

μ -paracompact and g_μ -paracompact generalized topological spaces

A. Deb Ray* and Rakesh Bhowmick†

Abstract

This paper defines generalizations of paracompactness on generalized topological spaces (GTS) and establishes that paracompactness, near paracompactness and several other paracompact-like properties follow as special cases, by choosing the GT suitably. Also, the generalizations of locally finite and closure preserving collections in a GTS, have been studied, pointing out their interrelations. Finally, it has been observed that the celebrated theorem of E. Michael in the context of regular paracompact spaces follow as a corollary to a result achieved in this paper.

Keywords: γ_μ -closure, μ -locally finite, g_μ -locally finite, μ -paracompact, g_μ -paracompact, γ_μ -regular.

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1. Introduction & Preliminaries

Paracompactness [2] is a very natural and perhaps the most successful generalization of compactness. Various eminent mathematicians of different times have studied several stronger as well as weaker forms of paracompactness, the most widely investigated one being near paracompactness [5]. The main purpose of this paper is to define a generalization of paracompactness on generalized topological spaces (GTS) which is a wider framework than topological spaces; and establish that by choosing the GT suitably paracompactness as well as near paracompactness follow as special cases. Also, it has been observed that by suitably choosing the generalized topology one may think of various paracompact-like spaces other than the two mentioned above.

*Department of Mathematics, West Bengal State University, Barasat, North 24 Parganas, Kolkata - 700126, INDIA;
Email : atasi@hotmail.com

†Department of Mathematics, West Bengal State University, Barasat, North 24 Parganas, Kolkata - 700126, INDIA;
Email : r.bhowmick88@gmail.com

In section 2, we introduce a closure operator γ_μ on a GTS (X, μ) and find certain relationships among the generalized closure operator on (X, μ) and the newly defined one. We have also generalized and studied local finite and closure preserving collections of sets with respect to the GT μ and the operator γ_μ .

In section 3, we define and investigate generalization of paracompactness which we have called μ -paracompactness and g_μ -paracompactness. The celebrated theorem of E. Michael in the context of regular paracompact spaces follow as a corollary to a result achieved in this paper for more general setting what we have called γ_μ -regular g_μ -paracompact GTS. Let X be a nonempty set and μ be a collection of subsets of X (i.e. $\mu \subseteq \mathcal{P}(X)$). μ is called a generalized topology (briefly GT) [1] on X iff $\phi \in \mu$ and $G_\lambda \in \mu$ for $\lambda \in \Lambda (\neq \phi)$ implies $\cup_{\lambda \in \Lambda} G_\lambda \in \mu$. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ -open sets and their complements are called μ -closed sets. The generalized closure of a subset S of X , denoted by $c_\mu(S)$, is the intersection of all μ -closed sets containing S . The set of all μ -open sets containing an element $x \in X$ is denoted by $\mu(x)$. The set of all open, δ -open [7] and θ -open [7], subsets of X are denoted respectively by $\tau(X)$ (or τ), $\Delta(X)$ (or Δ) and $\Theta(X)$ (or Θ). In what follows we shall denote the set of all natural numbers, integers and real numbers respectively by \mathbb{N}, \mathbb{Z} and \mathbb{R} .

2. Generalized local finite and Generalized closure preserving collection

Before generalizing locally finite and closure preserving collections we introduce a new operator on a GTS (X, μ) and show that such operator actually give rise to a topology on X .

2.1. Definition. Let (X, μ) be a GTS. Then for each $x \in X$ we define $\mu^*(x) = \{\cap_{i=1}^n W_i : W_i \in \mu(x), \forall i = 1, 2, \dots, n; n \in \mathbb{N}\}$

2.1. Remark. For any $x \in X$, $\mu(x) \subseteq \mu^*(x)$ and $\mu^*(x)$ is closed under finite intersection.

2.2. Definition. Let (X, μ) be a GTS. Then γ_μ -closure of a subset S of X , denoted by $\gamma_\mu(S)$ is defined by $\gamma_\mu(S) = \{x \in X : V \cap S \neq \phi \text{ for all } V \in \mu^*(x)\}$

The table below shows that how γ_μ -closure operator unifies several closure type operator.

μ	γ_μ
$P(X)$	identity operator
τ	closure operator
Δ	δ -closure operator [7]
Θ	θ -closure operator [7]

In a GTS (X, μ) γ_μ -closure operator satisfies the following properties (i) $\gamma_\mu(\phi) = \phi$, (ii) $S \subseteq X \Rightarrow S \subseteq \gamma_\mu(S) \subseteq c_\mu(S)$ and $\gamma_\mu(\gamma_\mu(S)) = \gamma_\mu(S)$, (iii) $A \subseteq B \subseteq X \Rightarrow \gamma_\mu(A) \subseteq \gamma_\mu(B)$ and $\gamma_\mu(A \cup B) = \gamma_\mu(A) \cup \gamma_\mu(B)$. Clearly γ_μ is a closure operator on X and hence give rise to a topology on X , denoted by μ^* and given by $\mu^* = \{S \subseteq X : \gamma_\mu(X \setminus S) = X \setminus S\}$. The elements of μ^* are called μ^* -open sets and the complements are called μ^* -closed sets. In fact for every $x \in X$, $W \in \mu^*(x)$ is a μ -open set. From now we may call the elements of $\mu^*(x)$ the open neighbourhoods of x .

In particular, if μ itself is a topology on X then $\mu = \mu^*$. Otherwise μ^* is finer than GT μ .

2.1. Example. Let us consider the set $X = \{a, b, c\}$. Then $\mu = \{\phi, \{a, b\}, \{a, c\}, X\}$ is clearly a GT on X . Let $S = \{b, c\}$. Now for any $V \in \mu(a)$, $V \cap S \neq \phi$ i.e. $a \in c_\mu(S)$, but $a \notin S$. So $c_\mu(S) \neq S$. Therefore S is not a μ -closed set. Again if we take $V_1 = \{a, b\}$ and $V_2 = \{a, c\}$ then $(V_1 \cap V_2) \cap S = \phi$. i.e. $a \notin \gamma_\mu(S)$. This implies that $S = \gamma_\mu(S)$ (using $S \subseteq \gamma_\mu(S)$ and $X = \{a, b, c\}$). Therefore S is a μ^* -closed set.

With the help of μ -open and μ^* -open sets we generalize the known concepts of local finite and closure preserving collections.

2.3. Definition. A family $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of sets in a GTS (X, μ) is called
 (i) μ -locally finite (resp. g_μ -locally finite) if for each $x \in X$ there exists $V \in \mu(x)$ (resp. $V \in \mu^*(x)$) such that V intersects at most finitely many members of \mathcal{U} i.e. $V \cap U_\alpha \neq \phi$ for at most finitely many indices α .
 (ii) μ -closure preserving (resp. γ_μ -closure preserving) if for any subcollection \mathcal{V} of \mathcal{U} , $c_\mu[\cup\{V : V \in \mathcal{V}\}] = \cup\{c_\mu V : V \in \mathcal{V}\}$ (resp. $\gamma_\mu[\cup\{V : V \in \mathcal{V}\}] = \cup\{\gamma_\mu V : V \in \mathcal{V}\}$).

In general, every μ -locally finite family on a GTS (X, μ) is a g_μ -locally finite family but not conversely.

2.2. Example. Let $X = \mathbb{Z}$. Then $\mu = \{A \subseteq \mathbb{Z} : A \text{ is infinite}\} \cup \{\phi\}$ forms a GT on X . Let us construct $I_n = \{x \in X : x \geq n\}$, $n \in \mathbb{N}$ and $J_n = \{x \in X : x \leq -n\}$, $n \in \mathbb{N}$. Now consider the family $\mathcal{U} = \{I_n\} \cup \{J_n\}$. Then for any $x \in X$, $V \in \mu(x)$ intersects infinitely many members of \mathcal{U} . Therefore \mathcal{U} is not a μ -locally finite family. Again for any $x \in X$ if we take $V_1 = \{y \in X : y \geq x\}$ and $V_2 = \{y \in X : y \leq x\}$ then $V_1, V_2 \in \mu(x)$ and $V_1 \cap V_2 (= \{x\}) \in \mu^*(x)$. If $x > 0$ then $V_1 \cap V_2$ intersects only I_1, I_2, \dots, I_x . If $x < 0$ then $V_1 \cap V_2$ intersects only J_1, J_2, \dots, J_x . If $x = 0$ then $V_1 \cap V_2$ intersects no members of \mathcal{U} . It follows that \mathcal{U} is a g_μ -locally finite family.

But when we take μ as τ then both coincide with locally finite [2]. Moreover, when we take μ as τ then both of μ -closure preserving and γ_μ -closure preserving property coincide with closure preserving.

2.1. Theorem. If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a μ -locally finite (resp. g_μ -locally finite) family on a GTS (X, μ) . Then

- (i) any subcollection of \mathcal{U} is also μ -locally finite (resp. g_μ -locally finite).
- (ii) $c_\mu \mathcal{U} = \{c_\mu(U) : U \in \mathcal{U}\}$ (resp. $\gamma_\mu \mathcal{U} = \{\gamma_\mu(U) : U \in \mathcal{U}\}$) is also μ -locally finite (resp. g_μ -locally finite).

Proof. (i) Straightforward.

(ii) Let $x \in X$. Then since $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is μ -locally finite (resp. g_μ -locally finite), there exists $V \in \mu(x)$ (resp. $V \in \mu^*(x)$) such that $V \cap U_\alpha \neq \phi$ for at most finitely many α 's. Now we show that $V \cap c_\mu(U_\alpha) \neq \phi$ (resp. $V \cap \gamma_\mu(U_\alpha) \neq \phi$) for at most finitely many α 's. Let $y \in V$, then $V \in \mu(y)$ (resp. $V \in \mu^*(y)$) is such that V intersects at most finitely many U_α 's. From the definition of $c_\mu(U_\alpha)$ (resp. $\gamma_\mu(U_\alpha)$), $y \in c_\mu(U_\alpha)$ (resp. $\gamma_\mu(U_\alpha)$) for at most finitely many U_α 's. This implies that $V \cap c_\mu(U_\alpha) \neq \phi$ (resp. $V \cap \gamma_\mu(U_\alpha) \neq \phi$) for at most finitely many α 's, as desired. \square

2.2. Theorem. If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a g_μ -locally finite family on a GTS (X, μ) , then \mathcal{U} is γ_μ -closure preserving.

Proof. Let \mathcal{B} be any subcollection of \mathcal{U} . We show that $\gamma_\mu[\cup\{B : B \in \mathcal{B}\}] = \cup\{\gamma_\mu(B) : B \in \mathcal{B}\}$. Since $\gamma_\mu(B) \subseteq \gamma_\mu[\cup\{B : B \in \mathcal{B}\}]$ for all $B \in \mathcal{B}$, $\cup\{\gamma_\mu(B) : B \in \mathcal{B}\} \subseteq \gamma_\mu[\cup\{B : B \in \mathcal{B}\}]$. Next let $x \notin \cup\{\gamma_\mu(B) : B \in \mathcal{B}\}$. Since \mathcal{B} is a subcollection of a g_μ -locally finite collection \mathcal{U} , \mathcal{B} is also g_μ -locally finite and so there exists $V \in \mu^*(x)$ such that V intersects at most finitely many members of \mathcal{B} , say B_1, B_2, \dots, B_n . Again since $x \notin \cup\{\gamma_\mu(B) : B \in$

$\mathcal{B}\}$, $x \notin \gamma_\mu(B_i)$ for $i = 1, 2, \dots, n$ and so there exist $W_i \in \mu^*(x)$ for $i = 1, 2, \dots, n$ such that $W_i \cap B_i = \phi$. Let $W = V \cap W_1 \cap W_2 \cap \dots \cap W_n$. Since $V, W_1, W_2, \dots, W_n \in \mu^*(x)$, $W \in \mu^*(x)$. So we have $W \in \mu^*(x)$ such that $W \cap [\cup\{B : B \in \mathcal{B}\}] = \phi$. This implies that $x \notin \gamma_\mu[\cup\{B : B \in \mathcal{B}\}]$. Therefore $\gamma_\mu[\cup\{B : B \in \mathcal{B}\}] \subseteq \cup\{\gamma_\mu(B) : B \in \mathcal{B}\}$, consequently $\gamma_\mu[\cup\{B : B \in \mathcal{B}\}] = \cup\{\gamma_\mu(B) : B \in \mathcal{B}\}$. \square

2.1. Corollary. The arbitrary union of μ^* -closed sets from a g_μ -locally finite family in a GTS (X, μ) is also μ^* -closed.

Let (X, μ) be a GTS. Then a family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ on X is said to be a covering of X if $X = \cup_{\alpha \in \Lambda} U_\alpha$. Moreover, if each U_α is μ -open (resp. μ -closed, μ^* -open, μ^* -closed) then \mathcal{U} is called μ -open (resp. μ -closed, μ^* -open, μ^* -closed) covering of X .

Let \mathcal{U} and \mathcal{V} be two covering of X , then \mathcal{V} is said to be subcovering of \mathcal{U} if each member of \mathcal{V} is also a member of \mathcal{U} . Moreover if \mathcal{V} contains finite (resp. countable) number of members, then \mathcal{V} is called finite (resp. countable) subcovering of \mathcal{U} .

Let (X, μ) be a GTS. Then a family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ on X is said to be a point finite covering of X if for each $x \in X$, there exists at most finitely many indices $\alpha \in \Lambda$ such that $x \in A_\alpha$. Moreover, if each member of \mathcal{U} is μ -open then \mathcal{U} is called point finite μ -open covering of X .

Let (X, μ) be a GTS. Let \mathcal{U} and \mathcal{V} be two covering of X , then \mathcal{V} is said to refine (or be a refinement of) \mathcal{U} if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$. If $\mathcal{W} \prec \mathcal{U}$ and $\mathcal{W} \prec \mathcal{V}$ then \mathcal{W} is called common refinement of \mathcal{U} and \mathcal{V} .

2.2. Remark. Each subcovering of a covering is a refinement of that covering.

2.3. Theorem. Let (X, μ) be a GTS. Let $\mathcal{A} = \{A_\alpha : \alpha \in A\}$ and $\mathcal{B} = \{B_\beta : \beta \in B\}$ be two covering of X . Then

- (1) $\mathcal{A} \wedge \mathcal{B} = \{A_\alpha \cap B_\beta : (\alpha, \beta) \in A \times B\}$ is a covering of X , refining both \mathcal{A} and \mathcal{B} .
Furthermore if both \mathcal{A} and \mathcal{B} are μ -locally finite (resp. g_μ -locally finite), so also is $\mathcal{A} \wedge \mathcal{B}$.
- (2) any common refinement of \mathcal{A} and \mathcal{B} is also a refinement of $\mathcal{A} \wedge \mathcal{B}$.

Proof. Straightforward. \square

2.4. Definition. Let (X, μ) be a GTS. A refinement $\{B_\beta : \beta \in \mathcal{B}\}$ of $\{A_\alpha : \alpha \in \mathcal{A}\}$ is called a precise refinement if $\mathcal{A} = \mathcal{B}$ and $B_\alpha \subseteq A_\alpha$, for each α .

2.4. Theorem. Let (X, μ) be a GTS. If a covering $\{A_\alpha : \alpha \in \mathcal{A}\}$ of X has a μ -locally finite (resp. g_μ -locally finite) refinement $\{B_\beta : \beta \in \mathcal{B}\}$ that covers X , then it has a precise μ -locally finite (resp. g_μ -locally finite) refinement $\{C_\alpha : \alpha \in \mathcal{A}\}$ that covers X . Furthermore, if each B_β is μ -open then each C_α can be chosen to be μ -open also.

Proof. Define a map $\phi : \mathcal{B} \rightarrow \mathcal{A}$ by assigning each $\beta \in \mathcal{B}$ to some $\alpha \in \mathcal{A}$ such that $B_\beta \subseteq A_\alpha$. For each α , let $C_\alpha = \cup\{B_\beta : \phi(\beta) = \alpha\}$, some C_α may be empty. Clearly $C_\alpha \subseteq A_\alpha$ for each α i.e $\{C_\alpha : \alpha \in \mathcal{A}\}$ is a refinement of $\{A_\alpha : \alpha \in \mathcal{A}\}$. Also since $\{B_\beta : \beta \in \mathcal{B}\}$ is a covering of X , each B_β appears somewhere $\{C_\alpha : \alpha \in \mathcal{A}\}$ and so $\{C_\alpha : \alpha \in \mathcal{A}\}$ is a covering of X . Again since $\{B_\beta : \beta \in \mathcal{B}\}$ is μ -locally finite (resp. g_μ -locally finite), for each $x \in X$ there exists $V \in \mu(x)$ (resp. $V \in \mu^*(x)$) such that V intersects at most finitely many B_β 's and consequently finitely many C_α 's. This implies that $\{C_\alpha : \alpha \in \mathcal{A}\}$ is μ -locally finite (resp. g_μ -locally finite). Hence the first part follows. For the second part, if each B_β is μ -open then clearly each C_α is also μ -open. \square

2.3. Remark. In the above theorem μ -locally finite (resp. g_μ -locally finite) can be replaced by point finite.

2.5. Theorem. Let $\{E_\alpha : \alpha \in \Lambda\}$ be any family of sets on a GTS (X, μ) and $\{B_\beta : \beta \in \mathcal{B}\}$ be any g_μ -locally finite μ^* -closed covering of X . If each B_β intersects at most finitely many sets E_α , then each E_α is contained in a μ^* -open set $U(E_\alpha)$ such that the family $\{U(E_\alpha) : \alpha \in \Lambda\}$ is g_μ -locally finite.

Proof. For each α define $U(E_\alpha) = X \setminus \cup \{B_\beta : B_\beta \cap E_\alpha = \phi\}$. Since, $\{B_\beta\}$ is g_μ -locally finite family of μ^* -closed sets $U(E_\alpha)$ is μ^* -open (since, $\cup \{B_\beta : B_\beta \cap E_\alpha = \phi\}$ is μ^* -closed, by corollary 2.1). Also $E_\alpha \subseteq U(E_\alpha)$ (since, $x \notin U(E_\alpha) \Rightarrow \exists x \in B_{\beta_0}$ for some $\beta_0 \in \mathcal{B}$ such that $B_{\beta_0} \cap E_\alpha = \phi$. Again $x \in B_{\beta_0}$ and $B_{\beta_0} \cap E_\alpha = \phi \Rightarrow x \notin E_\alpha$. i.e. $x \notin U(E_\alpha) \Rightarrow x \notin E_\alpha$).

We now prove that $\{U(E_\alpha) : \alpha \in \Lambda\}$ is g_μ -locally finite. Since, $\{B_\beta : \beta \in \mathcal{B}\}$ is g_μ -locally finite, for any given $x \in X$ there exists $V \in \mu^*(x)$ such that V intersects at most finitely many B_β 's say $B_{\beta_1}, B_{\beta_2}, \dots, B_{\beta_n}$. Obviously V is contained in $\cup_{i=1}^n B_{\beta_i}$, as $\{B_\beta\}$ forms a covering of X . Since $B_\beta \cap U(E_\alpha) \neq \phi$ iff $B_\beta \cap E_\alpha \neq \phi$ (since, $B_\beta \cap E_\alpha \neq \phi$ iff $B_\beta \not\subseteq \cup \{B_\beta : B_\beta \cap E_\alpha = \phi\}$ iff $B_\beta \cap (X \setminus \cup \{B_\beta : B_\beta \cap E_\alpha = \phi\}) \neq \phi$ iff $B_\beta \cap U(E_\alpha) \neq \phi$) and each $B_{\beta_i}, i = 1, 2, \dots, n$ intersects at most finitely many E_α , $\cup_{i=1}^n B_{\beta_i}$ intersects at most finitely many $U(E_\alpha)$. Thus we have $V \in \mu^*(x)$ such that V intersects at most finitely many $U(E_\alpha)$ (since, $V \subseteq \cup_{i=1}^n B_{\beta_i}$) and so $\{U(E_\alpha) : \alpha \in \Lambda\}$ is g_μ -locally finite. \square

2.2. Corollary. Let $\{E_\alpha : \alpha \in \Lambda\}$ be any family of sets on a GTS (X, μ) with $\mu = \mu^*$ and $\{B_\beta : \beta \in \mathcal{B}\}$ be any μ -locally finite μ -closed covering of X . If each B_β intersects at most finitely many sets E_α , then each E_α is contained in a μ -open set $U(E_\alpha)$ such that the family $\{U(E_\alpha) : \alpha \in \Lambda\}$ is μ -locally finite.

3. μ -paracompactness and g_μ -paracompactness

In this section we define generalized paracompactness to unify the existing concept of paracompact and nearly paracompact spaces. We see that many more paracompact-like properties may also be obtained by choosing the generalized topology suitably.

3.1. Definition. A GTS (X, μ) is said to be μ -paracompact (resp. g_μ -paracompact) if every μ -open covering of X has a μ -locally finite (resp. g_μ -locally finite) μ -open refinement that covers X .

3.1. Remark. g_μ -paracompactness is a generalization of μ -paracompactness, since every μ -paracompact GTS is a g_μ -paracompact GTS, but not conversely in general. If we take μ as τ then both μ -paracompact and g_μ -paracompact coincide with paracompact. If we take μ as Δ then both coincide with nearly paracompact.

3.2. Definition. A GTS (X, μ) is said to be μ -compact [6] (resp. μ -Lindelöf) if every μ -open covering of X has a finite (resp. countable) subcovering.

3.2. Remark. In general, every μ -compact GTS (X, μ) is μ -Lindelöf, but not conversely.

3.1. Theorem. Let (X, μ) be a GTS. If (X, μ) is μ -compact then it is also μ -paracompact.

Proof. Straightforward. \square

The converse of above theorem is not true in general, which follows from the following example:

3.1. Example. Let $X = \mathbb{Z}$, $\mu =$ discrete topology on X . then $\{\{n\} : n \in \mathbb{Z}\}$ is a μ -open covering of X which has no finite subcover but every μ -open cover of X has a μ -locally finite μ -open refinement $\{\{n\} : n \in \mathbb{Z}\}$ that covers X (since, $\{\{n\} : n \in \mathbb{Z}\}$ is a refinement of every μ -open cover of X).

3.3. Definition. [3] Let (X, μ) be a GTS. Then (X, μ) is said to be μ -regular if for any $x \in X$ and a μ -closed set F not containing x , there exist two disjoint μ -open sets U and V such that $x \in U$ and $F \subseteq V$.

3.4. Definition. Let (X, μ) be a GTS. Then (X, μ) is said to be γ_μ -regular if for any $x \in X$ and a μ -open set U containing x , there exists a μ -open set V containing x such that $\gamma_\mu(V) \subseteq U$.

3.2. Theorem. Let (X, μ) be a GTS. If (X, μ) is μ -regular then it is also γ_μ -regular.

Proof. Straightforward. \square

The converse is not necessarily true. This is observed in the following example:

3.2. Example. Let $X = \{a, b, c\}$ and $\mu = \{\phi, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then μ be a GT on X . Here $c_\mu(U) = X$ and $\gamma_\mu(U) = U$ for every μ -open set containing $x \in X$. It is easy to check that X is γ_μ -regular but not μ -regular.

3.3. Theorem. For any γ_μ -regular GTS (X, μ) , (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) hold, where

- (1) (X, μ) is g_μ -paracompact.
- (2) Every μ -open cover of X has a μ -open refinement that covers X and can be decomposed into at most countable collection of g_μ -locally finite families of μ -open sets.
- (3) Each μ -open cover of X has a g_μ -locally finite refinement that cover X .
- (4) Each μ -open cover of X has a μ^* -closed g_μ -locally finite refinement that covers X .

Proof. (1) \Rightarrow (2) Straightforward.

(2) \Rightarrow (3) Let $\{U_\beta : \beta \in \mathcal{B}\}$ be any μ -open covering of X . By (2) there exists an μ -open covering $\{V_{n,\alpha} : (n,\alpha) \in \mathbb{N} \times \mathcal{A}\}$, which is a refinement of $\{U_\beta : \beta \in \mathcal{B}\}$, where for each $n_0 \in \mathbb{N}$, the family $\{V_{n_0,\alpha} : \alpha \in \mathcal{A}\}$ is g_μ -locally finite (not necessarily a covering). For each $n \in \mathbb{N}$, let $W_n = \cup_\alpha V_{n,\alpha}$, then $\{W_n, n \in \mathbb{N}\}$ is a μ -open covering of X . For each $i \in \mathbb{N}$ define $A_i = W_i \setminus \cup_{j=1}^{i-1} W_j$. We now show that $\{A_i\}$ is a g_μ -locally finite covering of X . For each $x \in X$, let W_{i_0} is the first member of $\{W_n, n \in \mathbb{N}\}$ such that $x \in W_{i_0}$. Then it is clear that $x \in A_{i_0}$, hence $\{A_i\}$ is a covering of X . Again $W_{i_0} \cap A_i = \phi$ for each $i > i_0$ i.e. we have $W_{i_0} \in \mu(x) \subseteq \mu^*(x)$ such that W_{i_0} intersects at most finitely many members of $\{A_i\}$. Hence $\{A_i\}$ is g_μ -locally finite.

We now show that $\mathcal{K} = \{A_n \cap V_{n,\alpha} : (n,\alpha) \in \mathbb{N} \times \mathcal{A}\}$ is g_μ -locally finite refinement of $\{U_\beta : \beta \in \mathcal{B}\}$ that covers X . For any $A_n \cap V_{n,\alpha} \in \mathcal{K}$, since $\{V_{n,\alpha} : (n,\alpha) \in \mathbb{N} \times \mathcal{A}\}$ is a refinement of $\{U_\beta : \beta \in \mathcal{B}\}$ there exists U_β such that $A_n \cap V_{n,\alpha} \subseteq V_{n,\alpha} \subseteq U_\beta$. Hence \mathcal{K} is a refinement of $\{U_\beta : \beta \in \mathcal{B}\}$. Again \mathcal{K} is obviously a covering of X (since for $x \in X$, $\exists A_n$ such that $x \in A_n \Rightarrow x \in W_n = \cup_\alpha V_{n,\alpha} \Rightarrow x \in V_{n,\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$ i.e. $x \in A_n \cap V_{n,\alpha_0}$ for some $(n,\alpha_0) \in \mathbb{N} \times \mathcal{A}$). Next let $x \in X$. Then since $\{A_n : n \in \mathbb{N}\}$ is g_μ -locally finite, there exists $W \in \mu^*(x)$ such that W intersects at most finitely many member of $\{A_n : n \in \mathbb{N}\}$ say, $A_{n_1}, A_{n_2}, \dots, A_{n_r}$. Again since $\{V_{n_j,\alpha} : \alpha \in \mathcal{A}\}$, (for $j = 1, 2, \dots, r$) is g_μ -locally finite we have $W_{n_j} \in \mu^*(x)$, (for $j = 1, 2, \dots, r$) such that W_{n_j} intersects at most finitely many $V_{n_j,\alpha}$'s. Let $V = W \cap W_{n_1} \cap W_{n_2} \dots \cap W_{n_r}$ then since $W, W_{n_j} \in \mu^*(x)$, $V \in \mu^*(x)$. So we have $V \in \mu^*(x)$ such that V intersects at most finitely many member of \mathcal{K} . Hence \mathcal{K} is g_μ -locally finite. Thus \mathcal{K} is the required g_μ -locally finite refinement of $\{U_\beta : \beta \in \mathcal{B}\}$ that covers X .

(3) \Rightarrow (4) Let \mathcal{U} be a μ -open covering of X . With each $y \in X$, associate a definite $U_y \in \mathcal{U}$ containing it and then since X is γ_μ -regular, there exists a μ -open set V_y such that $y \in V_y \subseteq \gamma_\mu(V_y) \subseteq U_y$. The family $\{V_y : y \in X\}$ is then a μ -open covering and by (2) and theorem 2.4 it has a precise g_μ -locally finite refinement $\{A_y : y \in X\}$. Since $\{\gamma_\mu(A_y) : y \in X\}$ is also g_μ -locally finite (by theorem 2.1) and $\gamma_\mu(A_y) \subseteq \gamma_\mu(V_y) \subseteq U_y$ for each y , $\{\gamma_\mu(A_y) : y \in X\}$ is the desired refinement. \square

3.3. Remark. For a g_μ -regular GTS (X, μ) , if we take μ as μ^* then all the conditions (1) - (4) stated in the above theorem become equivalent. We have already proved (1) \Rightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4). So if we show (4) \Rightarrow (1) then our purpose will be fulfilled.

3.4. Theorem. For any γ_μ -regular GTS (X, μ) with $\mu = \mu^*$, if each μ -open cover of X has a μ^* -closed g_μ -locally finite refinement that covers X then (X, μ) is g_μ -paracompact.

Proof. Let \mathcal{U} be any μ -open covering of X and ξ be any μ^* -closed g_μ -locally finite refinement of it. Since $\mu = \mu^*$, ξ is a μ -closed μ -locally finite refinement. Then for each $x \in X$, there exists a μ -open set V_x containing x such that V_x intersects at most finitely many sets E of ξ . Using the μ -open covering $\{V_x : x \in X\}$, by given hypothesis we get a μ^* -closed g_μ -locally finite and hence a μ -closed μ -locally finite refinement \mathcal{B} that covers X . Since each B of \mathcal{B} intersects at most finitely many sets E of ξ it follows from that we can enlarge each E to an μ -open set $G(E)$ such that $\{G(E)\}$ is μ -locally finite (by corollary 2.2). Associating with each E a single set $U(E) \in \mathcal{U}$ containing E , it is evident that $\{G(E) \cap U(E)\}$ is an μ -open μ -locally finite refinement of \mathcal{U} . \square

If we consider a regular topological space (X, τ) and choose in particular the GT as τ then from theorem 3.3 and theorem 3.4 we obtain E.Michael's theorem. On the other hand, if $\mu = \delta$ -open sets of (X, τ) then we obtain a characterization parallel to E.Michael's theorem for almost regular nearly paracompact spaces [4].

References

- [1] Á.Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar., 96(4)(2002), 351 – 357.
- [2] J.Dieudonné, *Une généralisation des espaces compacts*, J.Math. Pures et. Appl. 23(1944), 65 – 76.
- [3] T.Noiri and B.Roy, *Unification of generalized open sets on Topological spaces*, Acta Math. Hungar., 130(4)(2011), 349 – 357.
- [4] M.N.Mukherjee and A.Debray, *On nearly paracompact spaces via regular even covers*, Matematnykn Bechnk, 50(1998), 23 – 29.
- [5] M.K.Singal and S.P.Arya, *On nearly paracompact spaces*. Matematički Vesnik, 6(21)(1969), 3 – 16.
- [6] R.Shen, *Remarks on Products of generalized topology*, Acta Math Hunger., 124(4)(2009), 363 – 369.
- [7] N.V.Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., 78(2)(1968)103 – 118.

