Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (1) (2019), 75 – 86
DOI : 10.15672/HJMS.2017.515

Caristi type fixed point theorems in fuzzy metric spaces

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Abstract

In this paper, we extend the generalized Caristi's fixed point theorem proved by Bollenbacher and Hicks to p-orbitally complete fuzzy metric spaces by considering the fuzzy metric spaces in the sense of George and Veeramani. We also give some illustrative examples that support our results.

Mathematics Subject Classification (2010). 54A40, 47H10

Keywords. Caristi's fixed point theorem, fixed point, fuzzy metric space, p-complete fuzzy metric space

1. Introduction

In 1976, Caristi [8] proved the following fixed point theorem on a complete metric space, which is one of the most important generalization of famous Banach contraction principle and is equivalent the Ekeland's variational principle [13].

Let T is a self-mapping of a complete metric space (X, d) such that there is a lower semi continuous function φ from X into $[0, \infty)$ satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$, then T has a fixed point.

In this theorem, saying that φ is lower semi continuous at x if for any sequence $\{x_n\} \subset X$, we have $\lim x_n = x$ implies $\varphi(x) \leq \liminf \varphi(x_n)$.

Several authors have obtained various extensions and generalizations of Caristi's theorem by considering Caristi type mappings on many different spaces. For example, [1–7, 9, 23–25, 27, 28, 30, 31, 33, 38, 40], and others.

In this paper, we extend the results in [7] to fuzzy metric spaces.

Several notions of fuzzy metric spaces have been introduced and discussed in different directions by many mathematicians, see [10, 14, 29, 34, 39]. In particular, Kramosil and Michalek [34] introduced and studied the notion of fuzzy metric space which is closely related to a class of probabilistic metric spaces. In [15, 17] George and Veeramani modified the concept of fuzzy metric space of Kramosil and Michalek, and obtained a Hausdorff and first countable topology on the modified fuzzy metric space. In [16, 20], it was proved that the topology induced by a fuzzy metric space in George and Veeramani's sense is metrizable. Grabiec [18] obtained a fuzzy version of the Banach contraction principle

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Received: 21.04.2017; Accepted: 11.09.2017

in fuzzy metric spaces in Kramosil and Michalek's sense, and since then many author [11, 12, 22, 26, 32, 35, 41, 42] have proved fixed point theorems in fuzzy metric spaces in the sense of Kramosil and Michalek and George and Veeramani, using one of the two different types of completeness in Grabiec's sense [18] or George and Veeramani's sense [15].

In [36] Miheţ defined a concept weaker than convergence called p-convergence and proved a fixed point theorem for fuzzy contractive mappings. Then, in [19] Gregori et al. introduced the concept of p-Cauchy sequence and showed that p-Cauchy sequence and Cauchy sequence are two different concepts even in principal fuzzy metric spaces and they also defined the concept p-completeness.

In this paper, we consider $(X, M, *)$ fuzzy metric space in George and Veeramani's sense and prove some fixed point theorems for Caristi type mappings orbitally p-complete fuzzy metric spaces.

2. Preliminaries

In this section, we give some known basic notion of fuzzy metric space in the sense of George and Veeramani. Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers.

Definition 2.1 ([39])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for every $a \in [0,1]$,
(iv) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 ([15])**.** The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X. If we replace (iv) by

(vi) $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$,

then 3-tuple $(X, M, *)$ is called a non-Archimedean fuzzy metric space. Since (vi) implies (iv) then each non-Archimedean fuzzy metric space is a fuzzy metric space.

Example 2.3. Let (X, d) be a metric space. Denote by $a.b$ the usual multiplication for all $a, b \in [0,1]$, and let M_d be the function defined on $X \times X \times (0, +\infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space called standard fuzzy metric space and $(M_d, \cdot)$ is called the standard fuzzy metric of d (see [15]).

George and Veeramani proved in [15] that ever fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has as a base the family of sets of the form

$$\{B(x, r, t) : x \in X,\ 0 < r < 1, t > 0\},$$

where

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

for all $r \in (0,1)$ and $t > 0$. They proved also that (X, τ_M) is a Hausdorff first countable topological space.

Definition 2.4 ([21]). A fuzzy metric M on X is said to be stationary if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x,y,t)$ is constant. In this case we write $M(x,y)$ instead of $M(x,y,t)$.

Theorem 2.5 ([15]). *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to +\infty$.*

The following definition was given by Miheţ.

Definition 2.6 ([36]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p-convergent to $x_0 \in X$ (we write $x_n \to_p x_0$) if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$.

If $\{x_n\}$ is p-convergent to x_0, then
(1) $\{x_n\}$ in X has at most one limit.
(2) Every subsequence of $\{x_n\}$ is also convergent and has the same limit as the whole sequence, see [36].
Note that $\{x_n\}$ is convergent to x_0 if and only if $\{x_n\}$ is p-convergent to x_0 for all $t > 0$, see [19].
In [36] the author gave an example that there exist p-convergent but not convergent sequences.

Definition 2.7 ([18]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is G-Cauchy sequence iff $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p \in \mathbb{N}$. A fuzzy metric space $(X, M, *)$ is G-complete if every G-Cauchy sequence is convergent in X.

Definition 2.8 ([15]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is Cauchy sequence iff for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A fuzzy metric space $(X, M, *)$ is complete if every Cauchy sequence is convergent in X.

In [19] Gregori et al. gave the following definition of Cauchyness and completeness in a natural way from the p-convergence concept.

Definition 2.9 ([19]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p-Cauchy if there exists $t_0 > 0$ such that for each $\varepsilon \in (0,1)$ there exists $n_0 \in N$ such that $M(x_n, x_m, t_0) > 1 - \varepsilon$ for all $n, m \geq n_0$, or equivalently $\lim_{n,m\to\infty} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$. A fuzzy metric space $(X, M, *)$ is p-complete if every p-Cauchy sequence in X is p-convergent to some point of X.

Note that $\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ is p-Cauchy for all $t > 0$ and, obviously, p-convergent sequences are p-Cauchy.
p-completeness and completeness are equivalent concepts in stationary fuzzy metrics, see [19].

Remark 2.10 ([19]). Let $(X, M_d, *)$ be a standard fuzzy metric space as in Example 2.3. Then $(X, M_d, *)$ is p-complete if and only if the metric space (X, d) is complete.

Definition 2.11 ([12]). Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is triangular if it satisfies the condition

$$\frac{1}{M(x,y,t)} - 1 \leq \frac{1}{M(x,z,t)} - 1 + \frac{1}{M(z,y,t)} - 1$$

for every $x, y, z \in X$ and every $t > 0$.

Note that every standard fuzzy metric $(M_d, \cdot)$ is triangular.

Theorem 2.12 ([37]). *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, +\infty)$.*

Definition 2.13. Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ a mapping. The set $O_T(x, \infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x. If for an $x \in X$, every p-Cauchy sequence in $O_T(x, \infty)$ is p-converges to a point in X, then the fuzzy metric space $(X, M, *)$ is said to be (x, T)-orbitally p-complete.

Definition 2.14. Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ a mapping. A real-valued function $G : X \times (0, +\infty) \to [0, \infty)$ is said to be (x, T)- orbitally p-weak lower semi-continuous (p-w.l.s.c.) at u iff $\{x_n\}$ is a sequence in $O_T(x, \infty)$ and

$$x_n \to_p u \quad \text{implies} \quad G(u, t_0) \leq \lim_{n\to\infty} \sup G(x_n, t_0)$$

for some $t_0 > 0$. That is, $G(., t_0)$ is p-w.l.s.c on X in Ćirić's sense, see [9].

3. Main results

In this section, we state and prove our main results in orbitally p-complete fuzzy metric spaces. Now, we give the first main result as follows.

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space with M is triangular, $T : X \to X$ and $\Phi : X \times (0, +\infty) \to [0, \infty)$. Suppose there exist $x \in X$ and $t_0 > 0$ such that $(X, M, *)$ is (x, T)-orbitally p-complete, and*

$$\frac{1}{M(y, Ty, t_0)} - 1 \leq \Phi(y, t_0) - \Phi(Ty, t_0) \tag{3.1}$$

for all $y \in O_T(x, \infty)$. Then:

(i) *$T^n x \to_p x'$ exists,*
(ii) $\frac{1}{M(T^n x, x', t_0)} - 1 \leq \Phi(T^n x, t_0)$,
(iii) *$Tx' = x'$ if and only if $G(z, t_0) = \frac{1}{M(z, Tz, t_0)} - 1$ is (x, T)-orbitally p-w.l.s.c. at x',*
(iv) $\frac{1}{M(T^n x, x, t_0)} - 1 \leq \Phi(x, t_0)$ *and* $\frac{1}{M(x', x, t_0)} - 1 \leq \Phi(x, t_0)$.

Proof. (i) Using inequality (3.1) we have

$$\begin{aligned} S_n = \sum_{i=0}^{n} \left(\frac{1}{M(T^i x, T^{i+1} x, t_0)} - 1\right) &\leq \sum_{i=0}^{n} [\Phi(T^i x, t_0) - \Phi(T^{i+1} x, t_0)] \\ &= \Phi(x, t_0) - \Phi(T^{n+1} x, t_0) \leq \Phi(x, t_0) \end{aligned}$$

for $n = 0, 1, 2, \dots$. Therefore, $\{S_n\}$ is bounded from above and also non-decreasing and so convergent.

Let $m > n$. Since M is triangular, we have

$$\frac{1}{M(T^n x, T^m x, t_0)} - 1 \leq \sum_{k=n}^{m-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) \tag{3.2}$$

Since $\{S_n\}$ is convergent, for every $1 > \varepsilon > 0$, we can choose a sufficiently large $N \in \mathbb{N}$ such that

$$\sum_{k=n}^{\infty} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) < \varepsilon$$

for all $n \geq N$. Thus, we get from inequality (3.2) that

$$\frac{1}{M(T^n x, T^m x, t_0)} - 1 \leq \sum_{k=n}^{m-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) < \varepsilon$$

and so

$$\frac{1}{M(T^n x, T^m x, t_0)} < 1 + \varepsilon.$$

Since $1 - \varepsilon^2 < 1$, it follows that

$$M(T^n x, T^m x, t_0) > \frac{1}{1+\varepsilon} = \frac{1-\varepsilon}{1-\varepsilon^2} > 1 - \varepsilon$$

for all $n, m \geq N$. Hence, $\{T^n x\}$ is a p-Cauchy sequence in $O_T(x, \infty)$. Since $(X, M, *)$ is (x, T)- orbitally p-complete, $T^n x \to_p x' \in X$ exists.

(ii) Let $m > n$. Using inequalities (3.1) and (3.2) we have

$$\begin{aligned}\frac{1}{M(T^n x, T^m x, t_0)} - 1 &\leq \sum_{k=n}^{m-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) \\ &\leq \sum_{k=n}^{m-1} [\Phi(T^k x, t_0) - \Phi(T^{k+1} x, t_0)] \\ &= \Phi(T^n x, t_0) - \Phi(T^m x, t_0) \leq \Phi(T^n x, t_0).\end{aligned}$$

Letting m tend to infinity, we have from (i) and Theorem 2.12

$$\frac{1}{M(T^n x, x', t_0)} - 1 \leq \Phi(T^n x, t_0).$$

(iii) Assume that $Tx' = x'$ and $\{x_n\}$ is a sequence in $O_T(x, \infty)$ with $x_n \to_p x'$. Then

$$G(x', t_0) = \frac{1}{M(x', Tx', t_0)} - 1 = 0 \leq \limsup \left(\frac{1}{M(x'_n, Tx'_n, t_0)} - 1\right) = \limsup G(x_n, t_0),$$

and so G is (x, T)-orbitally p-w.l.s.c. at x'.

Now let $x_n = T^n x$ and G is (x, T)-orbitally p-w.l.s.c. at x'. Then from (i) we have

$$\begin{aligned}0 \leq \frac{1}{M(x', Tx', t_0)} - 1 = G(x', t_0) &\leq \limsup G(T^n x, t_0) \\ &= \limsup \left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right) = 0\end{aligned}$$

which implies $\frac{1}{M(x', Tx', t_0)} - 1 = 0$. Thus $M(x', Tx', t_0) = 1$ and so $Tx' = x'$.

(iv) We first of all prove by induction that

$$\frac{1}{M(T^n x, x, t_0)} - 1 \leq \sum_{k=0}^{n-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) \tag{3.3}$$

for all $n = 1, 2, 3, \ldots$.

Inequality (3.3) is trivial when $n = 1$ and so we will assume that inequality (3.3) holds for $n - 1$. Since M is triangular, it follows from inequality (3.1) we have

$$\begin{aligned}\frac{1}{M(T^n x, x, t_0)} - 1 &\leq \frac{1}{M(T^n x, T^{n-1} x, t_0)} - 1 + \frac{1}{M(T^{n-1} x, x, t_0)} - 1 \\ &\leq \sum_{k=0}^{n-2} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) + \frac{1}{M(T^n x, T^{n-1} x, t_0)} - 1 \\ &= \sum_{k=0}^{n-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right).\end{aligned}$$

It therefore follows by induction that inequality (3.3) holds.

Using inequalities (3.1) and (3.3) we have

$$\begin{aligned}\frac{1}{M(T^n x, x, t_0)} - 1 &\leq \sum_{k=0}^{n-1} \left(\frac{1}{M(T^k x, T^{k+1} x, t_0)} - 1\right) \\ &\leq \sum_{k=0}^{n-1} \left[\Phi(T^k x, t_0) - \Phi(T^{k+1} x, t_0)\right] \\ &= \Phi(x, t_0) - \Phi(T^n x, t_0) \leq \Phi(x, t_0).\end{aligned}$$

Letting n tend to infinity we have

$$\frac{1}{M(x', x, t_0)} - 1 \leq \Phi(x, t_0).$$

□

Corollary 3.2. *Let $(X, M, *)$ be a fuzzy metric space with M is triangular and T be a self-mapping of X. Suppose there exist $x \in X$ and $t_0 > 0$ such that $(X, M, *)$ is (x, T)-orbitally p-complete, and*

$$\frac{1}{M(Ty, T^2 y, t_0)} - 1 \leq k\left(\frac{1}{M(y, Ty, t_0)} - 1\right) \tag{3.4}$$

for all $y \in O_T(x, \infty)$. Then:

(i) *$T^n x \to_p x'$ exists,*

(ii) $\frac{1}{M(T^n x, x', t_0)} - 1 \leq k^n (1-k)^{-1} \left(\frac{1}{M(x, Tx, t_0)} - 1\right)$,

(iii) *$Tx' = x'$ if and only if $G(z, t_0) = \frac{1}{M(z, Tz, t_0)} - 1$ is (x, T)-orbitally p-w.l.s.c. at x',*

(iv) $\frac{1}{M(T^n x, x, t_0)} - 1 \leq \frac{1}{1-k}\left(\frac{1}{M(x, Tx, t_0)} - 1\right)$, $\frac{1}{M(x', x, t_0)} - 1 \leq \frac{1}{1-k}\left(\frac{1}{M(x, Tx, t_0)} - 1\right)$.

Proof. Put $\Phi(y, t) = (1-k)^{-1}\left(\frac{1}{M(y, Ty, t)} - 1\right)$ for $y \in O_T(x, \infty)$. Let $y = T^n x$ in (3.4). Then we have,

$$\frac{1}{M(T^{n+1} x, T^{n+2} x, t_0)} - 1 \leq k\left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right)$$

and

$$\left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right) - k\left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right) \leq \\ \left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right) - \left(\frac{1}{M(T^{n+1} x, T^{n+2} x, t_0)} - 1\right)$$

and so

$$\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1 \leq \\ (1-k)^{-1}\Big[\left(\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1\right) - \left(\frac{1}{M(T^{n+1} x, T^{n+2} x, t_0)} - 1\right)\Big].$$

Thus, we get

$$\frac{1}{M(y, Ty, t_0)} - 1 \leq \Phi(y, t_0) - \Phi(Ty, t_0)$$

so (i), (iii) and (iv) are immediate from Theorem 3.1.

Using inequality (3.4) we have

$$\frac{1}{M(T^n x, T^{n+1} x, t_0)} - 1 \leq k^n\left(\frac{1}{M(x, Tx, t_0)} - 1\right)$$

and then from Theorem 3.1 (ii) we get

$$\begin{aligned}\frac{1}{M(T^n x, x', t_0)} - 1 &\leq \Phi(T^n x, t_0)\\ &= (1-k)^{-1}\left(\frac{1}{M(T^n x, T^{n+1}x, t_0)} - 1\right)\\ &\leq k^n (1-k)^{-1}\left(\frac{1}{M(x, Tx, t_0)} - 1\right)\end{aligned}$$

and this gives (ii). □

In the following theorem, we will show that if $(M, *)$ is non-Archimedean fuzzy metric, where the continuous t-norm is defined as $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$, then (i) and (iii) of Theorem 3.1 can be obtained without the triangular property of M.

Theorem 3.3. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space, where the continuous t- norm is defined as $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $T : X \to X$ and $\Phi : X \times (0, +\infty) \to [0, \infty)$. Suppose there exist $x \in X$ and $t_0 > 0$ such that $(X, M, *)$ is (x, T)- orbitally p-complete, and satisfying the inequality (3.1) for all $y \in O_T(x, \infty)$. Then:*

(i) *$T^n x \to_p x'$ exists,*
(ii) *$Tx' = x'$ if and only if $G(z, t) = \frac{1}{M(z,Tz,t_0)} - 1$ is (x, T)-orbitally p-w.l.s.c. at x'.*

Proof. (i) Using the same procedure as in the proof of Theorem 3.1, we obtain that

$$S_n = \sum_{i=0}^{n} \left(\frac{1}{M(T^i x, T^{i+1}x, t_0)} - 1\right)$$

is convergent. Therefore we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{M(T^n x, T^{n+1}x, t_0)} - 1\right) < \infty \quad \text{and so} \quad \lim_{n\to\infty} \left(\frac{1}{M(T^n x, T^{n+1}x, t_0)} - 1\right) = 0$$

Thus, $\lim_{n\to\infty} M(T^n x, T^{n+1}x, t_0) = 1$. Hence for $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that $M(T^n x, T^{n+1}x, t_0) > 1 - \varepsilon$ for all $n > n_0$. Let $n_0 < n < m$. Using (vi) of Definition 2.2, we have

$$\begin{aligned}M(T^n x, T^m x, t_0) &\geq \overbrace{M(T^n x, T^{n+1}x, t_0) * \cdots * M(T^{m-1}, T^m x, t_0)}^{m-n}\\ &= \min\{M(T^n x, T^{n+1}x, t_0), \ldots, M(T^{m-1}x, T^m x, t_0)\} > 1 - \varepsilon\end{aligned}$$

and so the sequence $\{T^n x\}$ is a p-Cauchy sequence in $O_T(x, \infty)$. Since $(X, M, *)$ is (x, T)-orbitally p-complete, $T^n x \to_p x'$ exists.

Using the same procedure as in the proof of Theorem 3.1 (iii), we obtain (ii). □

Similarly, using the same procedure as in the proof of Corollary 3.2 (i) and (iii), we obtain the following result.

Corollary 3.4. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space, where the continuous t- norm is defined as $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let T be a self- mapping of X. Suppose there exists an $x \in X$ such that $(X, M, *)$ is (x, T)- orbitally complete, and satisfying the inequality (3.4), for all $y \in O_T(x, \infty)$. Then:*

(i) *$T^n x \to_p x'$ exists,*
(ii) *$Tx' = x'$ if and only if $G(z, t) = \frac{1}{M(z,Tz,t_0)} - 1$ is (x, T)-orbitally p-w.l.s.c. at x'.*

If we replace non-Archimedean fuzzy metric by stationary fuzzy metric in the Theorem 3.3, then using the same procedure as in the proof of Theorem 3.3 and Corollary 3.2 we obtain the following results.

Theorem 3.5. *Let $(X, M, *)$ be a stationary fuzzy metric space, where the continuous t-norm is defined as $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $T : X \to X$ and $\Phi : X \to [0, \infty)$. Suppose there exists an $x \in X$ such that $(X, M, *)$ is (x, T)- orbitally p-complete, and*

$$\frac{1}{M(y, Ty)} - 1 \leq \Phi(y) - \Phi(Ty)$$

for all $y \in O_T(x, \infty)$. Then:

(i) $T^n x \to_p x'$ *exists,*

(ii) $Tx' = x'$ *if and only if* $G(z) = \frac{1}{M(z,Tz)} - 1$ *is (x, T)-orbitally p-w.l.s.c. at x'.*

Corollary 3.6. *Let $(X, M, *)$ be a stationary fuzzy metric space, where the continuous t-norm is defined as $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let T be a self- mapping of X. Suppose there exists an $x \in X$ such that $(X, M, *)$ is (x, T)- orbitally p-complete, and*

$$\frac{1}{M(Ty, T^2y)} - 1 \leq k\left(\frac{1}{M(y, Ty)} - 1\right)$$

for all $y \in O_T(x, \infty)$. Then:

(i) $T^n x \to_p x'$ *exists,*

(ii) $Tx' = x'$ *if and only if* $G(z) = \frac{1}{M(z,Tz)} - 1$ *is (x, T)-orbitally p-w.l.s.c. at x'.*

Note that Theorem 3.5 and Corollary 3.6 are true for complete fuzzy metric spaces since p-completeness and completeness are equivalent concepts in stationary fuzzy metrics.

The following theorem is slight generalization of Theorem 3 in [7].

Theorem 3.7 ([7])**.** *Let (X, d) be a metric space, $T : X \to X$ and $\varphi : X \to [0, \infty)$. Suppose there exists an $x \in X$ such that*

$$d(y, Ty) \leq \varphi(y) - \varphi(Ty) \tag{3.5}$$

for all $y \in O_T(x, \infty)$, and (X, d) is (x, T)- orbitally complete. Then:

(i) $\lim_{n\to\infty} T^n x = x'$ *exists,*

(ii) $d(T^n x, x') \leq \varphi(T^n x)$,

(iii) $Tx' = x'$ *if and only if* $F(z) = d(z, Tz)$ *is (x, T)-orbitally w.l.s.c. at x',*

(iv) $d(T^n x, x) \leq \varphi(x)$ *and* $d(x', x) \leq \varphi(x)$.

Proof. We consider the $(M_d, .)$ standard fuzzy metric induced by d on X as in Example 2.3. By Remark 2.10 $(X, M_d, *)$ is (x, T)- orbitally p-complete since (X, d) orbitally complete. Also $(M_d, .)$ is triangular.

Since $M_d(x, y, t) = \frac{t}{t+d(x,y)}$, we have $d(x, y) = \frac{t}{M_d(x,y,t)} - t$ for all $x, y \in X$ and $t > 0$.

Define $\Phi(x, t_0) = \frac{1}{t_0}\varphi(x)$ for all $x \in X$. Then from inequality (3.5) we have

$$\frac{t_0}{M_d(y, Ty, t_0)} - t_0 \leq t_0(\Phi(y, t_0) - \Phi(Ty, t_0))$$

and so

$$\frac{1}{M_d(y, Ty, t_0)} - 1 \leq \Phi(y, t_0) - \Phi(Ty, t_0).$$

Thus T satisfies inequality (3.1) of Theorem 3.1.

(i) From Theorem 3.1 (i) we have $T^n x \to_p x'$ exists and so $\lim_{n\to\infty} T^n x = x'$ (in the metric space).

(ii) From Theorem 3.1 (ii) we have

$$\frac{1}{M(T^n x, x', t_0)} - 1 \le \Phi(T^n x, t_0),$$

and so

$$\frac{1}{\frac{t_0}{t_0 + d(T^x x, x')}} - 1 = \frac{t_0 + d(T^n x, x') - t_0}{t_0} = \frac{d(T^n x, x')}{t_0} \le \frac{1}{t_0}\varphi(T^n x).$$

Thus $d(T^n x, x') \le \varphi(T^n x)$.

(iii) From Theorem 3.1 (iii) we have

$$\frac{1}{M_d(x, Tx, t_0)} - 1 = \frac{d(x, Tx)}{t_0}.$$

If $G(x, t_0) = \frac{1}{M_d(x,Tx,t_0)} - 1$ is (x, T) orbitally p-w.l.s.c. at x', then $t_0 G(x, t_0) = d(x, Tx)$ is (x, T)- orbitally w.l.s.c. at x' too. Thus (iii) follows from Theorem 3.1 (iii).

(iv) From Theorem 3.1 (iv) we have

$$\frac{1}{M_d(T^n x, x, t_0)} - 1 \le \Phi(x, t_0) \quad \text{and so} \quad \frac{d(T^n x, x')}{t_0} \le \frac{1}{t_0}\varphi(x).$$

Thus $d(T^n x, x) \le \varphi(x)$. Similarly $\frac{1}{M_d(x',x,t_0)} - 1 \le \Phi(x, t_0)$ and so $d(x', x) \le \varphi(x)$. □

By considering the $(M_d, .)$ standard fuzzy metric induced by d on X in Corollary 3.2 we obtain the following corollary.

Corollary 3.8 ([7]). *Let (X, d) be a metric space and T be a self mapping of X. Suppose there exists an $x \in X$ such that*

$$d(Ty, T^2 y) \le d(y, Ty)$$

for all $y \in O_T(x, \infty)$, and (X, d) is (x, T)- orbitally complete. Then:

(i) $\lim_{n\to\infty} T^n x = x'$ *exists,*
(ii) $d(T^n x, x') \le \varphi(T^n x)$,
(iii) $Tx' = x'$ *if and only if* $F(z) = d(z, Tz)$ *is* (x, T)*-orbitally w.l.s.c. at* x',
(iv) $d(T^n x, x) \le \varphi(x)$ *and* $d(x', x) \le \varphi(x)$.

4. Some examples

We finally give some examples which illustrate our results.

Example 4.1. Let $X = [0, \infty)$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let

$$M(x, y, t) = \frac{t}{t + |x - y|},$$

for all $x, y \in X$, $t > 0$, then $(X, *)$ is triangular. Define $T : X \to X$ by

$$T(x) = \begin{cases} x/2 & \text{if } 0 \le x < 1, \\ x + 1 & \text{if } 1 \le x < \infty \end{cases}$$

for all $x \in X$.

If we take $0 \leq x_0 < 1$, then $O_T(x_0,\infty) = \{x_0, x_0/2, x_0/2^2, \ldots, x_0/2^{n-1}, \ldots\}$ for $n = 1, 2, \ldots$ and so $(X, M, *)$ is (x_0, T)- orbitally p-complete. Also define $\Phi(x,t) = x$ and put $t_0 = 1$. Then for all $y \in O_T(x_0,\infty)$ we have

$$\begin{aligned} \frac{1}{M(y,Ty,t_0)} - 1 &= \frac{1}{\frac{1}{1+|y-Ty|}} - 1 = |y - Ty| \\ &= \left|\frac{x_0}{2^{n-1}} - \frac{x_0}{2^n}\right| = \frac{x_0}{2^n} = \Phi(y,t_0) - \Phi(Ty,t_0). \end{aligned}$$

Moreover,

$$G(z,t_0) = \frac{1}{M(z,Tz,t_0)} - 1 = |z - Tz| = \begin{cases} z/2 & \text{if } 0 \leq z < 1, \\ 1 & \text{if } 1 \leq z < \infty \end{cases}$$

is (x_0, T)-orbitally p-w.l.s.c. at $x = 0$.

All the conditions of Theorem 3.1 are therefore satisfied and $x = 0$ is a fixed point of T.

Example 4.2. Let $X = (0,\infty)$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let

$$M(x,y,t) = \frac{t}{t + |x - y|},$$

for all $x, y \in X$, $t > 0$. Define $T : X \to X$ by

$$T(x) = \begin{cases} x/2 & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 \leq x < \infty \end{cases}$$

for all $x \in X$.

If we take $0 < x_0 < 1$, then $O_T(x_0,\infty) = \{x_0, x_0/2, x_0/2^2, \ldots, x_0/2^{n-1}, \ldots\}$ for $n = 1, 2, \ldots$. But $(X, M, *)$ is not (x_0, T)- orbitally p-complete.

Now we take $1 \leq x_0 < \infty$. Then $O_T(x_0,\infty) = \{x_0, 1, 1, 1, \ldots\}$. Thus $(X, M, *)$ is (x_0, T)- orbitally p-complete.

Define $\Phi(x,t) = x$ and put $t_0 = 1$. Then for $y = x_0 \neq 1$ we have

$$\frac{1}{M(y,Ty,t_0)} - 1 = |y - Ty| = |x_0 - 1| = x_0 - 1 = \Phi(y,t_0) - \Phi(Ty,t_0).$$

Also inequality (3.1) is satisfied for $y = 1$. Moreover,

$$G(z,t_0) = \frac{1}{M(z,Tz,t_0)} - 1 = |z - Tz| = \begin{cases} z/2 & \text{if } 0 < z < 1, \\ z - 1 & \text{if } 1 \leq z < \infty \end{cases}$$

is (x_0, T)-orbitally p-w.l.s.c. at $x = 1$.

All the conditions of Theorem 3.1 are therefore satisfied and $x = 1$ is a fixed point of T.

Example 4.3. Let $X = [0,\infty)$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let

$$M(x,y,t) = \begin{cases} \frac{1}{1+\max\{x,y\}}, & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$, $t > 0$. $(X, M, *)$ is non-Archimedean fuzzy metric space. Define $T : X \to X$ by $T(x) = x/2$ for all $x \in X$.

If we take $x_0 = 1$, then $O_T(1,\infty) = \{1, 1/2, 1/2^2, \ldots, 1/2^{n-1}, \ldots\}$ for $n = 1, 2, \ldots$ and so $(X, M, *)$ is $(1, T)$- orbitally p-complete. Also define $\Phi(x,t) = 2x$. Then for all $y \in O_T(1,\infty)$ we have

$$\frac{1}{M(y,Ty,t_0)} - 1 = y = \frac{1}{2^{n-1}} = \frac{2}{2^{n-1}} - \frac{2}{2^n} = \Phi(y,t_0) - \Phi(Ty,t_0).$$

Moreover, $G(z,t_0) = \begin{cases} z, & \text{if } z \neq Tz, \\ 0 & \text{if } z = Tz \end{cases}$ is $(1, T)$-orbitally p-w.l.s.c. at $x = 0$.

All the conditions of Theorem 3.3 are therefore satisfied and $x = 0$ is a fixed point of T.

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