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Approximations in a hyperlattice by using set-valued homomorphisms

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Abstract

In this paper, the concepts of set-valued homomorphism and strong set-valued homomorphism of a hyperlattice are introduced. The notions of generalized lower and upper approximation operators constructed by means of a set-valued mapping are provided. We also propose the notions of generalized lower and upper approximations with respect to a hyperideal of a hyperlattice which is an extended notion of rough hyperideal in a hyperlattice and discuss some significiant properties of them.

Keywords: Hyperlattice; Hypercongruence; Approximation space; Rough set; Lower and upper approximations; Set-valued mapping.

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1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory which were initiated by Marty [15]. In a classical algebraic structure, the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements is a set. Hundreds of papers and several books have been written on hyperstructure theory, see for instance [5,6]. Hyperlattices were first studied by Konstantinidou and Mittas [18]. Since the concept of hyperlattice is a generalization of the concept of lattice, hyperlattice theory was studied by Konstantinidou [19-21], Ashrafi [3], Rahnamai-Barghi [29-30] Guo and Xin [14], Han and Zhao [12], Zhao and Han [37].

Rough set theory was proposed by Pawlak [26]; see also [27-28]. The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of

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ordinary sets called the lower and upper approximations. A key concept in Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. However, the requirement of an equivalence relation in Pawlak rough set model seems to be a very restrictive condition that may limit the applications of rough set models. Thus, one of the main directions of research in rough set theory is naturally the generalization of Pawlak rough set approximations. For instance, the notion of approximations are extended to general binary relations, coverings, completely distributive lattices, fuzzy lattices and Boolean algebras. This research soon led to a natural question concerning the possible connection between rough sets and algebraic systems.

In [22], Kuroki introduced a rough ideal in a semigroup. Kuroki and Wang [23] presented some properties of the lower and upper approximations with respect to normal subgroups. Davvaz [8] investigated the relationship between rough sets and ring theory by considering a ring as a universal set and introducing the concepts of rough subrings and rough ideals with respect to an ideal of a ring. Kazancı and Davvaz [16] introduced the notions of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a ring and presented some properties of such ideals. Rough semigroups, rough modules, rough lattices, rough MV-algebras, rough hemirings and rough γ semihyperrings have been investigated by many authors(see also [1,2,4,7,8,11,17,19,24,25,31,34]). Davvaz and Mahdavipour [10] presented a framework for generalizing the standard notion of rough set approximation space. They proposed new definitions of the lower and upper approximations which are basic concepts of rough set theory. In [9], Davvaz introduced the concept of set-valued homomorphism for groups which is a generalization of an ordinary homomorphism. The concepts of set-valued homomorphism and strong set-valued homomorphism of a ring were introduced by Yamak et al.[35] and Hooshmandasl et al. [13] .

The initiation and majority of studies on rough sets for algebraic structures have been concentrated on a congruence relation. The congruence relation, however, seems to restrict the application of the generalized rough set model for algebraic sets. This may be by reason of incomplete information about the objects under consideration. Sometimes due to imprecise human knowledge about the elements of the universe set, an equivalence relation among these elements is difficult to find. To overcome this problem, we require set-valued maps instead of equivalence relations in generalized rough sets. This technique is useful where it is not easy to find a equivalence relation among the objects of the universe set. This paper is structured as follows. After an introduction, in Section 2, we present some basic definitions and results about approximation operators. In Section 3, we restrict the universe of the approximation space to a hyperlattice and we introduce the axiomatic form of this concept. In Section 4, the concepts of generalized lower and upper approximation operators constructed by means of a set-valued homomorphism with respect to a hyperideal of a hyperlattice is presented and we examine some properties of these operators in a hyperlattice.

2. Preliminaries

In this section, we recall some notions and results (see [5,6,14,15,20]) which will be used throughout this article. Let L be a non-empty set and $P^*(L)$ be the set of all nonempty subsets of L. A hyperoperation on L is a map $\circ : L \times L \to P^*(L)$ which associates a nonempty subset $a \circ b$ with any pair (a, b) of elements of $L \times L$. The couple $(L, \circ)$ is called a hypergroupoid. If A and B are nonempty subsets of L, then for $a, b, x \in L$, we denote

(1) $x \circ A = \{x\} \circ A = \bigcup_{a \in A} x \circ a$, $A \circ x = A \circ \{x\} = \bigcup_{a \in A} a \circ x$. (2) $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

2.1. Definition. [14] Let L be a non-empty set endowed with two hyperoperations $\otimes$ and $\oplus$. The triple $(L, \otimes, \oplus)$ is called a hyperlattice if the following conditions hold for all $a, b, c \in L$:

(1) (idempotent laws) $a \in a \otimes a$, $a \in a \oplus a$,
(2) (commutative laws) $a \otimes b = b \otimes a$, $a \oplus b = b \oplus a$,
(3) (associative laws)$(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
(4) (absorption laws) $a \in a \otimes (a \oplus b)$, $a \in a \oplus (a \otimes b)$.

2.2. Definition. [14] Let $L = (L, \otimes, \oplus)$ be a hyperlattice and $S \in P^*(L)$. Then S is called a subhyperlattice of L if $a \otimes b$ and $a \oplus b \in P^*(S)$ for all $a, b \in S$. That is to say, S is subhyperlattice of L if and only if S is closed under the two hyperoperation $\otimes$ and $\oplus$ on L.

2.3. Example. Let $L = \{a, b, c, d\}$ be a set. Define the hyperoperations $''\otimes''$ and $''\oplus''$ on L with the following Cayley table :

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	a	{a,b}
c	a	a	c	c
d	a	{a,b}	c	d

$\oplus$	a	b	c	d
a	a	b	{c,d}	d
b	b	b	d	d
c	{c,d}	d	{c,d}	d
d	d	d	d	d

It is easy to check that $(L, \otimes, \oplus)$ is a hyperlattice. Consider the subsets $S_1 = \{a, d\}$, $S_2 = \{c, d\}$. Then S_1 and S_2 are subhyperlattices of L. If we get $S_3 = \{a, c\}$, then S_3 is not a subhyperlattice of L. Because it isn't closed under the hyperoperation $\oplus$ on L.

2.4. Definition. [14] Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A map $\varphi : L_1 \to L_2$ is called a

(i) weak hyperlattice homomorphism if $\varphi(a \otimes_1 b) \subseteq \varphi(a) \otimes_2 \varphi(b)$ and $\varphi(a \oplus_1 b) \subseteq \varphi(a) \oplus_2 \varphi(b)$ for all $a, b \in L_1$,
(ii) strong hyperlattice homomorphism if $\varphi(a \otimes_1 b) = \varphi(a) \otimes_2 \varphi(b)$ and $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$ for all $a, b \in L_1$.

If such a homomorphism φ is surjective, injective or bijective, then φ is called an epimorphism, a monomorphism or an isomorphism from the hyperlattice $(L_1, \otimes_1, \oplus_1)$ to the hyperlattice $(L_2, \otimes_2, \oplus_2)$, respectively.

2.5. Definition. Let $L = (L, \otimes, \oplus)$ be a hyperlatice and $A \in P^*(L)$. Then A is called a hyperideal of L if and only if $a \otimes x \in P^*(A)$, $a \oplus x \in P^*(A)$ for all $a \in A, x \in L$.

Let $(L, \otimes, \oplus)$ be a hyperlattice. An equivalence relation θ is a reflexive, symmetric, and transitive binary relation on L. If θ is an equivalence relation on L, then the equivalence class of $a \in L$ is the set $\{y \in L \mid (a, y) \in \theta\}$. We write it as $[a]_\theta$.
Let θ be an equivalence relation on L. For any $A, B \in P^*(L)$, we write that $A\overline{\theta}B$ if the following two conditions are hold:

(1)$\forall a \in A, \exists b \in B$ such that $a\theta b$; (2) $\forall x \in B, \exists y \in A$ such that $x\theta y$.

We denote $A\overline{\overline{\theta}}B$ if for all $a \in A, b \in B$ we have $a\theta b$.

2.6. Definition. [32] An equivalence relation θ on a hyperlattice $L = (L, \otimes, \oplus)$ is called a regular (strongly regular) hypercongruence relation if for every $x \in L$, $(a, b) \in \theta$ implies

$$(a \otimes x)\overline{\theta}(b \otimes x) \quad \text{and} \quad (a \oplus x)\overline{\theta}(b \oplus x) \ ((a \otimes x)\overline{\overline{\theta}}(b \otimes y) \quad \text{and} \quad (a \oplus x)\overline{\overline{\theta}}(b \oplus y)).$$

Clearly, any strongly regular hypercongruence relation is a regular hypercongruence relation.

2.7. Example. Let $L = \{a, b, c, d\}$ and let the hyperoperations $''\otimes''$ and $''\oplus''$ on L be defined as follows:

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	{a,b}	a	{a,b}
c	a	a	c	c
d	a	{a,b}	c	{c,d}

$\oplus$	a	b	c	d
a	{a,b}	b	{c,d}	d
b	b	b	d	d
c	{c,d}	d	{c,d}	d
d	d	d	d	d

Then $(L, \otimes, \oplus)$ is a hyperlattice [14]. Let θ be a hypercongruence relation on the hyperlattice L with the following equivalence classes: $[a]_\theta = [b]_\theta = \{a, b\}$, $[c]_\theta = [d]_\theta = \{c, d\}$. Then θ is a strongly regular hypercongruence relation on L.

2.8. Definition. Let $L = (L, \otimes, \oplus)$ be a hyperlattice and θ be a regular hypercongruence relation on L. Then θ is called a complete hypercongruence relation if $[a\otimes b]_\theta = \{x\otimes y \mid x \in [a]_\theta, y \in [b]_\theta\}$, and $[a\oplus b]_\theta = \{x\oplus y \mid x \in [a]_\theta, y \in [b]_\theta\}$ for all $a, b \in L$.

2.9. Example. Let $L = \{0, a, b, c, 1\}$ be a lattice $(L, \wedge, \vee)$, where the partial order relation on L is defined as shown in Figure 1. For all $x, y \in L$, $x \otimes y = \{x \wedge y\}$, $x \oplus y = \{x \vee y\}$, then $L = (L, \otimes, \oplus)$ is a hyperlattice.

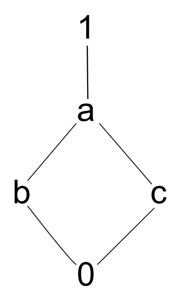

Figure 1. The lattice in Example 2.9.

(i) Let θ be a regular hypercongruence relation on the hyperlattice L with the following equivalence classes: $[1]_\theta = 1, [a]_\theta = [c]_\theta = \{a, c\}, [b]_\theta = [0]_\theta = \{b, 0\}$. Then θ is a complete hypercongruence relation.

(ii) Let θ be a regular hypercongruence relation on the hyperlattice L with the following equivalence classes: $[1]_\theta = [a]_\theta = \{1, a\}$, $[c]_\theta = \{c\}, [b]_\theta = \{b\}, [0]_\theta = \{0\}$. θ is not complete because $[c \oplus b]_\theta = \{1, a\}$, $[c]_\theta \oplus [b]_\theta = \{a\}$ and $[c \oplus b]_\theta \neq [c]_\theta \oplus [b]_\theta$.

2.10. Lemma. *Let $L = (L, \otimes, \oplus)$ be a hyperlattice and θ be a regular hypercongruence relation on L. Then for all $a, b, c, d \in L$,*

(i) *If $(a, b) \in \theta$ and $(c, d) \in \theta$, then $(a \otimes c)\overline{\theta}(b \otimes d)$ and $(a \oplus c)\overline{\theta}(b \oplus d)$,*
(ii) $\{x \otimes y \mid x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \otimes b]_\theta$,
(iii) $\{x \oplus y \mid x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \oplus b]_\theta$.

3. Rough subsets of a hyperlattice in the generalized approximation space

In this section, according to the notion of generalized approximation space presented in [9,35,36], we present some basic concepts about the generalized approximation space (U, W, T) and the associated lower and upper approximation operators. Let U and W be two non-empty universes. Let T be a set-valued mapping given by $T : U \to P(W)$. Then the triple (U, W, T) is referred to as a generalized approximation space. Any set-valued function from U to $P(W)$ defines a binary relation from U to W by setting $\rho_T = \{(x, y) \mid y \in T(x)\}$. Obviously, if ρ is an arbitrary relation from U to W, then it can be defined as a set-valued mapping $T_\rho : U \to P(W)$ by $T_\rho(x) = \{y \in W \mid (x, y) \in \rho\}$, where $x \in U$. For any set $X \subseteq W$, a pair of lower and upper approximations $\underline{T}(X)$ and $\overline{T}(X)$, are defined by
$\underline{T}(X) = \{x \in U \mid T(x) \subseteq X\}$ and $\overline{T}(X) = \{x \in U \mid T(x) \cap X \neq \emptyset\}$. The pair $(\underline{T}(X), \overline{T}(X))$ is referred to as a generalized rough set and $\underline{T}$ and $\overline{T}$ are referred to as lower and upper generalized approximation operators, respectively.

3.1. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A mapping $T : L_1 \to P(L_2)$ is called a set-valued homomorphism if for all $a, b \in L_1$,

(i) $T(a) \otimes_2 T(b) \subseteq T(a \otimes_1 b)$,
(ii) $T(a) \oplus_2 T(b) \subseteq T(a \oplus_1 b)$.

3.2. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A mapping $T : L_1 \to P(L_2)$ is called a strong set-valued homomorphism if for all $a, b \in L_1$,

(i) $T(a) \otimes_2 T(b) = T(a \otimes_1 b)$,
(ii) $T(a) \oplus_2 T(b) = T(a \oplus_1 b)$.

3.3. Example. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices.

(i) The set-valued map $T : L_1 \to P(L_2)$ defined by $T(a) = L_2$ is a set-valued homomorphism.
(ii) If θ is a regular hypercongruence relation on a hyperlattice L_1 then $T_\theta : L_1 \to P(L_1)$ defined by $T_\theta(a) = [a]_\theta$ is a set-valued homomorphism. If θ is a complete regular hypercongruence then T_θ is a strong set-valued homomorphism.
(iii) If $\varphi : L_1 \to L_2$ is a strong hyperlattice homomorphism, then the set-valued map $T : L_1 \to P(L_2)$ defined by $T(a) = \{\varphi(a)\}$ is a strong set-valued homomorphism.

Note that Example 3.3. (ii) indicates that every regular hyper congruence relations may be considered as a set-valued homomorphism. On the other hand, hypercongruence relations are important in hyperalgebraic systems. So set-valued homomorphisms are interesting for pure algebraic systems.

3.4. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $T : L_1 \to P(L_2)$ be a set valued homomorphism. If $X, Y \in P^*(L_2)$, then*

(i) $\overline{T}(X) \otimes_1 \overline{T}(Y) \subseteq \overline{T}(X \otimes_2 Y)$,
(ii) $\overline{T}(X) \oplus_1 \overline{T}(Y) \subseteq \overline{T}(X \oplus_2 Y)$.

Proof. (i) Assume that $x \in \overline{T}(X) \otimes_1 \overline{T}(Y)$. Then $x \in x_1 \otimes_1 x_2$ with $x_1 \in \overline{T}(X)$, $x_2 \in \overline{T}(Y)$. Hence $T(x_1) \cap X \neq \emptyset$ and $T(x_2) \cap Y \neq \emptyset$. Then there exist $a \in T(x_1) \cap X$ and $b \in T(x_2) \cap Y$ such that $a \in T(x_1), b \in T(x_2)$ and $a \in X, b \in Y$. Therefore $a \otimes_2 b \subseteq X \otimes_2 Y$. Since T is a set-valued homomorphism, we have $a \otimes_2 b \subseteq T(x_1) \otimes_2 T(x_2) \subseteq T(x_1 \otimes_1 x_2)$. Hence $T(x_1 \otimes_1 x_2) \cap (X \otimes_2 Y) \neq \emptyset$ which implies that $x \in \overline{T}(X \otimes_2 Y)$. So $\overline{T}(X) \otimes_1 \overline{T}(Y) \subseteq \overline{T}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

3.5. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice L and $X, Y \in P^*(L)$. Then*
(i) $\overline{T_\theta}(X) \otimes \overline{T_\theta}(Y) \subseteq \overline{T_\theta}(X \otimes Y)$,
(ii) $\overline{T_\theta}(X) \oplus \overline{T_\theta}(Y) \subseteq \overline{T_\theta}(X \oplus Y)$.

The following example shows that the inclusion symbol "$\subseteq$" in Propositions 3.4. may not be replaced by the equal sign.

3.6. Example. Consider the hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{a\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{b\}$ and $Y = \{d\}$, then $\overline{T}(X) \otimes \overline{T}(Y) = \emptyset$, $\overline{T}(X \otimes Y) = L$. Thus $\overline{T}(X) \otimes \overline{T}(Y) \neq \overline{T}(X \otimes Y)$. Further, if $T : L \to P(L)$ is a set-valued map defined as $T(x) = \{d\}$, then T is a set-valued homomorphism. If $X = Y = \{c\}$, then $\overline{T}(X) \oplus \overline{T}(Y) = \emptyset$, then $\overline{T}(X \oplus Y) = L$. Thus $\overline{T}(X) \oplus \overline{T}(Y) \neq \overline{T}(X \oplus Y)$.

3.7. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $T : L_1 \to P(L_2)$ be a strong set valued homomorphism. If $X, Y \in P^*(L_2)$, then*
(i) $\underline{T}(X) \otimes_1 \underline{T}(Y) \subseteq \underline{T}(X \otimes_2 Y)$,
(ii) $\underline{T}(X) \oplus_1 \underline{T}(Y) \subseteq \underline{T}(X \oplus_2 Y)$.

Proof. (i) Assume that $z \in \underline{T}(X) \otimes_1 \underline{T}(Y)$. Then $z \in x \otimes_1 y$ with $x \in \underline{T}(X)$, $y \in \underline{T}(Y)$. Hence $T(x) \subseteq X$ and $T(y) \subseteq Y$. Since T is a strong set-valued homomorphism, we have $T(x) \otimes_2 T(y) = T(x \otimes_1 x) \subseteq A \otimes_2 B$. Hence $z \in x \otimes_2 y \in \underline{T}(X \otimes_2 Y)$, that is $\underline{T}(X) \otimes_1 \underline{T}(Y) \subseteq \underline{T}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

3.8. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice L and $X, Y \in P^*(L)$. Then*
(i) $\underline{T_\theta}(X) \otimes \underline{T_\theta}(Y) \subseteq \underline{T_\theta}(X \otimes Y)$,
(ii) $\underline{T_\theta}(X) \oplus \underline{T_\theta}(Y) \subseteq \underline{T_\theta}(X \oplus Y)$.

The following example shows that the containment in the above proposition is proper.

3.9. Example. Consider the hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{a\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{d\}$, $Y = \{b\}$, then $\underline{T}(X) \otimes \underline{T}(Y) = \emptyset$, $\underline{T}(X \otimes Y) = L$.

Thus $\underline{T}(X) \otimes \underline{T}(Y) \neq \underline{T}(X \otimes Y)$. Further, if $T : L \to P(L)$ is a set-valued map defined as $T(x) = \{d\}$, then T is a set-valued homomorphism. If $X = Y = \{c\}$, then $\underline{T}(X) \oplus \underline{T}(Y) = \emptyset$, $\underline{T}(X \oplus Y) = L$. Thus $\underline{T}(X) \oplus \underline{T}(Y) \neq \underline{T}(X \oplus Y)$.

3.10. Proposition. *Let $T : L_1 \to P(L_2)$ be a (strong) set-valued homomorphism and $f : L_3 \to L_1$ be a weak (strong) hyperlattice homomorphism. Then $T \circ f$ is a (strong) set-valued homomorphism from $L_3 \to P(L_2)$ such that $\overline{T \circ f}(X) = f^{-1}(\overline{T}(X))$ and $\underline{T \circ f}(X) = f^{-1}(\underline{T}(X))$, for all $X \in P(L_2)$.*

Proof. The proof is straightforward.

□

3.11. Proposition. *Let $T : L_1 \to P(L_2)$ be a (strong) set-valued homomorphism and $f : L_2 \to L_3$ be a weak (strong) hyperlattice homomorphism. Then T_f is a (strong) set-valued homomorphism from $L_1 \to P(L_3)$ defined by $T_f(r) = f(T(r))$ such that $\underline{T_f}(X) = \underline{T}(f^{-1}(X))$ and $\overline{T_f}(X) = \overline{T}(f^{-1}(X))$, for all $X \in P(L_3)$.*

Proof. The proof is straightforward.

□

3.12. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and let $T : L_1 \to P(L_2)$ be a set-valued mapping. If $\overline{T}(X)$ and $\underline{T}(X)$ are subhyperlattices (resp. hyperideals) of L_1, then $(\overline{T}(X), \underline{T}(X))$ is called a generalized rough subhyperlattice (resp. hyperideal).

3.13. Example. Let $L = (L, \otimes, \oplus)$ be a hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{b\}$ and $X = \{a, b\}$. Then $\overline{T}(X)$ and $\underline{T}(X)$ are subhyperlattices (resp. hyperideals) of L. Hence $(\overline{T}(X), \underline{T}(X))$ is a generalized rough subhyperlattice (resp. hyperideal).

3.14. Theorem. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $X \in P^*(L_2)$.*

- (i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism and X is a subhyperlattice of L_2, then $\overline{T}(X)$ is a subhyperlattice of L_1.*
- (ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism and X is a subhyperlattice of L_2, then $\underline{T}(X)$ is, if it is non-empty, a subhyperlattice of L_1.*
- (iii) *If $T : L_1 \to P^*(L_2)$ is a set-valued homomorphism and X is a hyperideal of L_2, then $\overline{T}(X)$ is a hyperideal of L_1.*
- (iv) *If $T : L_1 \to P^*(L_2)$ is a strong set-valued homomorphism and X is a hyperideal of L_2, then $\underline{T}(X)$ is, if it is non-empty, a hyperideal of L_1.*

Proof. (i) Suppose that $x, y \in \overline{T}(X)$. Then $T(x) \cap X \neq \emptyset$ and $T(y) \cap X \neq \emptyset$. Hence there exist $a \in T(x) \cap X$ and $b \in T(y) \cap X$. Thus $a \otimes_2 b \subseteq T(x) \otimes_2 T(y) \subseteq T(x \otimes_1 y)$ and $a \oplus_2 b \subseteq T(x) \oplus_2 T(y) \subseteq T(x \oplus_1 y)$. Since X is a subhyperlattice of L_2, we have $a \otimes_2 b \subseteq X$ and $a \oplus_2 b \subseteq X$. So $T(x \otimes_1 y) \cap X \neq \emptyset$ and $T(x \oplus_1 y) \cap X \neq \emptyset$. Therefore $x \otimes_1 y, x \oplus_1 y \in \overline{T}(X)$. Consequently, $\overline{T}(X)$ is a subhyperlattice of L_1.

(ii) Suppose that $x, y \in \underline{T}(X)$. Then $T(x) \subseteq X$ and $T(y) \subseteq X$. Since X is a subhyperlattice of L_2 and T is a strong set-valued homomorphism, we have $T(x \otimes_1 y) =$

$T(x) \otimes_2 T(y) \subseteq X \otimes_2 X \subseteq X$ and $T(x \oplus_1 y) = T(x) \oplus_2 T(y) \subseteq X \oplus_2 X \subseteq X$. Thus $x \otimes_1 y, x \oplus_1 y \in \underline{T}(X)$. Therefore $\underline{T}(X)$ is a subhyperlattice of L_1.

(iii) By (i) $\overline{T}(X)$ is a subhyperlattice of L_1. Let $b \in L_1$. Since $T(b) \neq \emptyset$, there exist some $z \in L_2$ such that $z \in T(b)$. Let $x \in \overline{T}(X)$. Then $T(x) \cap X \neq \emptyset$ which implies that there exists $a \in T(x) \cap X$, that is $a \in T(x), a \in X$. Since X is a hyperideal of L_2 and T is a strong set-valued homomorphism, we have $a \otimes_2 z, a \oplus_2 z \subseteq X$ and $a \otimes_2 z \subseteq T(x) \otimes_2 T(b) = T(x \otimes_1 b)$, $a \oplus_2 z \subseteq T(x) \oplus_2 T(b) = T(x \oplus_1 b)$ which implies that $T(x \otimes_1 b) \cap X \neq \emptyset$ and $T(x \oplus_1 b) \cap X \neq \emptyset$. Thus $x \otimes_1 b, x \oplus_1 b \in \overline{T}(X)$. Therefore $\overline{T}(X)$ is a hyperideal of L_1.

(iv) Similarly, $\underline{T}(X)$ is a hyperideal of L_1.

□

The following example shows that the converse of the above theorem does not hold in general.

3.15. Example. Consider the hyperlattice defined Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{d\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{b, d\}$, then X is not a subhyperlattice (hyperideal) of L. But $\overline{T}(X) = L$ is a subhyperlattice (hyperideal) of L.

3.16. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice $L = (L, \otimes, \oplus)$.*

(i) *If X is a hyperlattice of L, then $\overline{T_\theta}(X)$ is a subhyperlattice of L.*
(ii) *If θ is a complete regular hypercongruence relation and X is a subhyperlattice of L, then $\underline{T_\theta}(X)$ is, if it is non-empty, a subhyperlattice of L.*
(iii) *If X is a hyperideal of L, then $\overline{T_\theta}(X)$ is a hyperideal of L.*
(iv) *If θ is a complete regular hypercongruence relation and X is a hyperideal of L, then $\underline{T_\theta}(X)$ is, if it is non-empty, a hyperideal of L.*

Now we give a counterexample which shows that the condition that θ is a complete regular hypercongruence relation in Corollary 3.16. is necessary.

3.17. Example. Consider the hyperlattice L and the congruence relation on L defined in Example 2.9.(ii). If $X = \{a, b, c, 0\}$, then X is a subhyperlattice of L. But $\underline{T_\theta}(X) = \{b, c, 0\}$ is not a subhyperlattice of L.

4. Generalized lower and upper approximation operators with respect to a hyperideal of a hyperlattice

4.1. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices, A be a hyperideal of L_2 and $T : L_1 \to P(L_2)$ be a set-valued mapping. Then we define $T_A : L_1 \to P(L_2)$ as $T_A(a) = T(a) \otimes_2 A$ for all $a \in L_1$. Then T_A is called the set-valued mapping with respect to a hyperideal A.

4.2. Definition. Let (L_1, L_2, T_A) be a generalized approximation space with respect to a hyperideal A and X be a non-empty subset of L_2. Then the sets
$\underline{T_A}(X) = \{a \in L_1 \mid T_A(a) \subseteq X\}$ and $\overline{T_A}(X) = \{a \in L_1 \mid T_A(a) \cap X \neq \emptyset\}$

are called generalized lower and upper approximations of X with respect to the hyperideal A, respectively.

4.3. Lemma. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and A, B be hyperideals of L_2. Let X be a subset of L_2 such that $A \subseteq B$. Then*

(i) $\overline{T_A}(X) \subseteq \overline{T_B}(X)$,

(ii) $\underline{T_B}(X) \subseteq \underline{T_A}(X)$.

Proof. (i) Suppose that $x \in \overline{T_A}(X)$. Then $(T(x) \otimes_2 A) \cap X \neq \emptyset$. So there exist $a \in (T(x) \otimes_2 A) \cap X$ such that $a \in (T(x) \otimes_2 A)$ and $a \in X$. Hence there exist $y \in T(x), z \in A$ such that $a = y \otimes_2 z$. Since $A \subseteq B$, we have $z \in B$. Thus $a = y \otimes_2 z \subseteq T(x) \otimes_2 B$ and $a \in X$. So $(T(x) \otimes_2 B) \cap X \neq \emptyset$. As a consequent, we obtain $\overline{T_A}(X) \subseteq \overline{T_B}(X)$.

(ii) The proof is similar to (i).

□

The following corollary follows from Lemma 4.3.

4.4. Corollary. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and A, B be hyperideals of L_2. Let X be a subset of L_2 such that $A \subseteq B$. Then*

(i) $\overline{T_{A\cap B}}(X) \subseteq \overline{T_A}(X) \cap \overline{T_B}(X)$,

(ii) $\underline{T_A}(X) \cap \underline{T_B}(X) \subseteq \underline{T_{A\cap B}}(X)$.

4.5. Proposition. *Let (L_1, L_2, T_A) be a generalized approximation with respect to a hyperideal A and X, Y be a non-empty subsets of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism, then $\overline{T_A}(X) \otimes_1 \overline{T_A}(Y) \subseteq \overline{T_A}(X \otimes_2 Y)$.*

(ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism, then $\underline{T_A}(X) \otimes_1 \underline{T_A}(Y) \subseteq \underline{T_A}(X \otimes_2 Y)$.*

Proof. (i) Suppose that $z \in \overline{T_A}(X) \otimes_1 \overline{T_A}(Y)$. Then there exist $x \in \overline{T_A}(X), y \in \overline{T_A}(Y)$ such that $z \in x \otimes_1 y$. Since $x \in \overline{T_A}(X)$ $y \in \overline{T_A}(Y)$ there exist $a \in T(x) \otimes_2 A$, $b \in T(y) \otimes_2 A$ such that $a \in T(x), b \in T(y), a \in X, b \in Y$. Since T is a set-valued homomorphism, we have $a \otimes_2 b \subseteq T(x) \otimes_2 T(y) \otimes_2 A \subseteq T(x \otimes_1 y) \otimes_2 A$ and $a \otimes_2 b \subseteq X \otimes_2 Y$. Hence $a \otimes_2 b \subseteq T(x \otimes_1 y) \otimes_2 A \cap (X \otimes_2 Y)$. So $z \in x \otimes_1 y \subseteq \overline{T_A}(X \otimes_2 Y)$.Therefore, we obtain $\overline{T_A}(X) \otimes_1 \overline{T_A}(Y) \subseteq \overline{T_A}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

4.6. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices, A, B be hyperideals of L_2 and X be a subhyperlatice of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism, then $\overline{T_A}(X) \otimes_1 \overline{T_B}(X) \subseteq \overline{T_{A\otimes_2 B}}(X)$.*

(ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism, then $\underline{T_A}(X) \otimes_1 \underline{T_B}(X) = \underline{T_{A\otimes_2 B}}(X)$.*

Proof. The proof is straightforward.

□

4.7. Theorem. *Let (L_1, L_2, T_A) be a generalized approximation space with respect to a hyperideal A and X be a non-empty subset of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism and X is a subhyperlattice of L_2, then $\overline{T_A}(X)$ is a subhyperlattice of L_1.*

(ii) *If* $T : L_1 \to P(L_2)$ *is a strong set-valued homomorphism and* X *is a subhyperlattice of* L_2*, then* $\underline{T_A}(X)$ *is, if it is non-empty, a subhyperlattice of* L_1.

(iii) *If* $T : L_1 \to P^*(L_2)$ *is a set-valued homomorphism and* X *is a hyperideal of* L_2*, then* $\overline{T_A}(X)$ *is a hyperideal of* L_1.

(iv) *If* $T : L_1 \to P^*(L_2)$ *be a strong set-valued homomorphism and* X *is a hyperideal of* L_2*, then* $\underline{T_A}(X)$ *is, if it is non-empty, a hyperideal of* L_1.

Proof. (i) Suppose that $x, y \in \overline{T_A}(X)$. Then, $(T(x) \otimes_2 A) \cap X \neq \emptyset$ and $(T(y) \otimes_2 A) \cap X \neq \emptyset$. Hence there exist $a \in (T(x) \otimes_2 A) \cap X$ and $b \in (T(y) \otimes_2 A) \cap X$. Since X is a subhyperlattice of L_2 , we have $a \otimes_2 b \subseteq X$ and $a \oplus_2 b \subseteq X$. On the other hand, $a \otimes_2 b \subseteq (T(x) \otimes_2 A) \otimes_2 (T(y) \otimes_2 A) \subseteq T(x) \otimes_2 T(y) \otimes_2 A \subseteq T(x \otimes_1 y) \otimes_2 A$ and $a \oplus_2 b \subseteq (T(x) \otimes_2 A) \oplus_2 (T(y) \otimes_2 A) \subseteq T(x) \oplus_2 T(y) \otimes_2 A \subseteq T(x \oplus_1 y) \otimes_2 A$. So $T(x \otimes_1 y) \otimes_2 A \cap X \neq \emptyset$ and $T(x \oplus_1 y) \otimes_2 A \cap X \neq \emptyset$. Thus $x \otimes_1 y, x \oplus_1 y \in \overline{T_A}(X)$. Therefore, $\overline{T_A}(X)$ is a subhyperlattice of L_1.

(ii) Similarly, $\underline{T_A}(X)$ is a subhyperlattice of L_1.

(iii) Using (i), $\overline{T_A}(X)$ is a subhyperlattice of L_1. Let $x \in \overline{T_A}(X)$ and $c \in L_1$. Then $(T(x) \otimes_2 A) \cap X \neq \emptyset$. So there exist $a \in (T(x) \otimes_2 A) \cap X$. Since $\overline{T_A}(X)$ is non-empty set, we can choose $z \in T(c)$. Since X is a hyperideal of L_2, we have $a \otimes_2 z, a \oplus_2 z \subseteq X$. On the other hand, $a \otimes_2 z \subseteq (T(x) \otimes_2 A) \otimes_2 T(c) \subseteq T(x \otimes_1 c) \otimes_2 A$, $a \oplus_2 z \subseteq (T(x) \otimes_2 A) \oplus_2 T(c) \subseteq T(x \oplus_1 c) \otimes_2 A$. So $(T(x \otimes_1 c) \otimes_2 A) \cap X \neq \emptyset$, $(T(x \oplus_1 c) \otimes_2 A) \cap X \neq \emptyset$ which implies $x \otimes_1 c, x \oplus_1 c \in \overline{T_A}(X)$. Therefore $\overline{T_A}(X)$ is a hyperideal of L_1.

(iv)The proof is straightforward.

□

The following example shows that the converse of the above theorem does not hold in general.

4.8. Example. Consider the hyperlattice defined in Example 2.9. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{d\}$. Then it is easy to see that T is a set-valued homomorphism. If $A = L$, $X = \{a, b, c\}$, then A is a hyperideal and X is not a subhyperlattice (hyperideal) of L. But $\overline{T_A}(X) = L$ is a subhyperlattice (hyperideal) of L.

5. Conclusion

The Pawlak rough sets on algebraic sets such as semigroups, groups, rings, modules and lattices were mainly studied by congruence relations. In this paper, a definition of set-valued homomorphism which was introduced for groups by Davvaz [9], for rings and modules by Yamak et al. [35-36], respectively, is considered as a regular hypercongruence relation for hyperlattices. We obtain some new properties of a set-valued homomorphism to provide opportunity for putting reasonable interpretations on the theory and applications of rough sets and adhering to the set-valued homomorphism and exploring the features of generalized rough approximations on hyperlattices. So, in this paper we propose a definition of set-valued homomorphism and explore the properties of generalized rough approximations on hyperlattices. Some new properties of set-valued homomorphisms which shall be very practical in the theory and applications of rough sets are obtained. Moreover, a new algebraic structure called generalized lower and upper approximations of a set with respect to a hyperideal is presented.

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