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A new characterization of $\mathbf{L_2(2^m)}$

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Abstract

Let G be a group and $\pi(G)$ be the set of primes p such that G contains an element of order p. Let $nse(G)$ be the set of numbers of elements of G of the same order. In this paper, we prove that the simple group $L_2(2^m)$ is uniquely determined by $nse(L_2(2^m))$, where $|\pi(L_2(2^m))| = 4$.

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1. Introduction

Let G be a group. By $\pi(G)$, we denote the set of primes p such that G contains an element of order p and by $\pi_e(G)$ we mean the set of element orders of G. If $k \in \pi_e(G)$, then m_k denotes the number of elements of order k in G and we define the set $nse(G) = \{m_k \mid k \in \pi_e(G)\}$.

During the classification of the finite simple groups, it has been observed that some of the known simple groups are characterizable by some of their properties and up to now, different characterizations are investigated for the finite simple groups. For instance, in

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[16], motivated by one of the Thompson's problem, the authors introduced a new characterization for the finite simple group G, by $nse(G)$ and $|G|$. In fact, they proved that if G is a finite simple K_4-group, then G is characterizable by $nse(G)$ and $|G|$ (The simple group G is called simple K_n-group if $|\pi(G)| = n$). Following this result, in [7] and [17], it is proved that the group $L_2(q)$, where $q \in \{3, 4, 5, 7, 8, 9, 11, 13\}$ is determined only by $nse(G)$. Up to the present time, it has been investigated that some other simple groups can be characterized by $nse(G)$ and $|G|$ or only by $nse(G)$ (see for instance [9]-[12]). In this paper, our aim is to show that the simple K_4-group $L_2(2^m)$ is characterizable by $nse(L_2(2^m))$. In fact, we improve the results of [16] in the following main theorem:

Main Theorem. Let G be a group. If $nse(G) = nse(L_2(2^m))$, where m, $2^m - 1$ and $(2^m + 1)/3$ are primes greater than 3, then $G \cong L_2(2^m)$.

2. Notation and Preliminaries

For a natural number n, by $\pi(n)$, we mean the set of all prime divisors of n, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow p-subgroup of G is denoted by G_p and by $n_p(G)$, we mean the number of Sylow p-subgroups of G. Also, the largest element order of G_p is denoted by $exp(G_p)$. Moreover, we denote by φ, the Euler totient function and by (a, b) the greatest common divisor of integers a and b.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

2.1. Lemma. [2, 6, 15, 20] *Let G be a finite simple K_n-group.*

(1) *If $n = 3$, then G is isomorphic to one of the following groups:*
A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

(2) *If $n = 4$, then G is isomorphic to one of the following groups:*
  (a) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_2(97)$, $L_2(243)$, $L_2(577)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;
  (b) $L_2(r)$, *where r is a prime, $r^2 - 1 = 2^a.3^b.v$, $v > 3$ is a prime, $a, b \in \mathbb{N}$;*
  (c) $L_2(2^m)$, *where m, $2^m - 1$ and $(2^m + 1)/3$ are primes greater than 3;*
  (d) $L_2(3^m)$, *where m, $(3^m - 1)/2$ and $(3^m + 1)/4$ are odd primes.*

2.2. Lemma. [4] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

2.3. Lemma. [17] *Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

2.4. Lemma. [13] *Let G be a finite group and $p \in \pi(G) \setminus \{2\}$. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

2.5. Lemma. [18, Theorem 3] *Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of order G that is prime to n.*

2.6. Lemma. [14] *Let the finite group G acts on the finite set X. If the action is semiregular, then $|G| \mid |X|$.*

2.7. Lemma. [5] *Let G be a solvable group and π be any set of primes. Then*

(1) *G has a Hall π-subgroup.*
(2) *If H is a Hall π-subgroup of G and V is any π-subgroup of G, then $V \leq H^g$ for some $g \in G$. In particular, the Hall π-subgroups of G form a single conjugacy class of subgroups of G.*

2.8. Lemma. *Let G be an unsolvable finite group. Then there is a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$, such that N is a solvable normal subgroup of G and M/N is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups.*

Proof. Since G is a finite group, it has a chief series $1 = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_{n-1} \trianglelefteq M_n = G$. Also, since G is unsolvable, there is a maximal $i < n$, such that M_{i-1} is solvable. According to the maximality of i, we can easily conclude that the chief factor $\frac{M_i}{M_{i-1}}$ is unsolvable. Since each chief factor is a simple group or the direct product of isomorphic simple groups, it is enough to set $N := M_{i-1}$ and $M := M_i$. □

The following number theoretic lemmas play a role in the proof of the main theorem:

2.9. Lemma. [19] *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;
(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise;} \end{cases}$
(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise;} \end{cases}$

In particular, for every $q \geq 2, k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

2.10. Lemma. *Let m be a natural number. Then*

(1) *3 divides $2^m - 1$ if and only if m is even.*
(2) *3 divides $2^m + 1$ if and only if m is odd.*

Proof. On account of Lemma 2.9, the proof is straightforward. □

2.11. Lemma. [3, Remark 1] *The only solution of the equation $p^m - q^n = 1$, where p, q are primes and $m, n > 1$, is $3^2 - 2^3 = 1$.*

2.12. Lemma. [1] *Let p be a prime number.*

(1) *If $p \neq 3$, then $x^2 \equiv -3 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 3$.*
(2) *The equation $x^2 \equiv -1 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 4$.*

2.13. Lemma. [8] *Let $p \neq 3$ be a prime number.*

(1) *If the diophantine equation $3x^2 + 1 = tp^k$ has a solution, then $p \equiv 1 \pmod 3$.*
(2) *If the diophantine equation $x^{2n} + x^n + 1 = tp^k$ or $x^{2n} - x^n + 1 = tp^k$ is solvable, then $p \equiv 1 \pmod 3$.*

2.14. Lemma. *Let m be a natural number such that*

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

with $m \geq 2$, u and t are primes, $t > 3$. Then the following hold:

(a) $(u - 1, t) = 1,\ (u - 1, t - 1) = t - 1,\ (u - 1, 2^m) = 2,\ (u + 1, t) = 1$;
(b) $(t - 1, u) = 1,\ (t - 1, 2^m) = 2,\ (t + 1, u) = 1$;
(c) $(u, t) = 1,\ (u, 3) = 1,\ (u, 2) = 1,\ (t, 3) = 1,\ (t, 2) = 1$;
(d) $\pi(t - 1) \setminus \{2, 3, t, u\} \neq \emptyset$;

(e) $3 \mid (1+2^m u)$ *but* $9 \nmid (1+2^m u)$.

Proof. (a) Since t is a prime, $(u-1,t)=1$ or t. If $(u-1,t)=t$, then $t \mid (u-1)$. Hence $(2^m+1) \mid 3(2^m-2)=3(2^m+1)-9$. Therefore $(2^m+1) \mid 9$ which implies that $m \in \{1,3\}$ but this contradicts $t>3$. So $(u-1,t)=1$. We have $(u-1,t-1)=(2^m-2,\frac{2^m-2}{3})=\frac{2^m-2}{3}=(t-1)$. Since $(2^{m-1}-1,2^{m-1})=1$, we conclude that $(2^m-2,2^m)=2$ and hence, $(u-1,2^m)=2$. Since t is odd, $(2^m,t)=1$ which implies that $(u+1,t)=1$.
(b) Since u is a prime, $(t-1,u)=1$ or u. If $(t-1,u)=u$, then $u \mid (t-1) \mid (u-1)$, which is a contradiction. So $(t-1,u)=1$. Since $(2^{m-1}-1,2^{m-1})=1$, we have $(2^m-2,2^m)=2$ and hence $(t-1,2^m)=2$. According to the hypothesis, u is a prime number and hence, $(t+1,u)=1$ or u. If $(t+1,u)=u$, then $(2^m-1) \mid (2^{m-2}+1)$ because u is odd. Thus $(2^m-1) \leq (2^{m-2}+1)$, which is a contradiction. So $(u,t+1)=1$.
(c) It is obvious.
(d) By (b), $(t-1,u)=1$. Thus $u \notin \pi(t-1)$. Also, it is obvious that $t \notin \pi(t-1)$. If $\pi(t-1)=\{2,3\}$, then $2^m-2=2.3^k$. Thus $2^{m-1}-1=3^k$. Therefore $2^{m-1}-3^k=1$, that by Lemma 2.11, is a contradiction. If $\pi(t-1)=\{2\}$, then $\frac{2^m-2}{3}=2$. Hence $2^{m-1}-1=3$. Therefore $m=3$, which is a contradiction. If $\pi(t-1)=\{3\}$, then $t-1$ is odd but we have $2 \mid (t-1)$, which is a contradiction. So there is a prime $p \in \pi(t-1)$ such that $p \neq 2,3,t,u$.
(e) Since $2^m+1=3t$, $3 \mid (2^m+1)$ and hence $3 \mid (2^{2m}-1)$. Thus $3 \mid (2^{2m}-1-2^m-1+3)=(2^{2m}-2^m+1)=(1+2^m u)$. Now, we are going to prove that $9 \nmid (1+2^m u)$. First we claim that $(m,3)=1$. If not, then $(m,3)=3$ and since $3 \mid (2^m+1)$, according to Lemma 2.10(2), we have m is odd and hence, $m=3k$, where k is an odd number. Thus $u=(2^m-1)=(2^{3k}-1)=(8^k-1)=(8-1)(8^{k-1}+8^{k-2}+\ldots+8+1)$ and since $u=2^m-1$ is a prime number, we conclude that $k=1$ and $m=3$, which contradicts $t>3$. Therefore $(m,3)=1$. If $9 \mid (1+2^m u)=(2^{2m}-2^m+1)$, then $27 \mid (2^m+1)(2^{2m}-2^m+1)=(2^{3m}+1)$. Thus $27 \mid (2^{3m}+1,2^{18}-1)$. Since $(m,3)=1$, we have $(18,3m)=3$ and hence Lemma 2.9 (3) implies that $(2^{3m}+1,2^{18}-1)=9$, which is a contradiction. □

2.15. Lemma. *Assume that the hypotheses of Lemma 2.14 are fulfilled. Further let* $x=2^m$ *and let* p *be a prime number such that* $p \notin \{2,3,t,u\}$ *and* $(p,u-1)=1$.

(1) *Let* $p \mid x^3-3x^2+2x+3$.
  (a) *If* $p \mid x+4$, *then* $p=13$;
  (b) *If* $p \mid x^2+x-4$, *then* $p=101$;
  (c) *If* $p \mid x^2+x+3$, *then* $p=23$;
  (d) *If* $p \mid x^2+4x+6$, *then* $p=43$;
  (e) *If* $p \mid x^2-2$, *then* $p=23$;
  (f) $p \nmid 2x+1$.
(2) *Let* $p \mid x^2-4x+6$.
  (a) *If* $p \mid 2x+1$, *then* $p=11$;
  (b) *If* $p \mid x+4$, *then* $p=19$;
  (c) *If* $p \mid x^2+x-4$, *then* $p=5$;
  (d) *If* $p \mid x^2+x+3$, *then* $p=11$;
  (e) $p \nmid x^2+4x+6$ *and* $p \nmid x^2-2$.
(3) *Let* $p \mid x^2-2$.
  (a) *If* $p \mid 2x+1$, *then* $p=7$;
  (b) *If* $p \mid x+4$, *then* $p=7$ *and* $p \mid 2x+1$;
  (c) $p \nmid x^2+x-4$.

Proof.
• Let $p \mid x^3-3x^2+2x+3$.
If $p \mid x+4$, then $p \mid (x^3-3x^2+2x+3)-(x^2-7x)(x+4)=3(10x+1)$ and since $(p,3)=1$,

we conclude that $p \mid 10x+1$. Therefore, $p \mid (10x+1)-10(x+4) = -3(13)$ which implies that $p = 13$. If $p \mid x^2+x-4$, then $p \mid (x^3-3x^2+2x+3)-(x-4)(x^2+x-4) = 10x-13$. Thus $p \mid -13(x^2+x-4)+4(10x-13) = -x(13x-27)$ and since $(p, x) = 1$, we conclude that $p \mid 13x-27$. Therefore, $p \mid 10(13x-27)-13(10x-13) = -101$ which implies that $p = 101$. If $p \mid x^2+x+3$, then $p \mid (x^3-3x^2+2x+3)-(x-4)(x^2+x+3) = 3(x+5)$. Thus $p \mid (x^2+x+3)-(x-4)(x+5) = 23$ and hence, $p = 23$. If $p \mid x^2+4x+6$, then $p \mid -2(x^3-3x^2+2x+3)+(x^2+4x+6) = x^2(-2x+7)$. Thus $p \mid -2x+7$. On the other hand, $p \mid (x^3-3x^2+2x+3)-(x-7)(x^2+4x+6) = 24x+45$. Therefore, $p \mid (24x+45)+12(-2x+7) = 3(43)$ which implies that $p = 43$. If $p \mid x^2-2$, then $p \mid (x^3-3x^2+2x+3)-(x-3)(x^2-2) = 4x-3$. On the other hand, $p \mid (x^2-2)+(4x-3) = (x-1)(x+5)$ and since $(p, x-1) = 1$, we conclude that $p \mid x+5$. Thus $p \mid -4(x+5)+(4x-3) = -23$ which implies that $p = 23$. If $p \mid 2x+1$, then $p \mid (x^3-3x^2+2x+3)-3(2x+1) = x(x+1)(x-4)$ and since $(p, x) = (p, x+1) = 1$, we conclude that $p \mid x-4$. Thus $p \mid (2x+1)-2(x-4) = 9$, which is a contradiction to the fact that $(p, 3) = 1$.

• Let $p \mid x^2-4x+6$.

If $p \mid 2x+1$, then $p \mid -2(x^2-4x+6)+x(2x+1) = 3(3x-4)$ and since $(p, 3) = 1$, we conclude that $p \mid 3x-4$. Therefore, $p \mid 3(2x+1)-2(3x-4) = 11$ which implies that $p = 11$. If $p \mid x+4$, then $p \mid (x^2-4x+6)-x(x+4) = -2(4x-3)$ and since $(p, 2) = 1$, we conclude that $p \mid 4x-3$. Thus $p \mid (4x-3)-4(x+4) = -19$ and hence, $p = 19$. If $p \mid x^2+x-4$, then $p \mid 4(x^2-4x+6)+6(x^2+x-4) = 10x(x-1)$ and since $(p, x-1) = (p, 2) = 1$, we conclude that $p = 5$. If $p \mid x^2+x+3$, then $p \mid -(x^2-4x+6)+(x^2+x+3) = (5x-3)$. Thus $p \mid (x^2+x+3)+(5x-3) = x(x+6)$ and since $(p, 2) = 1$, we conclude that $p \mid x+6$. Therefore, $p \mid 5(x+6)-(5x-3) = 3(11)$ which implies that $p = 11$. If $p \mid x^2+4x+6$, then $p \mid -(x^2-4x+6)+(x^2+4x+6) = 8x$. Thus $p \mid 2$ which is a contradiction to the fact that $(2, p) = 1$. If $p \mid x^2-2$, then $p \mid (x^2-4x+6)-(x^2-2) = -4(x-2)$. Since $(p, 2) = (p, x-2) = 1$, we get a contradiction.

• Let $p \mid x^2-2$.

If $p \mid 2x+1$, then $p \mid -2(x^2-2)+x(2x+1) = (x+4)$. Therefore, $p \mid (2x+1)-2(x+4) = -7$ which implies that $p = 7$. If $p \mid x+4$, then $p \mid -(x^2-2)+x(x+4) = 2(2x+1)$ and since $(p, 2) = 1$, we conclude that $p \mid 2x+1$. Thus $p \mid (2x+1)-2(x+4) = -7$ and hence, $p = 7$. If $p \mid x^2+x-4$, then $p \mid -(x^2-2)+(x^2+x-4) = (x-2)$. Since $(p, x-2) = 1$, we get a contradiction. □

3. Proof of the Main Theorem

We know that $nse(G) = nse(L_2(2^m))$, where m satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

$m \geq 2$, u and t are primes, $t > 3$. Denote $x = 2^m$. According to [16], we know that $\pi(L_2(2^m)) = \{2, 3, t, u\}$ and

$$nse(L_2(2^m)) = \{1, 3tu, 2^m u, (t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}.$$

We have divided the proof into a sequence of lemmas.

3.1. Lemma. *The group G is finite. If $i \in \pi_e(G)$, then*

$$\begin{cases} \varphi(i) \mid m_i \\ i \mid \sum_{d \mid i} m_d \end{cases} \tag{3.1}$$

and if $i > 2$, then m_i is even.

Proof. Since $nse(G) = nse(L_2(2^m))$, according to Lemma 2.3, G is a finite group. Now, if $i \in \pi_e(G)$, then Lemma 2.2 implies that $i \mid \sum_{d|i} m_d$. We know that the number of elements of order i in a cyclic group of order i is equal to $\varphi(i)$. Thus $m_i = \varphi(i)k$, where k is the number of cyclic subgroups of order i in G and hence, $\varphi(i) \mid m_i$. Also, it is known that if $i > 2$, then $\varphi(i)$ is even and since $\varphi(i) \mid m_i$, we conclude that m_i is even as well. □

3.2. Lemma. $|\pi(G)| \geq 2$.

Proof. Since $3tu \in nse(G)$, Lemma 3.1 yields $2 \in \pi(G)$ and $m_2 = 3tu$. Let $\pi(G) = \{2\}$. Then $|G| = 2^k$. If $exp(G_2) > 2^{m+2}$, then $2^{m+3} \in \pi_e(G)$ and hence $2^{m+2} = \varphi(2^{m+3}) \mid m_{2^{m+3}}$, which is a contradiction. Thus $exp(G_2) \leq 2^{m+2}$ and we have

$$|G| = 1 + 3tu + k_1 2^m u + k_2(t-1)2^m u + k_3 1/2(t-1)2^m u + k_4 1/2(u-1)2^m 3t \tag{3.2}$$

where k_1, k_2, k_3 and k_4 are natural numbers and $k_1+k_2+k_3+k_4 \leq m+1$. Since $u = x-1$ and $t = (x+1)/3$, we can conclude that $|G|$ divides

$$(2k_2 + k_3 + 3k_4)x^3 + (6 + 6k_1 - 6k_2 - 3k_3 - 3k_4)x^2 + (-6k_1 + 4k_2 + 2k_3 - 6k_4)x.$$

Moreover, since $1 + m_2 = 2^{2m}$, we conclude that $2^{2m} < 2^k$ and hence $x^2 \mid |G|$. Thus x^2 divides

$$(2k_2 + k_3 + 3k_4)x^3 + (6 + 6k_1 - 6k_2 - 3k_3 - 3k_4)x^2 + (-6k_1 + 4k_2 + 2k_3 - 6k_4)x$$

which implies that $x \mid 6k_1 - 4k_2 - 2k_3 + 6k_4$. Since

$$6k_1 - 4k_2 - 2k_3 + 6k_4 < 6(k_1 + k_2 + k_3 + k_4) \leq 6(m+1),$$

we conclude that $2^m \leq (6m+6)$. Thus $m = 5$ which implies that $u = 31$ and $t = 11$. From (3.2) we have

$$2^k = 1 + 1023 + 992k_1 + 9920k_2 + 4960k_3 + 15840k_4,$$

where $k_1 + k_2 + k_3 + k_4 \leq 6$ and it is easy to check that this equation has no solution. □

3.3. Lemma. $\pi(G) \neq \{2, 3\}$.

Proof. Let $\pi(G) = \{2, 3\}$. If G_3 is a cyclic group of order 3^k, then $n_3(G) = \frac{m_{3^k}}{\varphi(3^k)} = \frac{m_{3^k}}{2(3^{k-1})}$ and hence, according to $nse(G)$ and Lemma 2.14(c), we can conclude that t or u divides $n_3(G)$. On the other hand, since $n_3(G)$ divides $|G|$, we can get a contradiction. Thus G_3 is not cyclic and according to Lemmas 2.2 and 2.4, we have $9 \mid 1 + m_3$. If $m_3 = 2^m u$, then since by Lemma 2.14, $9 \nmid 1 + 2^m u$, we can get a contradiction. Also, since $(3, m_3) = 1$, we conclude that $m_3 \neq 1/2(u-1)2^m 3t$. Thus $m_3 \in \{(t-1)2^m u, 1/2(t-1)2^m u\}$ which implies that

$$(3, t-1) = 1. \tag{3.3}$$

If $6 \notin \pi_e(G)$, then by Lemma 2.6, $|G_3| \mid m_2$. According to Lemma 3.2, $m_2 = 3tu$ and hence Lemma 2.14 implies that G_3 is cyclic, which is a contradiction. Thus $6 \in \pi_e(G)$. Since $6 \mid 1 + m_2 + m_3 + m_6$ and $3 \mid 1 + m_2 + m_3$, we conclude that $3 \mid m_6$. Now according to $nse(G)$ and (3.3), we have $m_6 = 1/2(u-1)2^m 3t$ and hence, $9 \mid m_6$.

Now we have the following two cases:

Case 1. Let $exp(G_3) = 3$. Then by Lemma 2.5, $9 \mid \sum_{i \geq 2} m_{2^i} + \sum_{i \geq 2} m_{2^i 3}$ and $9 \mid \sum_{i \geq 1} m_{2^i} + \sum_{i \geq 1} m_{2^i 3}$. Thus $9 \mid m_2 + m_6$ and since $9 \mid m_6$, we conclude that $9 \mid m_2$, which is a contradiction.

Case 2. Let $exp(G_3) > 3$. If $18 \notin \pi_e(G)$, then similar to Case 1, we can get a contradiction. If $18 \in \pi_e(G)$, then according to Lemma 2.4, $9 \mid m_{2^i 3^j}$, where $i \geq 0,\ j \geq 2$. Since $18 \in \pi_e(G)$, we have $18 \mid 1 + m_2 + m_3 + m_6 + m_9 + m_{18}$. On the other hand, $9 \mid m_6$ and according to Lemma 3.1, $9 \mid 1 + m_3 + m_9$ and hence, $9 \mid m_2$, which is a contradiction. □

3.4. Lemma. $\pi(G) \subseteq \{2, 3, t, u\}$.

Proof. Suppose, contrary to our claim, that $p \in \pi(G) \setminus \{2, 3, t, u\}$. To obtain a contradiction, in the following six steps we will prove that there is no choice for m_p in $nse(G)$.
Step 1. $m_p \neq 2^m u$ and $(p, t-1) = 1$.
If $m_p = 2^m u$, then according to (3.1), $p \mid (1 + m_p) = (2^{2m} - 2^m + 1)$. Thus Lemma 2.13 implies that $3 \mid (p-1)$. On the other hand, by (3.1), we have $p - 1 \mid m_p$ and hence, $3 \mid m_p$, which is impossible according to Lemma 2.14. Therefore, $m_p \in \{(t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$. Since $(p, m_p) = 1$, we conclude that $(p, t-1) = 1$.
Step 2. $exp(G_p) = p$.
If $exp(G_p) > p$, then $p^2 \in \pi_e(G)$. Since $p(p-1) = \varphi(p^2) \mid m_{p^2}$, we conclude that p divides one of the numbers $2, 3, t, u, (t-1)$, which is a contradiction. So $exp(G_p) = p$.
Step 3. If $q \in \pi_e(G) \setminus \{1\}$ and $(q, p) = 1$, then $qp \in \pi_e(G)$ and $p \mid m_q + m_{qp}$.
If $qp \notin \pi_e(G)$, then Lemma 2.6 implies that $|G_p| \mid m_q$. Now according to $nse(G)$, we conclude that p divides one of the numbers $2, 3, t, u, (t-1)$, which is a contradiction. Thus $qp \in \pi_e(G)$. Let $q = q_1^{s_1} \ldots q_k^{s_k}$, where $q_1, \ldots, q_k$ are distinct prime numbers and $k, s_1, \ldots, s_k$ are natural numbers. We prove $p \mid m_q + m_{qp}$ by induction on $s = s_1 + \ldots + s_k$. Let $s = 1$. Then q is a prime number and according to (3.1), we have $p \mid 1 + m_p + m_q + m_{qp}$ and since $p \mid 1 + m_p$, we can easily conclude that $p \mid m_q + m_{qp}$. Let $s = 2$. Then there exist $1 \leq i < j \leq k$ such that $q = q_i q_j$ or $q = q_i^2$. If $q = q_i q_j$, then we have $p \mid 1 + m_p + m_{q_i} + m_{q_j} + m_{q_i p} + m_{q_j p} + m_{q_i q_j} + m_{q_i q_j p}$ and since $p \mid 1 + m_p,\ m_{q_i} + m_{q_i p},\ m_{q_j} + m_{q_j p}$, we conclude that $p \mid m_{q_i q_j} + m_{q_i q_j p}$, as desired. The case $q = q_i^2$ is similar and we omit the details for the sake of convenience. Now, assume the statement is true for the values less than s. We have

$$p \mid \sum_{d \mid qp} m_d = \sum_{\substack{d \mid qp \\ d \neq q, qp}} m_d + m_q + m_{qp}.$$

Moreover, according to induction hypothesis, $p \mid \sum_{\substack{d \mid qp \\ d \neq q, qp}} m_d$. Therefore, $p \mid m_q + m_{qp}$.
Step 4. There is $q \in \pi_e(G)$ such that $(q, p) = 1$, $m_q = 2^m u$ or $m_{qp} = 2^m u$. Moreover, we have $p \mid m_q + m_{pq}$.
According to $nse(G)$, there exists $i \in \pi_e(G)$ such that $m_i = 2^m u$. If $(i, p) = 1$, then according to Step 3, we have $p \mid m_i + m_{ip}$. So it is enough to assume $q := i$. If $(i, p) \neq 1$, then since according to Step 2, $exp(G_p) = p$, we have $i = qp$, where $(q, p) = 1$ and $q \in \pi_e(G) \setminus \{1\}$. According to Step 3, we have $p \mid m_i + m_{ip}$.
Step 5. $m_p \neq (t-1)2^m u$.
If $m_p = (t-1)2^m u$, then since $p \mid 1 + m_p$, we have $p \mid x^3 - 3x^2 + 2x + 3$. By using Step 4, we have the following five cases:
Case 1. If $\{m_q, m_{qp}\} = \{2^m u, 3tu\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid 2x + 1$, which is impossible according to Lemma 2.15(1).
Case 2. If $\{m_q, m_{qp}\} = \{2^m u, 2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or u, which is contradiction.
Case 3. If $\{m_q, m_{qp}\} = \{2^m u, (t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or t or u, which is contradiction.
Case 4. If $\{m_q, m_{qp}\} = \{2^m u, 1/2(t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x + 4$.

Thus Lemma 2.15(1) implies that $p = 13$. On the other hand, in this case $q \neq 2$ and hence Step 3 implies that $p \mid m_2 + m_{2p}$. Thus p divides one of the numbers $(2x+1)$, (x^2+x+3), (x^2+4x+6) or (x^2-2). Lemma 2.15 now yields $p \in \{23, 43\}$, a contradiction.
Case 5. If $\{m_q, m_{qp}\} = \{2^m u, 1/2(u-1)2^m 3t\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x^2+x-4$. Thus Lemma 2.15(1) implies that $p = 101$. On the other hand, similar to Case 4, $p \mid m_2 + m_{2p}$ and hence $p = 23$ or 43, which is a contradiction.
Step 6. $m_p \notin \{1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$.
If $m_p = 1/2(t-1)2^m u$ or $m_p = 1/2(u-1)2^m 3t$, then since $p \mid 1+m_p$, we have $p \mid x^2-4x+6$ or $p \mid x^2-2$, respectively. In the former case, similar argument as stated in Step 5 leads us to a contradiction. So, it is enough to consider the case $p \mid x^2-2$ for $m_p = 1/2(u-1)2^m 3t$. According to Step 4, we have the following five cases:
Case 1. If $\{m_q, m_{qp}\} = \{2^m u, 3tu\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid 2x+1$. Thus Lemma 2.15(3) implies that $p = 7$. On the other hand, $p \mid 2x+1$, hence Lemma 2.12 implies that $4 \mid (p-1) = 6$, which is contradiction.
Case 2. If $\{m_q, m_{qp}\} = \{2^m u, 2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or u, which is contradiction.
Case 3. If $\{m_q, m_{qp}\} = \{2^m u, (t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p = 2$ or t or u, which is contradiction.
Case 4. If $\{m_q, m_{qp}\} = \{2^m u, 1/2(t-1)2^m u\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x+4$. Thus Lemma 2.15(3) implies that $p = 7$. On the other hand, $p \mid 2x+1$, hence Lemma 2.12 implies that $4 \mid (p-1) = 6$, which is contradiction.
Case 5. If $\{m_q, m_{qp}\} = \{2^m u, 1/2(u-1)2^m 3t\}$, then $p \mid m_q + m_{qp}$ and hence $p \mid x^2+x-4$. Thus Lemma 2.15(3) implies a contradiction. □

3.5. Lemma. *If $t \in \pi(G)$, then $u \in \pi(G)$.*

Proof. The proof will be divided into the following four steps.
Step 1. $m_t = 1/2(t-1)2^m u$.
According to Lemma 3.1, we have $m_t \neq 1$ and $(m_t, t) = 1$ and hence $m_t \neq 3tu, 1/2(u-1)2^m 3t$. If $m_t = 2^m u$, then Lemma 3.1 implies that $t \mid 1 + m_t$ and hence $x+1 \mid 3x^2 - 3x + 3 = (x+1)(3x-6) + 9$. Thus $x+1 \mid 9$. So $m = 3$, which is a contradiction. If $m_t = (t-1)2^m u$, then $t \mid 1+m_t$ and hence $x+1 \mid x^3-3x^2+2x+3 = (x+1)(x^2-4x+6)-3$. Thus $x+1 \mid 3$. So $m = 1$, which is a contradiction. Therefore, $m_t = 1/2(t-1)2^m u$.
Step 2. $t^2 \notin \pi_e(G)$.
If $t^2 \in \pi_e(G)$, then by (3.1), we have $t(t-1) = \varphi(t^2) \mid m_{t^2}$. Hence Lemma 2.14 implies that $m_{t^2} = 1/2(u-1)2^m 3t$. Since $t^2 \mid 1 + m_t + m_{t^2}$, we conclude that $(x+1)^2 \mid (x+1)^2(6x-21) + 30(x+1)$. So $(x+1) \mid 30$, which is a contradiction.
Step 3. $|G_t| = t$ and $n_t(G) = \frac{m_t}{\varphi(t)} = 1/2(2^m u)$.
Since $t^2 \notin \pi_e(G)$, Lemma 2.2 implies that $|G_t| \mid 1 + m_t$. If $t^2 \mid |G_t|$, then $2(x+1)^2 \mid (x+1)^2(3x-15) + 33(x+1)$. Thus $(x+1) \mid 33$ which implies that $m = 5,\ t = 11$ and $nse(G) = \{1, 992, 1023, 4960, 9920, 15840\}$. Since $2 \in \pi(G)$, there is the largest element $2 \leq i$ of $\pi_e(G)$ such that $(i, 11) = 1$. By Step 2, $11^2 \notin \pi_e(G)$. Thus $\sum_{i|d} m_d = m_i + m_{11i}$ or m_i and hence Lemma 2.5 implies that $11^2 \mid |G_{11}| \mid m_i + m_{11i}$ or m_i. But according to $nse(G)$, we can get a contradiction. Therefore, $|G_t| = t$ which implies that $n_t(G) = \frac{m_t}{\varphi(t)} = 1/2(2^m u)$.
Step 4. $u \in \pi(G)$.
According to Step 3, since $n_t(G) = 1/2(2^m u)$ and $n_t(G) \mid |G|$, we conclude that $u \in \pi(G)$.
□

3.6. Lemma. $\pi(G) = \{2, 3, t, u\}$.

Proof. According to Lemmas 3.2-3.5, we can conclude that $\{2,u\} \subseteq \pi(G) \subseteq \{2,3,t,u\}$. In the following three steps, we show $n_u(G) = 2^{m-1}3t$ which completes the proof.
Step 1. $m_u = 1/2(u-1)2^m 3t$.
According to Lemma 3.1, we have $m_u \neq 1$ and $(m_u, u) = 1$ and hence, according to $nse(G)$, it is obvious that $m_u = 1/2(u-1)2^m 3t$.
Step 2. $u^2 \notin \pi_e(G)$.
If $u^2 \in \pi_e(G)$, then by (3.1), $u(u-1) = \varphi(u^2) \mid m_{u^2}$. But according to Lemma 2.14 and $nse(G)$ we can easily see that there is no choice for m_{u^2}. Therefore, $u^2 \notin \pi_e(G)$.
Step 3. $|G_u| = u$.
Since $u^2 \notin \pi_e(G)$, Lemma 2.2 implies that $|G_u| \mid 1 + m_u$. If $u^2 \mid 1 + m_u$, then $(x-1)^2 \mid (x-1)^2(x+1) - (x-1)$ which implies that $(x-1) \mid 1$, a contradiction. So $|G_u| = u$ and $n_u(G) = \frac{m_u}{\varphi(u)} = 2^{m-1}3t$. □

3.7. Lemma. $m_3 = 2^m u$.

Proof. According to Lemma 3.1, we have $m_3 \neq 1$ and $(m_3, 3) = 1$ and hence, $m_3 \neq 3tu,\ 1/2(u-1)2^m 3t$. If $m_3 = 1/2(t-1)2^m u$, then by (3.1), we have $3 \mid 1 + m_3$. Thus $18 \mid (x+1)(x^2 - 4x + 6)$. Lemma 2.14 now yields $3 \mid (x^2 - 4x + 6)$ and hence, $3 \mid (x-4)$ which implies that $3 \mid (2^{m-2} - 1)$. Thus according to Lemma 2.10, $3 \mid (2^m - 1) = u$, which contradicts Lemma 2.14(c). Also, if $m_3 = (t-1)2^m u$, then by (3.1), we have $3 \mid 1 + m_3$ and hence, $9 \mid 3 + (x-2)x(x-1)$. This implies that $3 \mid (x-2)x(x-1)$ and $9 \nmid (x-2)x(x-1)$. Since according to Lemma 2.14(c), we have $(2,3) = (u,3) = 1$, so $3 \mid (x-2)$ and $9 \nmid (x-2)$. Now we claim that $3t \notin \pi_e(G)$. Indeed, if $3t \in \pi_e(G)$, then $m_{3t} = \varphi(3t) n_t(G) k$, where k is the number of cyclic subgroups of order 3 in $C_G(G_t)$. Actually, this follows from the fact that all centralizers of Sylow t-subgroups of G in G are conjugate in G. So we have $(t-1)2^m u = \varphi(3t) n_t(G) \mid m_{3t}$ which implies that $m_{3t} = (t-1)2^m u$. Since by (3.1), $3t \mid 1 + m_3 + m_t + m_{3t}$ and $t \mid 1 + m_t$ and $m_3 = m_{3t}$, we conclude that $t \mid (2m_3) = (t-1)2^{m+1}u$, which is a contradiction according to Lemma 2.14(c). Therefore, $3t \notin \pi_e(G)$ which implies that G_3 acts fixed point freely on the set of elements of order t by conjugation. Lemma 2.6 now leads to $|G_3| \mid m_t$. Now, according to Lemma 2.14(c), we conclude that $|G_3| \mid 1/3(x-2)$. Since $3 \mid (x-2)$ but $9 \nmid (x-2)$, we conclude that $|G_3| = 1$, which is a contradiction. □

3.8. Lemma. $9 \notin \pi_e(G)$.

Proof. If $9 \in \pi_e(G)$, then according to (3.1), we have $6 = \varphi(9) \mid m_9$ and by Lemma 2.14 and $nse(G)$, we conclude that $m_9 \in \{(t-1)2^m u, 1/2(t-1)2^m u, 1/2(u-1)2^m 3t\}$. So we have the following two cases:
Case 1. If $m_9 = 1/2(u-1)2^m 3t = 1/2(t-1)2^m 9t$, then $9 \mid m_9$. On the other hand, (3.1) implies that $9 \mid 1 + m_3 + m_9$ and hence, $9 \mid 1 + m_3$, which contradicts Lemma 2.14(e).
Case 2. If $m_9 = (t-1)2^m u$ or $1/2(t-1)2^m u$, then by (3.1), $9 \mid 1 + m_3 + m_9$. Since by Lemma 2.14(e), $3 \mid 1 + m_3$ and $9 \nmid 1 + m_3$, we conclude that $3 \mid m_9$ and $9 \nmid m_9$ and hence $3 \mid (t-1)$ and $9 \nmid (t-1)$. Lemma 2.4 yields G_3 is a cyclic group of order 3^k, where $k \geq 2$. Thus by (3.1), $n_3(G) = \frac{m_{3^k}}{\varphi(3^k)} = \frac{m_{3^k}}{2(3^{k-1})}$ and also, from (3.1) and Lemma 2.14, we conclude that $m_{3^k} \in \{(t-1)2^{m-1}9t, (t-1)2^{m-1}u, (t-1)2^m u\}$. Therefore, $n_3(G) \in \{\frac{(t-1)2^{m-2}9t}{3^{k-1}}, \frac{(t-1)2^{m-2}u}{3^{k-1}}, \frac{(t-1)2^{m-1}u}{3^{k-1}}\}$. Moreover, according to Lemma 2.14(d), there is a prime $p \in \pi(t-1) \setminus \{2,3,t,u\}$ which implies that $p \mid n_3(G)$. But since $n_3(G) \mid |G|$, we conclude that $p \in \pi(G)$, a contradiction. □

3.9. Lemma. $|G_u| = u,\ |G_t| = t,\ |G_2| \mid 2^m,\ |G_3| = 3$ *and hence,* $|G| = 2^k 3tu$*, where* $k \leq m$*.*

Proof. According to Lemmas 3.5 and 3.6, we have $|G_u| = u$ and $|G_t| = t$. Since $9 \notin \pi_e(G)$, Lemma 2.2 implies $|G_3| \mid 1 + m_3$ and hence, Lemma 2.14(e) leads to $|G_3| = 3$. We know that $2u \notin \pi_e(G)$. Actually, this follows by the same method as in Lemma 3.7. Therefore, G_2 acts fixed point freely on the set of elements of order u by conjugation and Lemma 2.6 implies that $|G_2| \mid m_u$ and hence, according to Lemma 2.14, we have $|G_2| \mid 2^m$. □

3.10. Lemma. *G is unsolvable.*

Proof. If G is solvable, then by Lemma 2.7, G has a Hall π-subgroup H, where $\pi = \{3, t, u\}$ and all the Hall π-subgroups of G are conjugate and hence, $|G : N_G(H)| \mid 2^m$. Since $|H| = 3tu$, we conclude that $n_u(H) \in \{1, 3, t, 3t\}$ and according to Sylow theorem, we have $n_u(H) \equiv 1 \pmod u$ and hence Lemma 2.14 implies that $n_u(H) = 1$. On the other hand, we can easily see that

$$n_u(G) \mid n_u(H).|G : N_G(H)|.|N_G(H) : H| \mid 2^{m+k}.$$

Also, since the Sylow u-subgroups of G are cyclic, we have $m_u = (u-1).n_u(G)$ and hence, $m_u \mid 2^{m+k}(u-1)$, but according to Lemma 3.6, Step 1, we have $m_u = 1/2(u-1)2^m 3t$, which is a contradiction. □

3.11. Lemma. $G \cong L_2(2^m)$.

Proof. Since G is a finite unsolvable group, according to Lemma 2.8, there is a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$, such that N is a normal solvable subgroup of G and M/N is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups. Let $M/N \cong S_1 \times \ldots \times S_r$, where S_1 is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$. According to $|G| = 2^k.3.t.u$, where $k \leq m$ and the structure of M/N, we can easily conclude that $r = 1$ and M/N is a simple K_3-group or a simple K_4-group.

Case 1. If M/N is a simple K_3-group, then according to Lemma 2.1, we have $\pi(M/N) \cap \{5, 7, 13, 17\} \neq \emptyset$. But since $\pi(M/N) \subseteq \pi(G)$ and $|G| = 2^k.3.t.u$, where $k \leq m$, we can get a contradiction.

Case 2. If M/N is a simple K_4-group, then by Lemma 2.1, M/N is isomorphic to one of the following groups:

- If $M/N \cong A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(81),\ L_2(243),\ L_2(577),$ $L_3(4),\ L_3(7),\ L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7),\ S_4(9),\ S_6(2),$ $O_8^+(2),\ G_2(3),\ U_3(5),\ U_3(8),\ U_3(9),\ U_4(3),\ U_5(2),\ {}^3D_4(2),\ {}^2F_4(2)'$ or $L_2(3^m)$, where m, $(3^m-1)/2$ and $(3^m+1)/4$ are odd primes, then $3^2 \mid |M/N|$, a contradiction.
- If $M/N \cong L_2(25),\ L_2(49),\ L_3(5),\ U_3(4),\ Sz(32)$, then $5^2 \mid |M/N|$, a contradiction.
- If $M/N \cong L_2(97),\ U_3(7)$, then $7^2 \mid |M/N|$, a contradiction.
- If $M/N \cong Sz(8)$, then $3 \nmid |M/N|$, a contradiction.
- If $M/N \cong L_2(16)$, then $t = 5$, a contradiction.
- If $M/N \cong L_2(r)$, where r is a prime, $r^2 - 1 = 2^a.3^b.v$, $v > 3$ is a prime, $a, b \in \mathbb{N}$, then $|M/N| = |L_2(r)| = \frac{1}{(r-1,2)}r(r^2-1) = \frac{1}{(r-1,2)}r.2^a.3^b.v$ and hence, $\pi(M/N) = \{2, 3, r, v\}$. Since $\pi(M/N) \subseteq \pi(G)$, we have $v = t,\ r = u$ or $v = u,\ r = t$. But since v is a prime number which divides $r^2 - 1$, according to Lemma 2.14(a-b) we can get a contradiction.
- If $M/N \cong L_2(2^{m'})$, where m' satisfies

$$\begin{cases} 2^{m'} - 1 = u' \\ 2^{m'} + 1 = 3t' \end{cases}$$

with $m' \geq 2,\ u', t'$ are primes, $t' > 3$, then $|M/N| = 2^{m'}.3.t'.u'$. Since $|M/N| \mid |G|$ and $|G| = 2^k.3.t.u$, where $k \leq m$, we conclude that $m' \leq m$ and $t' = t$ or u. If $t' = u$, then

$\frac{2^{m'}+1}{3} = 2^m - 1$. Thus $2^{m'}(3.2^{m-m'} - 1) = 4$, which is a contradiction . So we conclude $t' = t$ and this implies that $m = m'$ and $u' = u$. Therefore, $M/N \cong L_2(2^m)$, where m satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t \end{cases}$$

with $m \geq 2$, u, t are primes, $t > 3$.

Since $2^m.3tu = |M/N| \mid |G| = 2^k.3.t.u$, where $k \leq m$, we conclude that $N = 1$ and $M = G = L_2(2^m)$. $\square$

According to the main theorem, we pose the following problem:

Problem: Is a group G isomorphic to $L_2(2^m)(m \geq 2)$ if and only if $nse(G) = nse(L_2(2^m))$?

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