}g(x+k) + (\theta+1)(2\theta+1)^{-1}g(x), \quad \forall x \in \mathbb{R}, \\ h(x+2k) &= \theta h(x+k) + (\theta+1)\,h(x), \quad \forall x \in \mathbb{R}, \\ w(x+2k) &= \mathcal{A}w(x+k) + (\mathcal{A}+1)w(x), \quad \forall x \in \mathbb{R}, \end{aligned}$$

where $\theta = \mathcal{A} + \sqrt{\mathcal{A}(\mathcal{A}+1)}$. If we let $\mathcal{A} = 1$, then we obtain the following equations,

$$g(x+2k) = \sigma(2\sigma+1)^{-1}g(x+k) + (\sigma+1)(2\sigma+1)^{-1}g(x), \quad \forall x \in \mathbb{R}, \tag{5.11}$$

$$h(x+2k) = \sigma h(x+k) + (\sigma+1)\,h(x), \quad \forall x \in \mathbb{R}, \tag{5.12}$$

$$w(x+2k) = w(x+k) + 2w(x), \quad \forall x \in \mathbb{R}, \tag{5.13}$$

where $\sigma = 1 + \sqrt{2}$ is the well known silver ratio. Suprisingly, equation (5.13)appears to be a Jacobsthal function. Since g and h are f-even and f-odd functions, respectively, we see that the function defined by $w(x) := (gh)(x) = (x - \lfloor x \rfloor)\sin(\pi x)$ with g and h satisfying equation (5.11)and (5.12) is indeed a Jacobsthal function by Corollary 5.6.

5.8. Theorem. *Let g be a recurrent function with period k ($k \in \mathbb{N}$) and h be an odd recurrent function, also, with period k satisfying*

$$g(x+2k) = Ag(x+k) + Bg(x), \quad \forall x \in \mathbb{R}, \tag{5.14}$$

$$h(x+2k) = -Uh(x+k) + Vh(x), \quad \forall x \in \mathbb{R}, \tag{5.15}$$

respectively, where $A, B, U,$ and V are non-negative real numbers. Suppose condition (C1) *is satisfied and*

(C3) *g and h are both f-even functions, or*
(C4) *g and h are both f-odd functions.*

Then, $w(x) := (gh)(x)$ is an odd recurrent function with period k satisfying the equation

$$w(x+2k) = -\mathcal{A}w(x+k) + \mathcal{B}w(x). \tag{5.16}$$

Proof. The proof is similar to the proof of Theorem 5.1 so we omit it. □

5.9. Example. Let k be an even natural number and $t \in \mathbb{R}^+$. Consider the functions $g(x) = x - \lfloor x \rfloor$ satisfying equation (5.7) and $h(x) = \cos(\pi x)$. We show that h is an odd recurrent function with period k satisfying the equation

$$h(x+2k) = -th(x+k) + (t+1)h(x). \tag{5.17}$$

We have,

$$\begin{aligned} h(x+2k) &= \cos(\pi(x+2k)) = \cos(\pi x)\cos(2k\pi) \\ &= -t\cos(\pi x) + (t+1)\cos(\pi x) \\ &= -t\cos(\pi x)\cos(k\pi) + (t+1)\cos(\pi x) \\ &= -t\cos(\pi(x+k)) + (t+1)\cos(\pi x) \\ &= -th(x+k) + (t+1)h(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Clearly, $AV = BU$. Since g and h are both f-even functions, then by Theorem 5.8, $w(x) := (x - \lfloor x \rfloor)\cos(\pi x)$ is an odd recurrent function with period k satisfying the equation given by

$$w(x+2k) = -\left(\frac{t^2}{2t+1}\right)w(x+k) + \left(\frac{(t+1)^2}{2t+1}\right)w(x), \quad \forall x \in \mathbb{R}. \tag{5.18}$$

5.10. Example. Let $k \in \mathbb{N}$ and $A, B, U, V \in \mathbb{R}^+$. Consider f-odd functions g and h satisfying equations (5.14) and (5.15), respectively. Then we have

$$\begin{aligned} g(x+2k) &= Ag(x+k) + Bg(x) = (A-B)g(x+k) \\ &= (B-A)g(x+2k) \end{aligned}$$

and

$$\begin{aligned} h(x+2k) &= -Uh(x+k) + Vh(x) = -(U+V)h(x+k) \\ &= (U+V)h(x+2k). \end{aligned}$$

These imply that $B - A = 1$ and $U + V = 1$. If $AV = BU$, then $AV = (B-1)V = (1-V)B = BU$, which implies that $B = V/(2V-1)$. Hence, we have the following:

$$g(x+2k) = (1-V)(2V-1)^{-1}g(x+k) + V(2V-1)^{-1}g(x), \tag{5.19}$$

$$h(x+2k) = -(1-V)h(x+k) + Vh(x). \tag{5.20}$$

Since $A \in \mathbb{R}^+$, we get $w(x) := (gh)(x)$ satisfying the equation

$$w(x+2k) = -\left(\frac{(1-V)^2}{2V-1}\right)w(x+k) + \left(\frac{V^2}{2V-1}\right)w(x), \quad \forall x \in \mathbb{R}, \tag{5.21}$$

is an odd recurrent function with period k if and only if $V \in \left(\frac{1}{2}, 1\right]$.

5.11. Corollary. *Let g be a recurrent function with period k and h be an odd recurrent function, also, with period k satisfying equations* (5.14) *and* (5.15)*. Suppose conditions* (C1)*, and* (C3) *or* (C4) *are satisfied. Then, $w(x) := (gh)(x)$ is never an odd Fibonacci nor an odd Pell function with period k except possibly when $g \equiv 0$ or $h \equiv 0$ for all $x \in \mathbb{R}$.*

F

5.12. Corollary. *Let $g(x)$ be a recurrent function with period k and h be an odd recurrent function, also, with period k satisfying equations* (5.14) *and* (5.15)*. Suppose $AU = 1, BV = 2, AV = BU$, and the functions g and h satisfies condition* (C3) *or* (C4)*, then $w(x) := (gh)(x)$ is an odd Jacobsthal function with period k.*

We leave the verification of Corollary 5.11 and Corollary 5.12 to the reader.

5.13. Example. Let k be an even natural number, $t \in \mathbb{R}^+$. Consider the functions $g(x) = x - \lfloor x \rfloor$ and $h(x) = \cos(\pi x)$. We note that g and h are both f-even functions. We claim that if these two functions satisfy the following equations

$$g(x+2k) = \left(\frac{\sigma}{2\sigma+1}\right)g(x+k) + \left(\frac{\sigma+1}{2\sigma+1}\right)g(x), \quad \forall x \in \mathbb{R}, \tag{5.22}$$

and

$$h(x+2k) = -\sigma h(x+k) + (\sigma+1)h(x), \quad \forall x \in \mathbb{R}, \tag{5.23}$$

respectively, where σ is the silver ratio, then the product $w(x) := (gh)(x) = (x - \lfloor x \rfloor)\cos(\pi x)$ forms an odd Jacobsthal function with period k which satisfies the equation

$$\begin{aligned} w(x+2k) &= -\left(\frac{\sigma^2}{2\sigma+1}\right)w(x+k) + \left(\frac{(\sigma+1)^2}{2\sigma+1}\right)w(x) \\ &= -w(x+k) + 2w(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

We know that g satisfies equation (5.22) from Example (5.2). Hence, we only need to show that $h(x) = \cos(\pi x)$ satisfies equation (5.23). In fact, we have shown this already in Example (5.9). Thus, by Corollary 5.12, $w(x) = (x - \lfloor x \rfloor)\cos(\pi x)$ is indeed an odd Jacobsthal function with period k.

5.14. Example. In Example (5.10), we have seen that for any arbitrary f-odd functions g and h satisfying equations (5.19) and (5.20) with $U \in \big[0, \frac{1}{2}, 0\big)$, the product $w(x) := (gh)(x)$ is an odd recurrent function with period k. If we let $V = 3 - \sigma \in \big(\frac{1}{2}, 1\big]$, where σ is the silver ratio, then the functions g and h satisfying the following equations

$$g(x + 2k) = \left(\frac{\sigma - 2}{5 - 2\sigma}\right) g(x + k) + \left(\frac{3 - \sigma}{5 - 2\sigma}\right) g(x), \quad \forall x \in \mathbb{R},$$
$$h(x + 2k) = -(\sigma - 2)h(x + k) + (3 - \sigma)h(x), \quad \forall x \in \mathbb{R}.$$

whose product is given by $w(x) := (gh)(x)$ satisfying the equation

$$w(x + 2k) = -\left(\frac{(\sigma - 2)^2}{5 - 2\sigma}\right) w(x + k) + \left(\frac{(3 - \sigma)^2}{5 - 2\sigma}\right) w(x), \quad \forall x \in \mathbb{R}, \tag{5.24}$$

is an odd Jacobsthal function with period k by Corollary 5.12.

5.15. Theorem. *Let g and h be two functions satisfying equation* (5.1) *and* (5.2)*, respectively. Suppose $\mathcal{A} = AU, \mathcal{B} = BV, AV = -BU$, and condition* (C3) *or* (C4) *is satisfied, then $w(x) := (gh)(x)$ is a recurrent function with period k satisfying equation* (5.16).

Proof. The proof is similar to the proof of Theorem 5.1 so we omit it. □

5.16. Corollary. *Let g and h be two functions satisfying equation* (5.1) *and* (5.2)*, respectively. Suppose $\mathcal{A} = AU, \mathcal{B} = BV, AV = -BU$, and condition* (C3) *or* (C4) *is satisfied, then, $w(x) := (gh)(x)$ si never a (or an odd) Fibonacci nor a (or an odd) Pell function with period k except possibly when $g \equiv 0$ or $h \equiv 0$ for all $x \in \mathbb{R}$. Moreover, if $\mathcal{A} = 1, \mathcal{B} = 2$, and g and h are both f-odd functions, then w is a Jacobsthal function with period k.*

Proof. Let g and h be two functions satisfying equation (5.1) and (5.2), respectively. Suppose $\mathcal{A} = AU, \mathcal{B} = BV, AV = -BU$, and g and h are both f-even functions, then we have the following:

$$\begin{aligned} g(x + 2k) &= Ag(x + k) + Bg(x) = (A + B)g(x + k) \\ &= (A + B)g(x + 2k), \quad \forall x \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} h(x + 2k) &= Uh(x + k) + Vh(x) = (U + V)h(x + k) \\ &= (U + V)h(x + 2k), \quad \forall x \in \mathbb{R}. \end{aligned}$$

These imply that $A + B = U + V = 1$. Since $AV = -BU$, we get $B = V/(2V - 1)$. Hence,

$$g(x + 2k) = (V - 1)(2V - 1)^{-1} g(x + k) + V(2V - 1)^{-1} g(x), \quad \forall x \in \mathbb{R},$$
$$h(x + 2k) = (1 - V)h(x + k) + Vh(x), \quad \forall x \in \mathbb{R}.$$

By assumption, $A \in \mathbb{R}^+$, then $w(x) := (gh)(x)$ satisfying the equation

$$w(x + 2k) = -\left(\frac{(1 - V)^2}{2V - 1}\right) w(x + k) + \left(\frac{V^2}{2V - 1}\right) w(x), \quad \forall x \in \mathbb{R},$$

is a recurrent function with period k if and only if $V \in \big[0, \frac{1}{2}\big)$. Now, suppose that w is a Fibonacci function with period k, then

$$\frac{V^2}{2V - 1} = 1 > 0 > \frac{-V^2 + 2V - 1}{2V - 1} = -\frac{(1 - V)^2}{2V - 1}. \tag{5.25}$$

So, it is impossible that w is a (or an odd) Fibonacci function with period k. Furthermore, it can also be seen in (5.25) that w cannot be a (nor an odd) Pell function with period

k. Similarly, suppose w is a Jacobsthal function with period k, then $V^2/(2V-1) = 2$, which implies that

$$-\frac{(1-V)^2}{2V-1} = \frac{-V^2+2V-1}{2V-1} = -1, \quad \text{or equivalently} \quad V = 2 \pm \sqrt{2}.$$

But $V \in \left[0, \frac{1}{2}\right)$, thus w cannot be a (nor an odd) Jacobsthal function with period k. On the other hand, if g and h are both f-odd functions, then we have

$$\begin{aligned} g(x+2k) &= Ag(x+k) + Bg(x) = (A-B)g(x+k) \\ &= (B-A)g(x+2k), \quad \forall x \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} h(x+2k) &= Uh(x+k) + Vh(x) = (U-V)h(x+k) \\ &= (V-U)h(x+2k), \quad \forall x \in \mathbb{R}. \end{aligned}$$

These imply that $B - A = V - U = 1$. Since $AV = -BU$, we obtain $B = V/(2V-1)$. Hence,

$$\begin{aligned} g(x+2k) &= \left(\frac{1-V}{2V-1}\right) g(x+k) + \left(\frac{V}{2V-1}\right) g(x), \\ h(x+2k) &= (V-1)h(x+k) + Vh(x). \end{aligned}$$

Because $A \in \mathbb{R}^+$, then $w(x) := (gh)(x)$ satisfying the equation

$$w(x+2k) = -\left(\frac{(1-V)^2}{2V-1}\right) w(x+k) + \left(\frac{V^2}{2V-1}\right) w(x), \quad \forall x \in \mathbb{R}, \tag{5.26}$$

is a recurrent function with period k if and only if $V \in \left(\frac{1}{2}, 1\right]$. It can easily be verified that there is no value for $V \in \left(\frac{1}{2}, 1\right]$ such that equation (5.26) is a Fibonacci or a Pell function with period k. However, we can find a value for $V \in \left(\frac{1}{2}, 1\right]$ such that (5.26) is a Jacobsthal function. In particular, we can choose $V = 3 - \sigma \in \left(\frac{1}{2}, 1\right]$ so that we have

$$g(x+2k) = \left(\frac{\sigma-2}{5-2\sigma}\right) g(x+k) + \left(\frac{3-\sigma}{5-2\sigma}\right) g(x), \tag{5.27}$$

$$h(x+2k) = (2-\sigma)h(x+k) + (3-\sigma)h(x). \tag{5.28}$$

Equations (5.27) and (5.28) imply that

$$\begin{aligned} w(x+2k) &= -\left(\frac{(\sigma-2)^2}{5-2\sigma}\right) w(x+k) + \left(\frac{(3-\sigma)^2}{5-2\sigma}\right) w(x) \\ &= -w(x+k) + 2w(x), \quad \forall x \in \mathbb{R}, \end{aligned}$$

is a Jacobsthal function with period k. This verifies the corollary. □

5.17. Theorem. *Let g and h be any two functions satisfying*

$$\begin{aligned} g(x+2k) &= -Ag(x+k) + Bg(x), \\ h(x+2k) &= -Uh(x+k) + Vh(x), \end{aligned}$$

respectively, where $A, B, U,$ and V are non-negative real numbers. Suppose that condition (C1) *is satisfied, and g is an f-even function whereas h is an f-odd function, then $w(x) := (gh)(x)$ forms another recurrent function with period k satisfying equation* (5.3)*. Furthermore, if $\mathcal{A} = AU, \mathcal{B} = BV, AV = -BU,$ and condition* (C3) *or* (C4) *is satisfied then, $w(x) := (gh)(x)$ is also a recurrent function with period k satisfying equation* (5.3)*.*

6. Quotients of recurrent functions

Here we discuss the limit of the quotients of recurrent functions with period k and provide proofs to two conjecture of Sroysang [19].

6.1. Theorem. *Let $k \in \mathbb{N}$ and $w\colon \mathbb{R} \to \mathbb{R}$ be a recurrent function with period k. Then, the limit of the quotient $w(x+k)/w(x)$ exists.*

Proof. Let $k, n \in \mathbb{N},\ r, s \in \mathbb{R}^+$, and consider the quotient $Q(x) := w(x+k)/w(x)$, where w is a recurrent function with period k. Then, we have two possibilities: (i) $Q(x) < 0$, and (ii) $Q(x) > 0$. First, suppose that $Q(x) < 0$ then (WLOG), $u := w(x) > 0$ and $v := w(x+k) < 0$. Hence,

$$\begin{aligned}
w(x+2k) &= -rw(x+k) + sw(x)\\
&= -rv + su,\\
w(x+3k) &= rw(x+2k) - sw(x+k) = r(-rv+su) - sv\\
&= -(r^2+s)v + rsu,\\
w(x+4k) &= rw(x+3k) + sw(x+2k) = r(-(r^2+s)v + rsu) + s(-rv+su)\\
&= -(r^3+2rs)v + s(r^2+s)u\\
&\ \vdots\\
w(x+nk) &= -W_n v + sW_{n-1}u, \quad \forall n \in \mathbb{N},
\end{aligned}$$

where W_n is the n^{th} Horadam number with initial conditions $W_0 = 0$ and $W_1 = 1$. Let $x' \in \mathbb{R}$. Then, we could find $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $x' = x + nk$. So we have

$$\frac{w(x'+k)}{w(x')} = \frac{w(x+(n+1)k)}{w(x+nk)} = \frac{-W_{n+1}v + sW_n u}{-W_n v + sW_{n-1}u} = \frac{-v\frac{W_{n+1}}{W_n} + su}{-v + su\frac{W_{n-1}}{W_n}}. \tag{6.1}$$

Since $x \to \infty$ as $n \to \infty$, then letting $n \to \infty$ equation (6.1) we get

$$\lim_{x\to\infty} \frac{w(x'+k)}{w(x')} = \lim_{n\to\infty} \frac{-v\frac{W_{n+1}}{W_n} + qu}{-v + qu\frac{W_{n-1}}{W_n}} = \frac{-v\left(\lim_{n\to\infty}\frac{W_{n+1}}{W_n}\right) + qu}{-v + qu\left(\lim_{n\to\infty}\frac{W_{n-1}}{W_n}\right)}.$$

Note that $\beta = \frac{r-\sqrt{r^2+4s}}{2} \in (-1, 0)$. So $\lim_{n\to\infty} \beta^n = 0$. Thus,

$$\lim_{x\to\infty} \frac{w(x'+k)}{w(x')} = \frac{-\alpha v + su}{-v + \alpha^{-1}su} = \alpha,$$

since $\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lim_{n\to\infty} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n - \beta^n} = \alpha$. On the other hand, suppose (WLOG), $w(x)$ and $w(x+k)$ are both positive and $x \gg k$. We can express $w(x+k)/w(x)$ as $w(2n+\delta+k)/w(2n+\delta)$ since any non-negative real number x can be written as $x = \delta + 2n$ for some $\delta \in \mathbb{R}$ and $n \in \mathbb{N}$. Now, we claim that

$$\lim_{n\to\infty} \frac{w(2n+\delta+k)}{w(2n+\delta)} = \alpha.$$

We show this by expressing both sides in terms of continued fractions. For the LHS, we have

$$\begin{aligned}\frac{w(2n+k+\delta)}{w(2n+\delta)} &= \frac{pw(2n+\delta)+qw(2n+\delta-k)}{w(2n+\delta)} = r+s\frac{w(2n+\delta-k)}{w(2n+\delta)}\\ &= r+s\cfrac{1}{r+s\,\cfrac{w(2n+\delta-2k)}{w(2n+\delta-k)}}\\ &= r+s\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{w(2n+\delta-3k)}{w(2n+\delta-2k)}}}\\ &\vdots\\ &= r+s\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{\ddots+s\frac{w(2n+\delta-(2n-1)k)}{w(2n+\delta-2(n-1)k)}}}}}.\end{aligned}$$

Hence,

$$\frac{w(2n+\delta+k)}{w(2n+\delta)} = r+s\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{\ddots+s\frac{w(2n+\delta-(2n-1)k)}{w(2n+\delta-2(n-1)k)}}}}. \tag{6.2}$$

For the RHS, we have $\alpha = r + (-r+\sqrt{r^2+4s})/2 = r + s/\alpha$. Thus, we have

$$\alpha = r+s\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{\ddots}}}}.$$

Now, taking the limit of equation (6.2) as $n\to\infty$, we get

$$\lim_{x\to\infty} Q(x) = \lim_{n\to\infty}\frac{w(2n+\delta+k)}{w(2n+\delta)} = r+s\cfrac{1}{r+s\,\cfrac{1}{r+s\,\cfrac{1}{\ddots}}} = \alpha.$$

This proves the theorem. □

6.2. Corollary. *Let $k\in\mathbb{N}$. If f, (resp. p and j) is a Fibonacci (resp. Pell function and Jacobsthal) function with period k, then the limit of the quotient $f(x+1)/f(x)$, (resp. $p(x+1)/p(x)$ and $j(x+1)/j(x)$) exists.*

6.3. Corollary. *Let $k\in\mathbb{N}$ and let w be a recurrent function with period k, then $\lim_{x\to\infty} w(x+k)/w(x) = \alpha$. In particular, if f (resp. p and j) is a Fibonacci (resp. Pell, and Jacobsthal) function with period k, then $\lim_{x\to\infty} f(x+k)/f(x) = \phi$ (resp. $\lim_{x\to\infty} p(x+k)/p(x) = \sigma$ and $\lim_{x\to\infty} j(x+k)/j(x) = 2$).*

Proof. Let $n, k \in \mathbb{N}$ and suppose that the quotient $w(x+k)/w(x)$ is positive. Furthermore, assume (WLOG) that $w(x)$ and $w(x+k)$ are both positive, then

$$\frac{w(x+(n+1)k)}{w(x+nk)} = \frac{W_{n+1}w(x+k) + sW_n w(x)}{W_n w(x+k) + sW_{n-1}w(x)} = \frac{\frac{W_{n+1}}{W_n}w(x+k) + sw(x)}{w(x+k) + \frac{W_{n-1}}{W_n}sw(x)}. \tag{6.3}$$

Letting $n \to \infty$ in equation (6.3), we get

$$\lim_{n\to\infty} \frac{w(x+(n+1)k)}{w(x+nk)} = \alpha.$$

If, on the other hand, $w(x+k)/w(x)$ is negative and suppose that (WLOG) $w(x) > 0$ and $w(x+k) < 0$ then, by the proof of Theorem 6.1, we see that $\lim_{x\to\infty} w(x+(n+1)k)/w(x+nk) = \alpha$. This proves the corollary. □

6.4. Remark. In [19, Conjecture 25], Sroysang conjectured that if f is a fibonacci function with period k, then $\lim_{x\to\infty} f(x+k)/f(x) = \phi$. Indeed, this is true by Corollary 6.3.

Sroysang's second conjecture found in [19, Conjecture 26] is also true as stated in the following results.

6.5. Theorem. *Let $k \in \mathbb{N}$ and $w := \mathbb{R} \to \mathbb{R}$ be an odd recurrent function with period k. Then, the limit of the quotient $w(x+k)/w(x)$ exists.*

6.6. Corollary. *Let $k \in \mathbb{N}$. If f, (resp. p and j) is an odd Fibonacci (resp. odd Pell function and odd Jacobsthal) function with period k, then the limit of the quotient $f(x+1)/f(x)$, (resp. $p(x+1)/p(x)$ and $j(x+1)/j(x)$) exists.*

6.7. Corollary. *Let $k \in \mathbb{N}$ and let w be an odd recurrent function with period k, then $\lim_{x\to\infty} w(x+k)/w(x) = -\alpha$. In particular, if f (resp. p and j) is an odd Fibonacci (resp. odd Pell, and odd Jacobsthal) function with period k, then $\lim_{x\to\infty} f(x+k)/f(x) = -\phi$ (resp. $\lim_{x\to\infty} p(x+k)/p(x) = -\sigma$ and $\lim_{x\to\infty} j(x+k)/j(x) = -2$).*

We omit the proof of these results since they can be proven in a similar fashion as in Theorem 6.1 and Corollary 6.3.

7. Summary

We were able to extend successfully the notion of Fibonacci functions [9] and Fibonacci functions with period k [19] by characterizing the concept of second-order linear recurrent functions with period k. We were also able to confirm the conjectures of Sroysang in [19] by proving a more general result about the asymptotic growth rate of Fibonacci functions with period k. In our next investigation [18], we will revisit Sroysang's conjecture and provide another proof using the results presented in [20] by Tollu et al. together with the concept of Cauchy sequences.

Acknowledgement

The author wishes to thank the referee for carefully handling and examining his manuscript.

References

[1] Amdeberhan, T. Chen, X. Moll, V. C. and Sagan, B. E. *Generalized Fibonacci polynomials and Fibonomial coefficients.* Retrieved from `arXiv:1306.6511v2 [math.CO] 26 Jul 2013`.

[2] Bacani, J. B., Rabago, J. F. T. *On two nonlinear difference equations*, submitted.

[3] Chandra, P. and Weisstein, E. W. *Fibonacci Number.* From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/FibonacciNumber.html`

[4] Cook, C. K. Hillman, R. A. and Shannon, A. G. *Some aspects of Fibonacci polynomial congruences*, Ann. Math. Inform. **41**, 211–217, 2013.

[5] Dunlap, R. A. *The golden ratio and Fibonacci numbers* (World Scientific, 1998).

[6] Hakami, A. *An application of Fibonacci sequence on continued fractions*, Int. Math. Forum **10** (2), 69–74, 2015.

[7] Han, J.S. Kim, H. S. and Neggers, J. *The Fibonacci norm of a positive integer n-observations and conjectures*, Int. J. Number Theory **6** (2), 371–385, 2010.

[8] Han, J. S. Kim, H. S. and Neggers, J. *Fibonacci sequences in groupoids*, Adv. Differ. Equ. **2012**, Article 19, 7 pages, 2012.

[9] Han, J. S. Kim, H. S. and Neggers, J. *On Fibonacci functions with Fibonacci numbers*, Adv. Differ. Equ. **2012**, Article 126, 7 pages, 2012.

[10] Horadam, A. F. *Basic properties of certain generalized sequence of numbers*, Fib. Quart. **3**, 161–176, 1965.

[11] Koshy, T. *Fibonacci and Lucas numbers with applications* (Pure and Applied Mathematics, Wiley-Interscience, New York, 2001).

[12] Larcombe, P. J. Bagdasar, O. D. and Fennessey, E. J. *Horadam sequences: a survey* Bulletin of the I.C.A. **67**, 49–72, 2013.

[13] Larcombe, P. J. *Horadam Sequences: a survey update and extension*, submitted.

[14] Ma, Y. and Zhang, T. *On Generalized Fibonacci Polynomials and Bernoulli Numbers*, J. Integer Seq. **8**, Article 05.5.3, 2005.

[15] Rabago, J. F. T. *On solving the second-order linear recurrence sequence*, Int. J. Math. Sci. Comp. **2** (1), 1–2, 2012.

[16] Rabago, J. F. T. *On second-order linear recurrent homogeneous differential equations with period k*, Hacet. J. Math. Stat. **43** (6), 923–933, 2014.

[17] Rabago, J. F. T. *On k-Fibonacci numbers with applications to continued fractions*, Journal of Physics: Conference Series **693** 012005, 2016.

[18] Rabago, J. F. T. *On the closed-form solution of a nonlinear difference equation and another proof to Sroysang's conjecture*, in preparation.

[19] Sroysang, B. *On Fibonacci functions with period k*, Discrete Dyn. Nat. Soc. **2013**, Article ID 418123, 4 pages, 2013.

[20] Tollu, D.T. Yazlik, Y. and Taskara, N. *On the solutions of two special types of Riccati difference equation via Fibonacci numbers*, Adv. Differ. Equ. **2013**, Article 174, 2013.

[21] Vajda, S.A. *Fibonacci and Lucas numbers, and the golden section* (Ellis-Horwood, 1989).