

Strong summation process in locally integrable function spaces

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Abstract

In this paper, using the concept of strong summation process, we give a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_{p,q}(loc)$ into itself. We also study modulus of continuity for $L_{p,q}(loc)$ approximation and give the rate of convergence of these operators.

Keywords: \mathcal{A} –summability, positive linear operators, locally integrable functions, Korovkin type theorem, modulus of continuity, rate of convergence.

2000 AMS Classification: 41A25, 41A36.

Received : 13.01.2015 *Accepted :* 14.05.2015 *Doi :* 10.15672/HJMS.20164513102

1. Introduction

The classical theorem of Korovkin [7] on approximation of continuous functions on a compact interval gives conditions in order to decide whether a sequence of positive linear operators converges to the identity operator. Some results concerning the Korovkin type approximation theorem in the space $L_p[a, b]$ of the Lebesgue integrable functions on a compact interval may be found in [4]. If the sequence of positive linear operators does not converge then it might be beneficial to use matrix summability methods.

Approximation theory has important applications in the theory of polynomial approximation, in functional analysis, numerical solutions of differential and integral equations [1], [8].

The purpose this paper is to study a Korovkin type approximation theorem of a function f by means of sequence of positive linear operators from the space of locally integrable functions into itself with the use of a matrix summability method which includes both convergence and almost convergence. We also obtain rate of convergence in $L_{p,q}(loc)$ approximation with positive linear operators by means of modulus of continuity.

Now we recall some information of locally integrable functions given in [6].

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Let $q(x) = 1 + x^2$; $-\infty < x < \infty$. For $h > 0$, by $L_{p,q}(loc)$ we will denote the space of measurable functions f satisfying the inequality,

$$(1) \quad \left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right)^{1/p} \leq M_f q(x) \quad , \quad -\infty < x < \infty$$

where $p \geq 1$ and M_f is a positive constant which depends on the function f .

It is known [6] that $L_{p,q}(loc)$ is a linear normed space with norm,

$$(2) \quad \|f\|_{p,q} = \sup_{-\infty < x < \infty} \frac{\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right)^{1/p}}{q(x)}$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. To simplify the notation, we need the following. For any real numbers a and b put

$$\begin{aligned} \|f; L_p(a, b)\|_{p,q} &:= \left(\frac{1}{b-a} \int_a^b |f(t)|^p dt \right)^{1/p}, \\ \|f; L_{p,q}(a, b)\|_{p,q} &= \sup_{a < x < b} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)}, \\ \|f; L_{p,q}(|x| \geq a)\|_{p,q} &= \sup_{|x| \geq a} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)}. \end{aligned}$$

With this notation the norm in $L_{p,q}(loc)$ may be written in the form

$$\|f\|_{p,q} = \sup_{x \in \mathbb{R}} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)}.$$

It is known [6] that $L_{p,q}^k(loc)$ is the subspace of all functions $f \in L_{p,q}(loc)$ for which there exists a constant k_f such that

$$\lim_{|x| \rightarrow \infty} \frac{\|f - k_f q; L_p(x-h, x+h)\|_{p,q}}{q(x)} = 0.$$

As usual, if T is a positive linear operator from $L_{p,q}(loc)$ into $L_{p,q}(loc)$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_{p,q}}{\|f\|_{p,q}}$.

2. Strong \mathcal{A} -summation process in $L_{p,q}(loc)$

The main aim of the present work is to study a Korovkin type approximation theorem for a sequence of positive linear operators acting on the weighted space $L_{p,q}(loc)$ by using matrix summability method which includes both convergence and almost convergence. We also give an example of positive linear operators which verifies our Theorem 2.5. but does not verify the classical one (see Theorem 2.1 below).

Let $\mathcal{A} := \{A^{(n)}\} = \{a_{k,j}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. If

$$(3) \quad \lim_k \sum_j a_{k,j}^n \|T_j f - f\|_{p,q} = 0, \quad \text{uniformly in } n,$$

then we say that $\{T_j f\}$ is strongly \mathcal{A} -summable to f for every f in $L_{p,q}(loc)$ where it is assumed that the series converges for each k, n and f . Some results concerning summation processes on some other spaces may be found in [2], [9] and [10].

We recall the following result of [6] that we need in the sequel.

2.1. Theorem. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself and satisfy the conditions

i) The sequence (T_j) is uniformly bounded, that is, $\|T_j\| \leq C < \infty$, where C is a constant independent of j ,

ii) $\lim_j \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$ where $f_i(y) = y^i$, $i = 0, 1, 2$. Then

$$\lim_j \|T_j f - f\|_{p,q} = 0$$

for each function $f \in L_{p,q}^k(loc)$, (see [6]).

The next result shows that Korovkin type theorem does not hold in the whole space $L_{p,q}(loc)$.

2.2. Theorem. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\limsup_k \sum_n a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Then there exists a function f^* in $L_{p,q}(loc)$ for which

$$(4) \quad \limsup_k \sum_n a_{kj}^{(n)} \|T_j f^* - f^*\|_{p,q} \geq 2^{1-\frac{1}{p}}.$$

Proof. We consider the sequence of operators T_j given in [6] :

$$T_j(f; x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h) & , x \in [2(j-1)h, (2j+1)h) \\ f(x) & , otherwise. \end{cases}$$

It is shown in [6] that

$$\|T_j f\|_{p,q} \leq 4 \|f\|_{p,q}.$$

Assume now that $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ is a sequence of infinite matrices defined by

$$a_{kj}^{(n)} = \begin{cases} \frac{1}{k+1} & , n \leq j \leq n+k \\ 0 & , otherwise. \end{cases}$$

Consider the following function f^* given in [6] :

$$f^*(x) = \begin{cases} x^2 & , if x \in \bigcup_{k=1}^{\infty} [(2k-1)h, 2kh) \\ -x^2 & , if x \in \bigcup_{k=1}^{\infty} [2kh, (2k+1)h) \\ 0 & , if x < 0. \end{cases}$$

Then $f^* \in L_{p,q}(loc)$ and it is shown in [6] that

$$\|T_j f^* - f^*\|_{p,q} \geq 2^{1-\frac{1}{p}} \frac{(2j-1)^2 h^2}{1+4j^2 h^2}.$$

Hence

$$\frac{1}{k+1} \sum_{j=n}^{k+n} \|T_j f^* - f^*\|_{p,q} \geq \frac{1}{k+1} \sum_{j=n}^{k+n} 2^{1-\frac{1}{p}} \frac{(2j-1)^2 h^2}{1+4j^2 h^2}.$$

On applying the operator $\limsup_{k \ n}$ on both sides one can see that

$$\limsup_{k \ n} \frac{1}{k+1} \sum_{j=n}^{k+n} \|T_j f^* - f^*\|_{p,q} \geq 2^{1-1/p}.$$

Therefore the theorem is proved.

Now we show that the above mentioned problem has a positive solution in the subspace $L_{p,q}^k(loc)$. First we give the following

2.3. Lemma. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\limsup_{k \ n} \sum_j a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Assume that

$$(5) \quad H' = \sup_{n,k} \sum_j a_{kj}^{(n)} < \infty.$$

Then, for any continuous and bounded function f on the real axis,

$$\limsup_{k \ n} \sum_j a_{kj}^{(n)} \|T_j(f; x) - f(x); L_{p,q}(a, b)\| = 0$$

holds, where a and b are any real numbers.

Proof. Since f is uniformly continuous function on any closed interval, given $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if $|t - x| < \delta$ implies that

$$(6) \quad |f(t) - f(x)| < \varepsilon, \text{ for all } x \in [a, b], t \in \mathbb{R}.$$

Also, setting $M = \sup_{x \in \mathbb{R}} |f(x)|$, we can write if $|t - x| \geq \delta$ that

$$(7) \quad |f(t) - f(x)| < 2M, \text{ for all } x \in [a, b], t \in \mathbb{R}.$$

Combining (6) and (7) we have

$$(8) \quad |f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2} (t - x)^2,$$

where $-\infty < t < \infty; x \in [a, b]$. Let $c := \max\{|a|, |b|\}$ and using the positivity and linearity of operators T_j we obtain from (8) that

$$\begin{aligned} & \sum_j a_{kj}^{(n)} \|T_j(f(t); x) - f(x); L_{p,q}(a, b)\| \\ & \leq \sum_j a_{kj}^{(n)} \|T_j(|f(t) - f(x)|; x)\|_{p,q} + |f(x)| \sum_j a_{kj}^{(n)} \|T_j(1; x) - 1\|_{p,q} \\ & < \sum_j a_{kj}^{(n)} \left\| T_j\left(\varepsilon + \frac{2M}{\delta^2} (t - x)^2; x\right) \right\|_{p,q} + M \sum_j a_{kj}^{(n)} \|T_j(1; x) - 1\|_{p,q} \\ & = \varepsilon \sum_j a_{kj}^{(n)} + \frac{2M}{\delta^2} \sum_j a_{kj}^{(n)} \|T_j(t^2; x) - x^2\|_{p,q} + \frac{4Mc}{\delta^2} \sum_j a_{kj}^{(n)} \|T_j(t; x) - x\|_{p,q} \\ & + \left(\frac{2Mc^2}{\delta^2} + \varepsilon + M \right) \sum_j a_{kj}^{(n)} \|T_j(1; x) - 1\|_{p,q}. \end{aligned}$$

Hence the proof is completed.

2.4. Theorem. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that

$$(9) \quad H := \sup_{n,k} \sum_j a_{kj}^{(n)} \|T_j\| < \infty$$

and

$$(10) \quad H' := \sup_{n,k} \sum_j a_{kj}^{(n)} < \infty.$$

Then $\{T_j\}$ is an A -strong summation process in $L_{p,q}^k(loc)$, i.e., for any function $f \in L_{p,q}^k(loc)$ we have

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f; x) - f(x)\|_{p,q} = 0$$

if and only if

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$.

Proof. We follow [6] up to a certain stage. If $f \in L_{p,q}^k(loc)$ then $f - k_{f,q} \in L_{p,q}^0(loc)$. So it is sufficient to prove the theorem for the function $f \in L_{p,q}^0(loc)$. For $\varepsilon > 0$, there exists a point x_0 such that the inequality

$$(11) \quad \left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right)^{1/p} < \varepsilon q(x)$$

holds for all x , $|x| \geq x_0$. By the well known Lusin Theorem, there exists a continuous function φ on the finite interval $[-x_0 - h, x_0 + h]$ such that the inequality

$$(12) \quad \|f - \varphi; L_p(-x_0, x_0)\| < \varepsilon$$

is fulfilled. Setting

$$(13) \quad \delta < \min \left\{ \frac{2h\varepsilon^p}{M^p(x_0)}, h \right\},$$

where $M(x_0) = \max \left\{ \max_{|x| \leq x_0+h} |\varphi(x)|, 1 \right\}$, we can define a continuous function g by

$$g(x) = \begin{cases} \varphi(x) & , \text{ if } |x| \leq x_0 + h \\ 0 & , \text{ if } |x| \geq x_0 + h + \delta \\ \text{linear} & , \text{ otherwise.} \end{cases}$$

Then by (11), (12), (13) and the Minkowski inequality, we obtain

$$(14) \quad \|f - g\|_{p,q} < \varepsilon$$

for any $\varepsilon > 0$ (see [6]).

Now we can find a point $x_1 > x_0$ such that

$$(15) \quad q(x_1) > \frac{M(x_0)}{\varepsilon} \text{ and } g(x) = 0 \text{ for } |x| > x_1,$$

where $M(x_0)$ is defined above. Then by (12), (13), (14) and the definition of g and Lemma 2.1. we get

$$\begin{aligned}
\sum_j a_{kj}^{(n)} \|T_j(f; x) - f(x)\|_{p,q} &\leq \sum_j a_{kj}^{(n)} \|T_j(f - g)\|_{p,q} + \sum_j a_{kj}^{(n)} \|T_j g - g\|_{p,q} \\
&\quad + \sum_j a_{kj}^{(n)} \|f - g\|_{p,q} \\
&\leq \varepsilon \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\
&\quad + \sum_j a_{kj}^{(n)} \|T_j g - g; L_{p,q}(-x_1, x_1)\| \\
&\quad + \sum_j a_{kj}^{(n)} \|T_j g - g; L_{p,q}(|x| \geq x_1)\| \\
&\leq \varepsilon \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} + 1 \right) \\
(16) \quad &\quad + \sum_j a_{kj}^{(n)} \|T_j g; L_{p,q}(|x| \geq x_1)\|.
\end{aligned}$$

Since $|g(x)| \leq M(x_0)$ for all $x \in \mathbb{R}$, we can write

$$\begin{aligned}
\sum_j a_{kj}^{(n)} \|T_j g; L_{p,q}(|x| \geq x_1)\|_{p,q} &\leq M(x_0) \sum_j a_{kj}^{(n)} \|T_j 1; L_{p,q}(|x| \geq x_1)\| \\
&\leq M(x_0) \sum_j a_{kj}^{(n)} \|T_j 1 - 1; L_{p,q}(|x| \geq x_1)\| \\
&\quad + M(x_0) \sum_j a_{kj}^{(n)} \|1; L_{p,q}(|x| \geq x_1)\| \\
&\leq M(x_0) \sum_j a_{kj}^{(n)} \|T_j 1 - 1\|_{p,q} \\
&\quad + \frac{M(x_0)}{q(x_1)} \sum_j a_{kj}^{(n)}.
\end{aligned}$$

Considering hypothesis and (15) we get by (16) that

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j f - f\|_{p,q} = 0.$$

In the whole space $L_{p,q}(loc)$ we have the following.

2.5. Theorem. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries for which (9) and (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Then for any functions $f \in L_{p,q}(loc)$ we have

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x \in \mathbb{R}} \frac{\|T_j f - f; L_p(x-h, x+h)\|_{p,q}}{q^*(x)} \right) = 0$$

where q^* is a weight function such that $\lim_{|x| \rightarrow \infty} \frac{1+x^2}{q^*(x)} = 0$.

Proof. By hypothesis, given $\varepsilon > 0$, there exists x_0 such that for all x with $|x| \geq x_0$ we have

$$(17) \quad \frac{1+x^2}{q^*(x)} < \varepsilon.$$

Let $f \in L_{p,q}(\text{loc})$. Then, for all n, k we get

$$\begin{aligned} \gamma_n &:= \sum_j a_{kj}^{(n)} \left\| B_k^{(n)} f - f; L_p(|x| > x_0) \right\| \\ &\leq \sum_j a_{kj}^{(n)} \|T_j f\|_{p,q} + \sum_j a_{kj}^{(n)} \|f\|_{p,q} \\ &\leq \|f\|_{p,q} \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) < N, \text{ say.} \end{aligned}$$

Hence we have $\sup_n \gamma_n < \infty$ is bounded. By Lusin's theorem we can find a continuous function φ on $[-x_0 - h, x_0 + h]$ such that

$$(18) \quad \|f - \varphi; L_p(-x_0 - h, x_0 + h)\| < \varepsilon.$$

Now we consider the following function G given in [6]

$$G(x) := \begin{cases} \varphi(-x_0 - h) & , x \leq -x_0 - h \\ \varphi(x_0) & , |x| < x_0 + h \\ \varphi(x_0 + h) & , x \geq x_0 + h. \end{cases}$$

We see that G is continuous and bounded on the whole real axis. Now let $f \in L_{p,q}(\text{loc})$ and we get for all n, k that

$$\begin{aligned} \beta_n &:= \sum_j a_{kj}^{(n)} \|T_j f - f; L_{p,q}(-x_0, x_0)\| \\ &\leq \sum_j a_{kj}^{(n)} \|T_j(f - G); L_{p,q}(-x_0, x_0)\| + \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| \\ &\quad + \sum_j a_{kj}^{(n)} \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \\ &\leq \sum_j a_{kj}^{(n)} \|T_j\|_{p,q} \|(f - G); L_{p,q}(-x_0 - h, x_0 + h)\| \\ &\quad + \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| \\ &\quad + \sum_j a_{kj}^{(n)} \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \\ &\leq \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\ &\quad + \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\|. \end{aligned}$$

Hence by the hypothesis and Lemma 2.1. we have

$$(19) \quad \limsup_k \limsup_n \beta_n = 0.$$

On the other hand, a simple calculation shows that

$$\begin{aligned}
 u_n &:= \sum_j a_{kj}^{(n)} \|T_j f - f\|_{p,q} \\
 &< \sum_j a_{kj}^{(n)} \sup_{|x| < x_0} \frac{\left(\frac{1}{2h} \int_{x-h}^{x+h} \left| \sum_j a_{kj}^{(n)} T_j f - f \right|^p dt\right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
 &+ \sum_j a_{kj}^{(n)} \sup_{|x| \geq x_0} \frac{\left(\frac{1}{2h} \int_{x-h}^{x+h} \left| \sum_j a_{kj}^{(n)} T_j f - f \right|^p dt\right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
 &= \beta_n \sup_{|x| < x_0} \frac{q(x)}{q^*(x)} + \gamma_n \sup_{|x| \geq x_0} \frac{q(x)}{q^*(x)} \\
 (20) \quad &< \beta_n q(x_0) + \varepsilon \gamma_n.
 \end{aligned}$$

It follows from (17), (18), (19), (20) and Lemma 2.1. that

$$\begin{aligned}
 u_n &< q(x_0) \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\
 &+ q(x_0) \left\| \sum_j a_{kj}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\| + \varepsilon N \\
 &= K\varepsilon + q(x_0) \left\| \sum_j a_{kj}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\|
 \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. By Lemma 2.1. we get

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x \in \mathbb{R}} \frac{\|T_j f - f; L_p(x - h, x + h)\|_{p,q}}{q^*(x)} \right) = 0.$$

2.6. Remark. We now present an example of a sequence of positive linear operators which satisfies Theorem 2.5 but does not satisfy Theorem 2.1. Assume now that $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ is a sequence of infinite matrices defined by

$$a_{kj}^{(n)} = \begin{cases} \frac{1}{k+1} & , n \leq j \leq n+k \\ 0 & , \text{otherwise.} \end{cases}$$

In this case \mathcal{A} -summability method reduces to almost convergence, ([8]).

Let $T_j : L_{p,q}(loc) \rightarrow L_{p,q}(loc)$ be given by

$$T_j(f; x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h) & , x \in [2(j-1)h, (2j+1)h] \\ f(x) & , \text{otherwise} \end{cases}$$

The sequence $\{T_j\}$ satisfies Theorem 2.1. (see [6]). It is shown that for all $j \in \mathbb{N}$, $\|T_j f\|_{p,q} \leq 4 \|f\|_{p,q}$. Hence $\{T_j\}$ is an uniformly bounded sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Also

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Now define $\{P_j\}$ by

$$P_j(f; x) = (1 + u_j) T_j(f; x)$$

where

$$u_j = \begin{cases} 1 & , j = 2^n, n \in \mathbb{N} \\ 0 & d.d. \end{cases}$$

It is easy to see that $\{u_j\}$ almost convergent to zero. Therefore the sequence of positive linear operators $\{P_j\}$ satisfies Theorem 2.5. but does not satisfy Theorem 2.1.

3. Rates of Convergence For Strong \mathcal{A} -Summation Process in $L_{p,q}(loc)$

In this section, using the modulus of continuity, we study rates of convergence in $L_{p,q}(loc)$.

We now turn to introducing some notation and basic definitions to obtain the rate convergence of the operators given in Theorem 2.5.

Also, we consider the following modulus of continuity:

$$w(f, \delta) = \sup_{|x-y| \leq \delta} |f(y) - f(x)|,$$

where δ is a positive constant, $f \in L_{p,q}(loc)$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(loc)$,

$$w(f, \delta) \leq (1 + [c]) w(f, \delta),$$

where $[c]$ is defined to be the greatest integer less than or equal to c , [3].

To obtain our main results we first need the following lemma.

3.1. Lemma. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Then for each $j \in \mathbb{N}$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have

$$\begin{aligned} \sum_j a_{kj}^{(n)} \|T_j f - f; L_{p,q}(a, b)\| &\leq w(f; \delta) \sum_j a_{kj}^{(n)} \|T_j f_0 - f_0\|_{p,q} \\ &+ 2w(f; \delta) \sum_j a_{kj}^{(n)} + C_1 \sum_j a_{kj}^{(n)} \|T_j f_0 - f_0\|_{p,q} \end{aligned}$$

where $f_0(t) = 1$, $\varphi_x(t) := (t-x)^2$, $C_1 = \sup_{a < x < b} |f(x)|$ and $\delta := \alpha_j = \sqrt{\|T_j \varphi_x\|_{p,q}}$.

Proof. Let f be any continuous and bounded function on the real axis, and let $x \in [a, b]$ be fixed. Using linearity and monotonicity of T_j and for any $\delta > 0$, by modulus of continuity, we get

$$\begin{aligned} |T_j(f; x) - f(x)| &\leq T_j \left(w \left(f, \frac{|t-x|}{\delta} \delta \right), x \right) \\ &+ |f(x)| |T_j(f_0; x) - f_0(x)| \\ &\leq w_q(f, \delta) |T_j(f_0; x) - f_0(x)| + w_q(f, \delta) \\ &+ \frac{w_q(f, \delta)}{\delta^2} |T_j \varphi_x| + |f(x)| |T_j(f_0; x) - f_0(x)|. \end{aligned}$$

Let $\delta := \alpha_j = \sqrt{\|T_j \varphi_x\|_{p,q}}$. Then we have

$$\begin{aligned} \|T_j f - f; L_{p,q}(a, b)\| &\leq w_q(f, \delta) \|T_j(f_0; x) - f_0(x)\|_{p,q} + \\ &\quad + w_q(f, \delta) \\ &\quad + \frac{w_q(f, \delta)}{\left(\sqrt{\|T_j \varphi_x\|_{p,q}}\right)^2} \|T_j \varphi_x\|_{p,q} \\ &\quad + \|T_j(f_0; x) - f_0(x)\|_{p,q} \sup_{a < x < b} |f(x)| \end{aligned}$$

Now let $C_1 = \sup_{a < x < b} |f(x)|$. Then we get

$$\begin{aligned} \sum_j a_{k,j}^{(n)} \|T_j f - f; L_{p,q}(a, b)\| &\leq \sum_j a_{k,j}^{(n)} w_q(f, \delta) \|T_j(f_0; x) - f_0(x)\|_{p,q} \\ &\quad + 2 \sum_j a_{k,j}^{(n)} w_q(f, \delta) \\ &\quad + C_1 \sum_j a_{k,j}^{(n)} \|T_j(f_0; x) - f_0(x)\|_{p,q}. \end{aligned}$$

3.2. Theorem. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries for which (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that for each continuous and bounded function f on the real line, the following conditions hold:

$$\begin{aligned} (i) \limsup_k \sum_n a_{k,j}^{(n)} \|T_j(f_0; x) - f_0(x)\|_{p,q} &= 0 \\ (ii) \limsup_k \sum_n a_{k,j}^{(n)} w_q(f, \delta) &= 0 \\ (iii) \limsup_k \sum_n a_{k,j}^{(n)} w_q(f, \delta) \|T_j(f_0; x) - f_0(x)\|_{p,q} &= 0 \end{aligned}$$

where $\delta = \alpha_j = \sqrt{\|T_j \varphi_x\|_{p,q}}$. Then we have

$$\limsup_k \sum_n a_{k,j}^{(n)} \|T_j f - f; L_{p,q}(a, b)\| = 0.$$

Proof. Using Lemma 3.1. and considering (i), (ii), (iii) and (10) we have

$$\limsup_k \sum_n a_{k,j}^{(n)} \|T_j f - f; L_{p,q}(a, b)\| = 0$$

for all continuous and bounded functions on the real axis.

3.3. Theorem. Let $\mathcal{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries for which (9) and (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that

$$\limsup_k \sum_n a_{k,j}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. If

$$(i) \limsup_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_0; x) - f_0(x)\|_{p,q} = 0$$

$$(ii) \limsup_k \sup_n \sum_j a_{kj}^{(n)} w_q(G, \delta) = 0$$

$$(iii) \limsup_k \sup_n \sum_j a_{kj}^{(n)} w_q(G, \delta) \|T_j(f_0; x) - f_0(x)\|_{p,q} = 0$$

where G is given as in the proof of Theorem 2.5. Then we have

$$\limsup_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x \in \mathbb{R}} \frac{\|T_j f - f; L_p(x-h, x+h)\|}{q^*(x)} \right) = 0$$

where q^* is a weight function such that $\lim_{|x| \rightarrow \infty} \frac{1+x^2}{q^*(x)} = 0$.

Proof. It is known from Theorem 2.5. that

$$\begin{aligned} u_n &< q(x_0) \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\ &+ q(x_0) \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| + \varepsilon N \\ &= K\varepsilon + q(x_0) \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. Then by Lemma 3.1. and Theorem 2.5. we get

$$\begin{aligned} u_k^{(n)} &\leq K\varepsilon + q(x_0) \sum_j a_{kj}^{(n)} w_q(G; \delta) \|T_j(f_0; x) - f_0(x)\|_{p,q} \\ &+ 2q(x_0) \sum_j a_{kj}^{(n)} w_q(G; \delta) \\ &+ q(x_0) C'_1 \sum_j a_{kj}^{(n)} \|T_j(f_0; x) - f_0(x)\|_{p,q} \end{aligned}$$

where $C'_1 := \sup_{-x_0 < x < x_0} |G(x)|$ and the proof is completed.

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