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On the P-interiors of submodules of Artinian modules

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Abstract

Let R be a commutative ring and M an Artinian R-module. In this paper, we study the dual notion of saturations (that is, P-interiors) of submodules of M and obtain some related results.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. We write $N \leq M$ to indicate that N is a submodule of an R-module M. Also $Spec(R)$ and $\mathbb{Z}$ will denote the set of all prime ideals of R and the ring of integers respectively.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism $S \xrightarrow{a} S$ is either surjective or zero (see [13]). A submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M. Thus, the intersection of all completely irreducible submodule of M is zero (see [6]).

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The *saturation* of $N \leq M$ with respect to $P \in Spec(R)$ is the contraction of N_P in M and designated by $S_P(N)$. It is well known that

$$S_P(N) = \{e \in M : es \in N \ for\ some\ s \in R - P\}.$$

In [1], H. Ansari-Toroghy and F. Farshadifar, introduced the dual notions of saturations of submodules, that is, P-interiors of submodules and investigated some related results (see [1] and [3]). Let N be a submodule of M. *The P-interior of N relative to M* is defined [1, 2.7] as the set

$$I_P^M(N) = \cap\{L \mid L\ is\ a\ completely\ irreducible\ submodule\ of\ M\ and$$
$$rN \subseteq L\ for\ some\ r \in R - P\}.$$

There are considerable results about saturation of a module with respect to a prime ideal in literature (see, for example, [7], [8], and [9]). It is natural to ask that to what extent the dual of these results hold. The purpose of this paper is to answer this question and provide more information about the P-interiors of submodules in case that our module is an Artinian module.

2. P-interiors of submodules and related properties

Recall that an R-module L is said to be *cocyclic* if L is a submodule of $E(R/m)$ for some maximal ideal m of R, where $E(R/m)$ is the injective envelope of R/m (see [14]).

2.1. Lemma. Let L be a completely irreducible submodule of an R-module M and $a \in R$. Then $(L :_M a)$ is a completely irreducible submodule of M.

Proof. This follows from the fact that a submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module by [6] and that $M/(L :_M a) \cong (aM + L)/L$.

□

We use the following basic fact without further comment.

2.2. Remark. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

2.3. Lemma. Let $P \in Spec(R)$ and N be a submodule of an R-module M. If $M/I_P^M(N)$ is a finitely cogenerated R-module, then there exists $r \in R - P$ such that $rN \subseteq I_P^M(N)$.

Proof. Since $M/I_P^M(N)$ is finitely cogenerated, there exists a finite number of completely irreducible submodules $L_1, L_2, ..., L_n$ of M such that $I_P^M(N) = \cap_{i=1}^n L_i$ and $r_iN \subseteq L_i$ for some $r_i \in R - P$. Set $r = r_1...r_n$. Then $rN \subseteq I_P^M(N)$. □

2.4. Theorem. Let $P \in Spec(R)$ and N be a submodule of an R-module M. Then we have the following.

(a) If M is an Artinian R-module, then $I_P^M(I_P^M(N)) = I_P^M(N)$.
(b) If M is an Artinian R-module, then $Hom_R(R_P, I_P^M(N)) = Hom_R(R_P, N)$.
(c) $Ann_R(N) \subseteq S_P(Ann_R(N)) \subseteq Ann_R(I_P^M(N))$.
(d) If M is an Artinian R-module, then $Ann_R(I_P^M(N)) = S_P(Ann_R(I_P^M(N)))$.

Proof. (a) Clearly, $I_P^M(I_P^M(N)) \subseteq I_P^M(N)$. To prove the opposite inclusion, let L be a completely irreducible submodule of M such that $I_P^M(I_P^M(N)) \subseteq L$. By Lemma 2.3, there exists $r \in R - P$ such that $rI_P^M(N) \subseteq I_P^M(I_P^M(N))$. Therefore, $rI_P^M(N) \subseteq L$. Again by Lemma 2.3, there exists $s \in R - P$ such that $sN \subseteq I_P^M(N)$. Hence $rsN \subseteq L$. It follows that $I_P^M(N) \subseteq L$, as required.

(b) By Lemma 2.3, there exists $r \in R - P$ such that $rN \subseteq I_P^M(N)$. Now $rN \subseteq I_P^M(N) \subseteq N$ implies that

$$Hom_R(R_P, rN) \subseteq Hom_R(R_P, I_P^M(N)) \subseteq Hom_R(R_P, N).$$

As $r \in R - P$, one can see that $Hom_R(R_P, rN) = Hom_R(R_P, N)$. Therefore,** $Hom_R(R_P, N) = Hom_R(R_P, I_P^M(N))$.

(c) Clearly, $Ann_R(N) \subseteq S_P(Ann_R(N))$. Now let $r \in S_P(Ann_R(N))$. Then there exists $s \in R - P$ such that $rs \in Ann_R(N)$ and so $rsN = (\mathbf{0})$. Thus for each $i \in I$, $rsN \subseteq L_i$, where $\{L_i\}_{i \in I}$ is the collection of all completely irreducible submodules of M. Hence $sN \subseteq (L_i :_M r)$ for each $i \in I$. This implies that $I_P^M(N) \subseteq (L_i :_M r)$ for each $i \in I$ because $(L_i :_M r)$ is a completely irreducible submodule of M by Lemma 2.1. Therefore, $rI_P^M(N) \subseteq \cap_{i \in I} L_i = (\mathbf{0})$. Thus $r \in Ann_R(I_P^M(N))$.

(d) Clearly, $Ann_R(I_P^M(N)) \subseteq S_P(Ann_R(I_P^M(N)))$. Now let $r \in S_P(Ann_R(I_P^M(N)))$. Then there exists $s \in R - P$ such that $rs \in Ann_R(I_P^M(N))$ and so $rsI_P^M(N) = (\mathbf{0})$. As M is an Artinian R-module, there exists $t \in R - P$ such that $tN \subseteq I_P^M(N)$ by Lemma 2.3. Therefore, $strN = (\mathbf{0})$. This implies that for each $i \in I$, $stN \subseteq (L_i :_M r)$, where $\{L_i\}_{i \in I}$ is the collection of all completely irreducible submodules of M. Hence $I_P^M(N) \subseteq (L_i :_M r)$. Therefore, $rI_P^M(N) \subseteq \cap_{i \in I} L_i = (\mathbf{0})$. Hence $r \in Ann_R(I_P^M(N))$, as required. □

2.5. Definition. We say that a submodule N of an R-module M is *cotorsion-free with respect to (w.r.t.)* P if $I_P^M(N) = N$, where $P \in Spec(R)$.

2.6. Lemma. Let N ba a submodule of an R-module M and $P \in Spec(R)$. If N is cotorsion-free w.r.t. P, then N is cotorsion-free w.r.t. Q for each $Q \in V(P)$.

Proof. Since $P \subseteq Q$, $I_P^M(N) \subseteq I_Q^M(N)$. Therefore, $N = I_P^M(N) \subseteq I_Q^M(N) \subseteq N$. Hence $N = I_P^M(N) = I_Q^M(N)$ for each $Q \in V(P)$. □

A non-zero R-module M is said to be *secondary* if for each $a \in R$, the endomorphism $M \xrightarrow{a} M$ is either surjective or nilpotent (see [10]). Clearly, every second module is a secondary module.

2.7. Example. (1) If $P \in Spec(R)$, then every P-secondary submodule of an R-module M is cotorsion-free w.r.t. P by [4, 2.8].
(2) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is cotorsion-free w.r.t. (0).

2.8. Corollary. Let $P \in Spec(R)$ and N be a submodule of an R-module M. If N is cotorsion-free w.r.t. P, then $Ann_R(I_P^M(N)) = S_P(Ann_R(I_P^M(N)))$.

Proof. The results follows from part (c) of Theorem 2.4 because $N = I_P^M(N)$. □

The *cosupport* of an R-module M [12] is denoted by $Cosupp(M)$ and it is defined by

$$Cosupp(M) = \{P \in Spec(R) | P \supseteq Ann_R(L) \ for\ some\ cocyclic$$
$$homomorphic\ image\ L\ of\ M\}.$$

2.9. Theorem. Let $P \in Spec(R)$ and N be a submodule of an Artinian R-module M. Then we have the following.

(1) $Ann_{R_P}(Hom_R(R_P, N)) = (Ann_R(I_P^M(N)))_P$.
(2) The following statements are equivalent.
  (a) $Hom_R(R_P, N) \neq (\mathbf{0})$.
  (b) $Ann_R(I_P^M(N)) \subseteq P$.
  (c) $I_P^M(N) \neq (\mathbf{0})$.
  (d) $P \in Cosupp_R(N)$.

Proof. (1) By Theorem 2.4 (b), $Hom_R(R_P, I_P^M(N)) = Hom_R(R_P, N)$. It is easy to see that

$$(Ann_R(I_P^M(N)))_P \subseteq Ann_{R_P}(Hom_R(R_P, I_P^M(N)).$$

To see the reverse inclusion, we note that $I_P^M(I_P^M(N)) = \phi(Hom_R(R_P, I_P^M(N)))$ by [2, 2.15], where $\phi : Hom_R(R_P, I_P^M(N)) \to I_P^M(N)$ is the natural homomorphism defined by $\phi(f) = f(1_{R_P})$ for any $f \in Hom_R(R_P, I_P^M(N))$. Now by Theorem 2.4 (a), $I_P^M(N) = \phi(Hom_R(R_P, I_P^M(N)))$. But always we have

$$Ann_R(Hom_R(R_P, I_P^M(N))) \subseteq Ann_R(\phi(Hom_R(R_P, I_P^M(N))).$$

Hence $Ann_R(Hom_R(R_P, I_P^M(N))) \subseteq Ann_R(I_P^M(N))$. Therefore,

$$Ann_{R_P}(Hom_R(R_P, I_P^M(N))) \subseteq (Ann_R(I_P^M(N)))_P,$$

as required.

(2) $(a) \Leftrightarrow (d)$. By [12, 2.3], $Cosupp_R(N) = V(Ann_R(N))$ and by [11, p. 130], $Cos_R(N) = V(Ann_R(N))$, where $Cos_R(N) = \{P \in Spec(R) : Hom_R(R_P, N) \neq (\mathbf{0})\}$. Hence we get the equivalence (a) and (d).

$(b) \Rightarrow (c)$. This is clear.

$(a) \Rightarrow (b)$. $Hom_R(R_P, N) \neq (\mathbf{0}) \Leftrightarrow Ann_{R_P}(Hom_R(R_P, N)) \neq R_P$. Thus by using part (1), we have

$$Hom_R(R_P, N) \neq (\mathbf{0}) \Leftrightarrow (Ann_R(I_P^M(N)))_P \neq R_P \Leftrightarrow Ann_R(I_P^M(N)) \subseteq P.$$

$(c) \Rightarrow (a)$. If $Hom_R(R_P, N) = (\mathbf{0})$, then $Hom_R(R_P, I_P^M(N)) = (\mathbf{0})$. Thus by [2, 2.15],

$$I_P^M(N) = I_P^M(I_P^M(N)) = \phi(Hom_R(R_P, I_P^M(N)) = (\mathbf{0}),$$

where $\phi : Hom_R(R_P, I_P^M(N)) \to I_P^M(N)$ is the natural homomorphism defined by $\phi(f) = f(1_{R_P})$ for any $f \in Hom_R(R_P, I_P^M(N))$. This contradiction completes the proof. □

We need the following lemma.

2.10. Lemma. [7, 2.2] Let I be an ideal of R and $P \in Spec(R)$. Then the following statements are equivalent.

(a) $S_P(I)$ is a P-primary ideal of R.
(b) $\sqrt{S_P(I)} = P$.
(c) P is a minimal prime ideal of I.

2.11. Theorem. Let $P \in Spec(R)$ and N be a submodule of an Artinian R-module M. Then the following statements are equivalent.

(a) $I_P^M(N)$ is a P-secondary submodule of M.
(b) $Ann_R(I_P^M(N))$ is a P-primary ideal of R.
(c) $\sqrt{Ann_R(I_P^M(N))} = P$.

In particular, $I_P^M(N)$ is P-second if and only if $Ann_R(I_P^M(N)) = P$.

Proof. $(a) \Rightarrow (b)$. This is clear.

$(b) \Rightarrow (a)$. Since $Ann_R(I_P^M(N))$ is a P-primary ideal of R and $I_P^M(I_P^M(N)) = I_P^M(N)$ by Theorem 2.4 (a), $I_P^M(N)$ is a P-secondary submodule of M by [4, 2.2].

$(b) \Rightarrow (c)$. This is elementary.

$(c) \Rightarrow (b)$. Put $I = Ann_R(I_P^M(N))$. Then by Theorem 2.4 (d), $S_P(I) = I$. Now, we have $\sqrt{I} = P = \sqrt{S_P(I)}$ by the hypothesis. It follows from Lemma 2.10 that $S_P(I)$ is a P-primary ideal of R. Hence $I = S_P(I) = Ann_R(I_P^M(N))$ is a P-primary ideal of R, as required. □

2.12. Definition. Let M be an R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$. We say the pair (N, P) satisfies *property* $(**)$ if $S_P(Ann_R(N)) = Ann_R(I_P^M(N)) \neq R$. We say the module M satisfies *property* $(**)$ if for every $(\mathbf{0}) \neq N \leq M$ and $P \in V(Ann_R(N))$ the pair (N, P) satisfies *property* $(**)$.

2.13. Remark. (a) For every $N \leq M$ and $P \in Spec(R)$, if $Ann_R(N) \not\subseteq P$, then $I_P^M(N) = (\mathbf{0})$ because there exists $r \in R - P$ such that $rN = (\mathbf{0})$. Hence for each $i \in I$, $rN \subseteq L_i$, where $\{L_i\}_{i \in I}$ is the set of all completely irreducible submodules of M. Therefore, $I_P^M(N) \subseteq \cap_{i \in I} L_i = (\mathbf{0})$. However, the converse is not true in general. As a counter example, take the $\mathbb{Z}$-module $\mathbb{Z}$ as M, $N = \mathbb{Z}$, and $P = (0)$.

(b) Let M be an R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$. If a pair (N, P) satisfies property $(**)$, then by part (a), we have $Ann_R(N) \subseteq P$.

2.14. Example. (a) The $\mathbb{Z}$-module $\mathbb{Z}$ does not satisfy property $(**)$ because $(\mathbb{Z}, (0))$ does not satisfy this property.

(b) Let N be a non-zero submodule of an R-module M and let P be a prime ideal of R. If N is cotorsion-free w.r.t. P, then (N, P) satisfies property $(**)$. This is because $I_P^M(N) = N \neq (\mathbf{0})$ implies that $Ann_R(I_P^M(N)) = Ann_R(N) \neq R$ and hence by Corollary 2.8, we have

$$Ann_R(N) = S_P(Ann_R(N)) = Ann_R(I_P^M(N)) \neq R.$$

Moreover, not only (N, P), but also (N, Q) for each $Q \in V(P)$ satisfies property $(**)$ by Lemma 2.6. In particular, every P-secondary submodule S of M and each $Q \in V(P) = V(Ann_R(S))$ satisfies property $(**)$ by Example 2.7.

2.15. Theorem. Every non-zero Artinian R-module M satisfies property $(**)$.

Proof. Let $(\mathbf{0}) \neq N \leq M$ and $P \in V(Ann_R(N))$. By Lemma 2.3, there exists $t \in R - P$ such that $tN \subseteq I_P^M(N)$. Now let $r \in Ann_R(I_P^M(N))$. Then $rtN = (\mathbf{0})$. Hence $r \in S_P(Ann_R(N))$. Thus $R \neq Ann_R(I_P^M(N)) \subseteq S_P(Ann_R(N))$. The reverse inclusion follows from Theorem 2.4 (c). □

2.16. Remark. Those modules M which satisfy property $(**)$ are not necessarily Artinian. For example, every vector space W satisfies property $(**)$ even it is of infinite dimensional. This is due to that every non-zero subspace U of W is (0)-second with $V(Ann_R(U)) = \{(0)\}$.

2.17. Corollary. Let M be an Artinian R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$.

(1) The following statements are equivalent.
  (a) $I_P^M(N)$ is a P-secondary submodule of M.
  (b) $\sqrt{S_P(Ann_R(N))} = P$.
  (c) P is a minimal prime ideal of $Ann_R(N)$.

(2) $I_P^M(N)$ is a P-second submodule of M if and only if $S_P(Ann_R(N)) = P$.

In particular, if $Ann_R(N) = P$, then $I_P^M(N)$ is a P-second submodule of M.

Proof. The proof is straightforward from Theorem 2.11, Lemma 2.10, and Theorem 2.4. □

3. Maximal second submodules

A submodule N of an R-module M is said to be a *maximal second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a second submodule L of M such that $N \subset L \subset K$ (see [1]).

3.1. Lemma. Let R be an integral domain and let M be an Artinian non-zero R-module.

(a) If $I^M_{(0)}(M) \neq (\mathbf{0})$, then $I^M_{(0)}(M)$ is a maximal (0)-second submodule of M and it contains every (0)-second submodule of M.
(b) $I^M_{(0)}(M) = M$ if and only if M is a (0)-second submodule of M.

Proof. (a) This follows from [1, 2.9] and [3, 2.10].
(b) This follows from part (a) and [3, 2.10]. □

3.2. Theorem. Let R be an integral domain of dimension 1, M be a non-zero Artinian R-module and $(0) \neq P \in V(Ann_R(M))$. Then $I^M_P((0 :_M P))$ is a maximal second submodule of M if and only if $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$.

Proof. Since $(0) \subset P \subseteq Ann_R((0 :_M P))$, $dimR = 1$, and R is a domain, it follows that if $Ann_R((0 :_M P)) \neq R$, then $Ann_R((0 :_M P)) = P$. Hence $I^M_P((0 :_M P))$ is a second submodule of M by [1, 2.8].
Suppose that $I^M_P((0 :_M P))$ is a maximal second submodule of M. Then there are two cases:

(i) $I^M_P((0 :_M P)) = M$ and
(ii) $I^M_P((0 :_M P)) \neq M$.

In case (i), M is a P-second submodule for $P \neq (0)$. Consequently, $I^M_{(0)}(M) \neq M$ by Lemma 3.1 (b). Hence $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$.
In case (ii), $I^M_P((0 :_M P))$ is a proper maximal second submodule of M. Hence M is not a second submodule, in particular, it is not a (0)-second submodule so that $I^M_{(0)}(M) \neq M$ by Lemma 3.1 (b) again. Thus if $I^M_{(0)}(M) \neq (\mathbf{0})$, then $I^M_{(0)}(M)$ is a proper maximal (0)-second submodule of M by Lemma 3.1 (a). Consequently, $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$ by the maximality of $I^M_P((0 :_M P))$ in M. On the other hand, if $I^M_{(0)}(M) = (\mathbf{0})$, then obviously, $I^M_P((0;_M P)) \not\subseteq I^M_{(0)}(M)$.

Conversely, suppose that $I^M_P((0;_M P)) \not\subseteq I^M_{(0)}(M)$. Then clearly $I^M_{(0)}(M) \neq M$. Thus by Lemma 3.1 (b), M is not a (0)-second submodule. To see that $I^M_P((0 :_M P))$ is a maximal second submodule of M, let K be a second submodule of M such that $I^M_P((0 :_M P)) \subseteq K \subseteq M$. Then

$$(0) \subseteq Ann_R(M) \subseteq Ann_R(K) \subseteq Ann_R(I^M_P((0 :_M P))) = P.$$

Since $dimR = 1$, the prime ideal $Ann_R(K) = (0)$ or P. If $Ann_R(K) = (0)$, then K is a (0)-second submodule. However, $K \neq M$ because M is not a (0)-second submodule as we have seen above. Since every proper (0)-second submodule contained in $I^M_{(0)}(M)$, we have that $I^M_P((0 :_M P)) \subseteq K \subseteq I^M_{(0)}(M) \neq (\mathbf{0})$ which contradicts to $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$. Therefore, $Ann_R(K) = P$, i.e., K is a P-second submodule. Thus $K = I^M_P(K) \subseteq I^M_P((0 :_M P))$. Therefore, $K = I^M_P((0 :_M P))$. This proves that $I^M_P((0 :_M P))$ is a maximal second submodule of M. □

3.3. Proposition. Let Y be a set of prime ideals of R which contains all the maximal ideals, M be an Artinian R-module, and N be a non-zero submodule of M. Then $N = \sum_{P \in Y} I^M_P(N)$.

Proof. Let L be a completely irreducible submodule of M such that $\sum_{P \in Y} I^M_P(N) \subseteq L$ so that $I^M_P(N) \subseteq L$ for every $P \in Y$. Hence by Lemma 2.3, we have $(L :_R N) \not\subseteq P$ for every $P \in Y$. This implies that $(L :_R N) \not\subseteq m$ for every maximal ideal $m \in Y$. This in turn implies that $(L :_R N) = R$ and hence $N \subseteq L$. Thus $N \subseteq \sum_{P \in Y} I^M_P(N)$. The reverse inclusion is clear. □

3.4. Corollary. Let (R, m) be a local ring, M an Artinian R-module, and $(\mathbf{0}) \neq N \leq M$. Then N is cotorsion-free w.r.t. m.

Proof. Take $Y = \{m\}$ in Proposition 3.3. Then we have $I^M_m(N) = N$. $\square$

Let N be a submodule of an R-module M. The (*second*) *socle* of N is defined as the sum of all second submodules of M contained in N and it is denoted by $soc(N)$ or $sec(N)$ (see [1] and [5]). In case N does not contain any second submodule, the socle of N is defined to be $(\mathbf{0})$.

3.5. Proposition. Let M be an Artinian R-module, $P \in Spec(R)$, and $(\mathbf{0}) \neq N \leq M$. If P is a minimal prime ideal of $Ann_R(N)$ and $I^M_P((0 :_N P)) \neq (\mathbf{0})$, then $I^M_P((0 :_N P))$ is a maximal second submodule of $K \leq M$ with $I^M_P((0 :_N P)) \subseteq K \subseteq N$. In particular $I^M_P((0 :_N P))$ is a maximal P-second submodule of $sec(N)$.

Proof. Since $I^M_P((0 :_N P)) \neq (\mathbf{0})$, $I^M_P((0 :_N P))$ is a maximal P-second submodule of $(0 :_N P)$ by [1, 2.9]. Now suppose that K is a submodule of M such that $I^M_P((0 :_N P)) \subseteq K \subseteq N$ and S is a Q-second submodule of M such that $I^M_P((0 :_N P)) \subseteq S \subseteq K \subseteq N$. Then as P is a minimal prime ideal of $Ann_R(N)$, we have $Q = P$. Thus $S \subseteq (0 :_N P)$. It follows that $S = I^M_P((0 :_N P))$ as desired. The last assertion follows from the fact that $I^M_P((0 :_N P)) \subseteq sec(N) \subseteq N$. So the proof is completed.

$\square$

The following example shows that the condition $I^M_P((0 :_N P)) \neq (\mathbf{0})$ in the statement of Proposition 3.5 can not be dropped.

3.6. Example. Consider $M = N = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module, where p is a prime number. Let $q \neq p$ be an another prime number. Then clearly, $q\mathbb{Z}$ is a minimal prime ideal of $Ann_{\mathbb{Z}}(M)$ and $I^M_{(q)}((0 :_N q\mathbb{Z})) = (\mathbf{0})$.

The next theorem gives an important information on the maximal second submodules of an Artinian R-modules.

3.7. Theorem. Let N be a non-zero submodule of an Artinian R-module M. Then every maximal second submodule of N must be of the form $I^M_P((0 :_N P))$ for some $P \in V(Ann_R(N))$.

Proof. Let S be a maximal P-second submodule of N. Then $S \subseteq N$ and $Ann_R(S) = P$ so that $S \subseteq (0 :_N P)$. Therefore, $S = I^M_P(S) \subseteq I^M_P((0 :_N P)) \subseteq N$ by [3, 2.10]. Since $P \in V(Ann_R(N))$, $I^M_P((0 :_N P))$ is a P-second submodule , as we have seen in the proof of Proposition 3.5. Thus $S = I^M_P((0 :_N P))$. $\square$

3.8. Corollary. Let M be an Artinian R-module and $(\mathbf{0}) \neq N \leq M$. Then $sec(N) = \sum_{P \in Y} I^M_P((0 :_N P))$, where Y is a finite subset of $V(Ann_R(N))$.

Proof. By [1, 2.6, 2.2], there exists $n \in \mathbb{Z}$ such that $sec(N) = \sum^n_{i=1} S_i$, where for $1 \leq i \leq n$, S_i is a maximal second submodule of N. Now the proof follows from Theorem 3.7. We remark that this corollary is also a direct consequence of [3, Proposition 2.7 (a)]. $\square$

3.9. Corollary. Let N be a non-zero submodule of an Artinian R-module M. If $I^M_P((0 :_N P)) \neq (\mathbf{0})$ and N is a P-secondary submodule of an R-module M for some $P \in Spec(R)$, then we have the following.

(a) $I^M_P((0 :_N P))$ is a maximal P-second submodule of $sec(N)$.
(b) If P is a maximal ideal of R, then $sec(N) = I^M_P((0 :_N P))$ so that $sec(N)$ is a P-second submodule of M.

Proof. (a) This follows from Proposition 3.5 because P is a minimal prime ideal of $Ann_R(N)$.

(b) By Corollary 3.8, $sec(N) = \sum_{Q \in V(Ann_R(N))} I_Q^M((0:_N Q))$. Since P is maximal and $\sqrt{Ann_R(N)} = P$, $V(Ann_R(N)) = \{P\}$. Thus $sec(N) = I_P^M((0:_N P))$ as required. $\square$

3.10. Corollary. Let I be an ideal of R and M be an Artinian R-module such that $(0:_M I) \neq (\mathbf{0})$. Then $sec((0:_M I)) = \sum_{P \in V(Ann_R((0:_M I)))} I_P^M((0:_M P))$.

Proof. Set $N = (0:_M I)$. Then this follows from Corollary 3.8 since, $(0:_{(0:_M I)} P) = (0:_M P)$ for every $P \in V(Ann_R((0:_M I)))$. $\square$

3.11. Example. For any prime integer p, let $M = (\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}_{p^\infty}$. Then M is an Artinian faithful $\mathbb{Z}$-module and $V(Ann_{\mathbb{Z}}(M)) = V((0)) = Spec(\mathbb{Z})$. Hence $sec(M) = \sum_{(q) \in V((0))} I_{(q)}^M((0:_M q\mathbb{Z}))$ by Corollary 3.10. Since $I_{(q)}^M((0:_M q\mathbb{Z})) = I_{(q)}^M(\mathbf{0}) = (\mathbf{0})$ for each prime number $p \neq q$,

$$\begin{aligned} sec(M) &= I_{(0)}^M(M) + I_{(p)}^M((0:_M p\mathbb{Z})) \\ &= ((0) \times \mathbb{Z}_{p^\infty}) + ((\mathbb{Z}/p\mathbb{Z}) \times < 1/p + \mathbb{Z} >) \\ &= M. \end{aligned}$$

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