Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 867 – 874

Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient

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Abstract

In this paper, a self adjoint boundary value problem with a piecewise continuous coefficient on the positive half line $[0, \infty)$ is considered. The resolvent operator is constructed and the expansion formula with respect to eigenfunctions or equivalently Parseval equality is obtained. The spectrum of the operator is discussed.

2000 AMS Classification: 34L10, 34L40.

Keywords: Sturm-Liouville operator, eigenvalue, eigenfunction, resolvent operator, expansion formula.

Received 20/06/2013 : Accepted 08/04/2014 Doi : 10.15672/HJMS.2015449435

1. Introduction

Here, we consider the boundary value problem on the half line $0 < x < \infty$ generated by the differential equation

$$(1.1) \qquad -y'' + q(x)y = \lambda^2 \rho(x) y$$

and the boundary condition

$$(1.2) \qquad y'(0) - hy(0) = 0,$$

where λ is a spectral parameter, $q(x)$ is a real valued function satisfying the condition

$$(1.3) \qquad \int_0^\infty (1+x)\,|q(x)|\,dx < \infty$$

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and

$$\rho(x) = \begin{cases} \alpha^2, & 0 \le x < a, \\ 1, & x \ge a, \end{cases}$$

where $0 < \alpha \neq 1$. It is not hard to verify that the function

$$f_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) e^{i\lambda\mu^-(x)}$$

is the solution of equation (1.1) when $q(x) \equiv 0$, where

$$\mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)}).$$

As it is known from [5, 8] that for λ from the closed upper half plane equation (1.1) has a unique solution $f(x,\lambda)$ which can be represented in the form

$$f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{\infty} K(x,t)e^{i\lambda t}dt, \tag{1.4}$$

where $K(x,\cdot) \in L_1(\mu^+(x), +\infty)$. The function $f(x,\lambda)$ is called the *Jost solution* of equation (1.1).

Note that, a singular Sturm-Liouville problem in the form of (1.1), (1.2) is encountered when applying separation of variables to mathematical physics problems in nonhomogeneous media, e. g. when $q(x) \equiv 0$ an application of electric prospecting problem, was given in [13, 15]. In this works, expansion formula was obtained by using Titchmarsh's [14] method with the help of integral representation (1.4), for the solution of equation (1.1). When $\rho(x) \equiv 1$ spectral expansion formula, for singular differantial operators on the interval $[0, \infty)$ was investigated with different methods in [14, 10], etc. When $\rho(x) \neq 1$, spectral properties of similar problems were considered in [4, 3, 5, 7, 8, 9]. Also, in this case the direct and inverse problem in a finite interval were examined in [1, 11].

Using (1.4) we have for real $\lambda \neq 0$ that the functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ form the fundamental system of solutions of equation (1.1) and the Wronskian of this system is equal to $2i\lambda$:

$$W\left\{f(x,\lambda), \overline{f(x,\lambda)}\right\} = f'(x,\lambda)\overline{f(x,\lambda)} - f(x,\lambda)\overline{f'(x,\lambda)} = 2i\lambda.$$

By $\omega(x,\lambda)$, we denote the solutions of equation (1.1) satisfying the initial data

$$\omega(0,\lambda) = 1,\ \omega'(0,\lambda) = h.$$

Proof of the following propositions can be done analoguously to [8].

1.1. Proposition. *For real $\lambda \neq 0$ the following identity*

$$2i\lambda \frac{\omega(x,\lambda)}{f'(0,\lambda) - hf(0,\lambda)} = \overline{f(x,\lambda)} - S(\lambda)f(x,\lambda) \tag{1.5}$$

holds, here

$$S(\lambda) = \frac{\overline{f'(0,\lambda) - hf(0,\lambda)}}{f'(0,\lambda) - hf(0,\lambda)} \qquad \textit{and} \qquad |S(\lambda)| = 1.$$

$S(\lambda)$ is called the *scattering function* of the boundary value problem (1.1), (1.2).

1.2. Proposition. *The function $\varphi(\lambda) \equiv f'(0,\lambda) - hf(0,\lambda) \neq 0$ may have only a finite number of zeros λ_k, $(k = 1, 2, ..., n)$ in the half plane $Im\lambda > 0$. These zeros are all simple and lie on the imaginary axis.* For $\lambda = i\lambda_j$ $(\lambda_j > 0)$, $j = \overline{1,n}$, we get

$$m_j^{-2} \equiv \int_0^{\infty} \rho(x)\left|f(x,i\lambda_j)\right|^2 dx = -\frac{1}{2i\lambda_j}\dot{\varphi}(i\lambda_j)f(0,i\lambda_j).$$

These values are called the *norming constants* of the boundary value problem (1.1), (1.2).

2. Spectrum

This section is devoted to examine the properties of the eigenvalues of the boundary value problem (1.1), (1.2).

2.1. Theorem. *The operator L has no eigenvalues on the positive half line.*

Proof. Let $\lambda_0^2 > 0$ be an eigenvalue of the operator L and $y_0(x) = y(x, \lambda_0)$ be the corresponding eigenfunction. Since $f(x, \lambda_0)$ and $\overline{f(x, \lambda_0)}$ form the fundamental system of solutions, the general solution of (1.1) can be written in the form

$$y_0(x) = c_1 f(x, \lambda_0) + c_2 \overline{f(x, \lambda_0)}.$$

As $x \to \infty$,

$$f(x, \lambda_0) \to e^{i\lambda_0 x} \quad \textit{and} \quad \overline{f(x, \lambda_0)} \to e^{-i\lambda_0 x},$$

hence

$$y_0(x) = c_1 e^{i\lambda_0 x} + c_2 e^{-i\lambda_0 x} + o(1).$$

Since, its principal part is periodic this function does not belong to $L_2(0, \infty)$ for any values of c_1 and c_2. □

2.2. Theorem. *For $-\lambda_0^2$ $(\lambda_0 \neq 0)$ to be an eigenvalue it is necessary and sufficient that $\varphi(\lambda_0) = 0$.*

Proof. Indeed, let $\varphi(\lambda_0) = 0$ $(Im\lambda_0 > 0)$. Thus, $f'(0, \lambda_0) - hf(0, \lambda_0) = 0$. Therefore, $f(x, \lambda_0)$ is a solution of the boundary value problem (1.1), (1.2). While $x \to \infty$ $f(x, \lambda_0)$ decreases exponentially. Hence, $f(x, \lambda_0) \in L_2(0, \infty)$ and for the corresponding eigenvalue $-\lambda_0^2$ $f(x, \lambda_0)$ is the eigenfunction of operator L. On the other hand, let $-\lambda_0^2$ $(\lambda_0 \neq 0)$ be an eigenvalue and $y(x, \lambda_0)$ be the suitable eigenfunction of operator L. Then $y'(0, \lambda_0) - hy(0, \lambda_0) = 0$. It is clear that, $y(0, \lambda_0) \neq 0$. Without loss of generality assume that $y(0, \lambda_0) = 1$, then $y'(0, \lambda_0) = h$. Since, $f(x, \lambda_0)$ and $\hat{f}(x, \lambda_0)$ form the fundamental system of solutions of equation (1.1) (see [12] p. 297), we can write

$$y(x, \lambda_0) = c_1 f(x, \lambda_0) + c_2 \hat{f}(x, \lambda_0).$$

As $x \to \infty$, we obtain $c_2 = 0$, then $c_1 \neq 0$. Substituting $x = 0$ in the last relation, we get

$$y'(0, \lambda_0) - hy(0, \lambda_0) = c_1$$

i.e.,

$$f'(0, \lambda_0) - hf(0, \lambda_0) = \varphi(\lambda_0) = 0.$$

Thus, for each eigenvalue $-\lambda_0^2$, there is one and only one adequate (up to a multiplicative constant) eigenfunction:

$$y(x, \lambda_0) = cf(x, \lambda_0), \ (c \neq 0).$$

□

The proof of the following theorem can be obtained directly form Theorem 2.1 and Theorem 2.2. **2.3. Theorem.** *The operator L has a finite number of eigenvalues: $-\lambda_1^2, -\lambda_2^2, ...,$
$-\lambda_n^2$.*

Therefore, it is appropriate at this point to note that the spectral problem (1.1), (1.2) has a finite number of negative eigenvalues and it fills positive half line with its continuous spectrum.

3. The Resolvent Operator and Expansion Formula for the Eigenfunctions

In the space $L_{2,\rho}(0,\infty)$, we define an inner product by

$$< f, g >:= \int_0^\infty f(x)\overline{g(x)}\rho(x)dx,$$

where $f(x), g(x) \in L_{2,\rho}(0,\infty)$.

Let us define

$$D(L) = \left\{ \begin{array}{c} f(x) \in L_{2,\rho}(0,\infty) : f(x), f'(x) \in AC\left[0,\infty\right), l(f) \in L_{2,\rho}(0,\infty), \\ f'(0) - hf(0) = 0 \end{array} \right\},$$

as $L : f \to l(f)$ where

$$l(f) = \frac{1}{\rho(x)} \left\{-f''(x) + q(x)f(x)\right\}.$$

The boundary value problem (1.1), (1.2) is equivalent to the equation $Ly = \lambda^2 y$ and the operator L is self-adjoint in the space $L_{2,\rho}(0,\infty)$.

Let us assume that λ^2 is not a spectrum point of operator $R_{\lambda^2}(L) = (L - \lambda^2 I)^{-1}$ and find the expression of the operator $R_{\lambda^2}(L)$ as all numbers λ^2 $(Im\lambda \geq 0, \varphi(\lambda) \neq 0)$ belong to the resolvent set of the operator L.

3.1. Theorem. *The resolvent* $R_{\lambda^2}(L)$ *is the integral operator*

$$R_{\lambda^2}(L) = \int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt$$

with the kernel,

$$G(x,t;\lambda) = -\frac{1}{\varphi(\lambda)} \left\{ \begin{array}{ll} \omega(x,\lambda) f(t,\lambda), & t \geq x, \\ f(x,\lambda)\omega(t,\lambda), & t \leq x. \end{array} \right. \tag{3.1}$$

Proof. Let $g(x) \in D(L)$ and assume that it is a finite function at infinity. To construct the resolvent operator of L we need to solve the boundary value problem

$$-y'' + q(x)y = \lambda^2\rho(x)y + g(x)\rho(x), \tag{3.2}$$

$$y'(0) - hy(0) = 0. \tag{3.3}$$

We know that the functions $w(x,\lambda)$ and $f(x,\lambda)$ are the solutions of homogeneous problem for $Im\lambda > 0$. Now let us find the solutions of the problem (3.2), (3.3) which has the form

$$y(x,\lambda) = c_1(x,\lambda)w(x,\lambda) + c_2(x,\lambda)f(x,\lambda). \tag{3.4}$$

By applying the method of variation of constants, we get the system of equations

$$\left\{ \begin{array}{ll} c_1'(x,\lambda)\, w(x,\lambda) + c_2'(x,\lambda)\, f(x,\lambda) & = 0, \\ c_1'(x,\lambda)\, w'(x,\lambda) + c_2'(x,\lambda)\, f'(x,\lambda) & = -\rho(x)\, g(x). \end{array} \right. \tag{3.5}$$

Since $y(x,\lambda) \in L_{2,\rho}(0,\infty)$, then $c_1(0,\infty) = 0$. By using this relation and the system equations (3.5), we obtain

$$c_1(x,\lambda) = -\frac{1}{\varphi(\lambda)} \int_x^\infty f(t,\lambda)g(t)\rho(t)dt,$$

$$(3.6)\qquad c_2(x,\lambda)=c_2(0,\lambda)-\frac{1}{\varphi(\lambda)}\int_0^x w(t,\lambda)g(t)\rho(t)dt.$$

Substituting (3.6) into (3.4) and taking (3.3) into consideration, the proof of Theorem 3.1 is completed. □

3.2. Lemma. *Let $g(x)$ be a twice continuously differential function vanishing outside of some finite interval and $g(x)\in D(L)$. Then, as $|\lambda|\to\infty$, $Im\lambda>0$ the following holds:*

$$(3.7)\qquad \int_0^\infty G(x,t;\lambda)g\left(t\right)\rho\left(t\right)dt=-\frac{g\left(x\right)}{\lambda^2}+\frac{Z(x,\lambda)}{\lambda^2},$$

where

$$Z(x,\lambda)=\int_0^\infty G(x,t,\lambda)\tilde{g}(t)\rho(t)dt$$

as $\tilde{g}(t)=-g''(t)+q(t)g(t)$.

Proof. The proof can be easily seen by using Theorem 3.1 and integrating by parts. □

Bounded solutions of boundary value problem (1.1), (1.2) are given in the following way:

$$u(x,\lambda)=\sqrt{\frac{1}{2\pi}}\left[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\right],\quad 0<\lambda^2<\infty,$$

$$u(x,i\lambda_j)=m_jf(x,i\lambda_j),\qquad j=1,2,...,n.$$

By using the contour integration, it can be shown that they form a complete system.

3.3. Theorem. *The expansion formula which is equivalent to Parseval equality*

$$(3.8)\qquad \delta(x-t)=\sum_{j=1}^{n}u(x,i\lambda_j)u(t,i\lambda_j)\rho(t)+\int_0^\infty u(x,\lambda)\overline{u(t,\lambda)}\rho(t)d\lambda$$

holds, where $\delta\left(x\right)$ is Dirac delta function, also when $x\to\infty$ the following asymptotic formulae are true:

$$(3.9)\qquad \begin{aligned} u(x,\lambda)&=e^{-i\lambda x}-S(\lambda)e^{i\lambda x}+o(1),\qquad (0<\lambda^2<\infty)\\ u(x,i\lambda_j)&=m_je^{-\lambda_jx}\left[1+o(1)\right],\qquad (j=1,...,n).\end{aligned}$$

Proof. Let Γ_R denote the circle of radius R and center zero which boundary contour is positive oriented. Assume $D=\{z:|z|\le R,|Imz|\ge\epsilon\}$, denote the positive oriented boundary contour of D as $\Gamma_{R,\epsilon}$ and take integration along this contour. By multiplying both sides of (3.7) by $\frac{1}{2\pi i}\lambda$ and integrating it with respect to λ, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\lambda d\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt=-\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\frac{g(x)}{\lambda}d\lambda+Z_{R,\epsilon}(x),$$

where

$$Z_{R,\epsilon}(x)=\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\frac{Z(x,\lambda)}{\lambda}d\lambda.$$

It can be shown from the properties of the functions $w(x,\lambda)$, $f(x,\lambda)$ that, as $R\to\infty$ and $\epsilon\to0$, $Z_{R,\epsilon}\to0$ holds for $\forall x\in[0,T]\subset[0,\infty)$ uniformly. From the last relation, as $R\to\infty$, $\epsilon\to0$ we can write

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\lambda d\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\to -g(x)+$$

$$+\frac{1}{2\pi i}\int_{-\infty}^\infty\lambda d\lambda\int_0^\infty[G(x,t;\lambda+i0)-G(x,t;\lambda-i0)]g(t)\rho(t)dt.$$

On the other hand, using the residue calculus, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}}\lambda d\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt &= \sum_{j=1}^{n}\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\\ &+\sum_{j=1}^{n}\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right].\end{aligned}$$

From the last two relations we obtain

$$\begin{aligned}g(x)= &-\sum_{j=1}^{n}\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]-\\ &-\sum_{j=1}^{n}\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda\int_0^\infty[G(x,t;\lambda+i0)-G(x,t;\lambda-i0)]g(t)\rho(t)dt.\end{aligned}$$

Let $\psi(x,\lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\psi(0,\lambda)=0,\qquad \psi'(0,\lambda)=1$$

and $W\{\omega(x,\lambda),f(x,\lambda)\}=1$. From here, we can write

$$f(x,\lambda)=f(0,\lambda)\omega(x,\lambda)-\varphi(\lambda)\psi(x,\lambda).$$

Therefore, from (3.1) we have

$$G(x,t;\lambda)=-\frac{f(0,\lambda)}{\varphi(\lambda)}\omega(x,\lambda)\omega(t,\lambda)-\begin{cases}\omega(x,\lambda)\psi(t,\lambda), & x\le t,\\ \psi(x,\lambda)\omega(t,\lambda), & t\le x.\end{cases}$$

Accordingly for $Im\lambda\ge 0$, we obtain

$$\begin{aligned}\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt= &-\frac{1}{\varphi(\lambda)}f(0,\lambda)\,\omega(x,\lambda)\int_0^\infty\omega(t,\lambda)\,g(t)\rho(t)dt-\\ &-\psi(x,\lambda)\int_0^x\omega(t,\lambda)\,g(t)\rho(t)dt-\\ &-\omega(x,\lambda)\int_x^\infty\psi(t,\lambda)\,g(t)\rho(t)dt.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\underset{\lambda=i\lambda_j}{Res}\left[\lambda\int_0^\infty G(x,t;\lambda)g(t)\rho(t)dt\right]+\underset{\lambda=-i\lambda_j}{Res}\left[\lambda\int_0^\infty \overline{G(x,t;\lambda)}g(t)\rho(t)dt\right]=\\ &=-\frac{2i\lambda_j}{\dot{\varphi}(i\lambda_j)}f(0,i\lambda_j)\omega(x,i\lambda_j)\int_0^\infty\omega(t,i\lambda_j)g(t)\rho(t)dt=\\ &=u(x,i\lambda_j)\int_0^\infty u(t,i\lambda_j)g(t)\rho(t)dt.\end{aligned}$$

We can write

$$\begin{aligned}G(x,t;\lambda+i0)-G(x,t;\lambda-i0) &= \left[-\frac{f(0,\lambda+i0)}{\varphi(\lambda+i0)}+\frac{f(0,\lambda-i0)}{\varphi(\lambda-i0)}\right]\omega(x,\lambda)\omega(t,\lambda)=\\ &=\frac{\varphi(\lambda)\overline{f(0,\lambda)}-\overline{\varphi(\lambda)}f(0,\lambda)}{|\varphi(\lambda)|^2}\omega(x,\lambda)\omega(t,\lambda)=\\ &=\frac{2i\lambda}{|\varphi(\lambda)|^2}\omega(x,\lambda)\omega(t,\lambda).\end{aligned}$$

It follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda d\lambda\int_0^{\infty}[G(x,t;\lambda+i0)-G(x,t;\lambda-i0)]g(t)\rho(t)dt=$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{\lambda^2}{|\varphi(\lambda)|^2}\omega(x,\lambda)\int_0^{\infty}\omega(t,\lambda)g(t)\rho(t)dtd\lambda=$$

$$=\int_0^{\infty}u(x,\lambda)\int_0^{\infty}u(t,\lambda)g(t)\rho(t)dtd\lambda.$$

Therefore, from (3.10) we get the expansion formula for the eigenfunctions:

$$g(x)=\sum_{j=1}^{n}u(x,i\lambda_j)\int_0^{\infty}u(t,i\lambda_j)g(t)\rho(t)dt+ \tag{3.11}$$

$$+\int_0^{\infty}u(x,\lambda)\int_0^{\infty}\overline{u(t,\lambda)}g(t)\rho(t)dtd\lambda$$

or we obtain (3.8) that is equivalent to the Parseval equality. Asymptotic expressions (3.9) can be obtained from (1.5) when $x\to\infty$. □

Writing the expansion formula (3.11) in the form of Stieltjes integral we have

$$g(x)=\int_{-\infty}^{\infty}\omega(x,\lambda)\left(\int_0^{\infty}\omega(t,\lambda)g(t)\rho(t)dt\right)d\sigma(\lambda),$$

where

$$d\sigma(\lambda)=\begin{cases}\frac{2}{\pi}\frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2}, & \lambda\geq 0,\\ \sum_{j=1}^{n}\frac{(2i\lambda_j)^2\delta(\lambda-i\lambda_j)}{m_j^2\dot{\varphi}(i\lambda_j)^2}, & \lambda<0\end{cases}$$

is the *spectral function* of operator L.
Now taking

$$G(\lambda)=\int_0^{\infty}\omega(x,\lambda)g(x)\rho(x)dx,$$

we get

$$g(x)=\int_{-\infty}^{\infty}G(\lambda)\omega(x,\lambda)d\sigma(\lambda).$$

Multiplying both sides of this equivalence by $g(x)$, we obtain the Parseval equality

$$\int_0^{\infty}g^2(x)dx=\int_{-\infty}^{\infty}G^2(\lambda)d\sigma(\lambda).$$

Acknowledgment. This work is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK).

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