Hacettepe Journal of Mathematics and Statistics
Volume 45 (3) (2016), 663 – 673

Crossed modules of hypergroups associated with generalized actions

Murat Alp*†and Bijan Davvaz ‡

Abstract

In this article, by using the notion of generalized action, we introduce the concept of crossed module of hypergroups, in the sense of Marty, and its related structures from the light of crossed polymodules. Hypergroups in the sense of Marty are more different than polygroups since they have not identity element or inverse element in general. Examples of crossed modules of hypergroups are originally presented. These examples illustrate the structure and behavior of crossed modules of hypergroups. Moreover, we obtain a crossed module in the sense of Whitehead from a crossed module of hypergroups by applying the notion of fundamental relation.

Keywords: action, crossed module, hypergroup, fundamental relation.

2000 AMS Classification: 13D99, 20N20, 18D35.

Received : 17.03.2015 *Accepted* : 24.06.2015 *Doi* : 10.15672/HJMS.20164513095

1. Introduction

The crossed module is a very powerful applications tools for mathematicians. The importance of crossed modules are: crossed modules may be thought of as 2-dimensional objects (Groups, polygroups, etc), a number of improvements in group theory are better seen from a crossed module point of view and crossed modules occur geometrically as $\pi_2(X, A) \to \pi_1 A$ when A is a subspace of X or as $\pi_1 F \to \pi_1 E$ where $F \to E \to B$ is a fibration.

Crossed modules were defined by J. H. C. Whitehead in [25]. The important constructions of crossed modules are induced crossed module [8], actor of a crossed module

*Department of Mathematics, Nigde University, Nigde, Turkey.
Email : muratalp@nigde.edu.tr
†Corresponding Author.
‡Department of Mathematics, Yazd University, Yazd, Iran.
Email : bdavvaz@yahoo.com, davvaz@yazd.ac.ir
The paper was essentially prepared during the second author's stay at the Department of Mathematics, Nigde University in 2015. The second author is greatly indebted to Professor Murat Alp for his hospitality and TUBITAK-BIDEB.

[23] and pullback crossed modules of algebroids [3]. A new application of crossed module is the crossed module of polygroups [4]. Polygroups application can be taught as generalization of crossed module on groups. Cat^1-structures are defined and proved that the category of crossed modules is equivalent to the category of cat^1- structures by Loday [20]. So, many crossed module applications related to cat^1-structure were given by mathematicians after the definition of cat^1-structures such as pullback cat^1-commutative algebra [2] and cat^1-polygroups[13]. Also computations of these two categories play very important role to solve specific problems and construct examples to well known theories. GAP [16] provides a high level programming language with so many kind advantages. A GAP share package XMOD [6] was improved by taking these advantages. As example, [5] and [1] can be considered to this share package usage. Another important application of crossed module is the crossed module of hypergroups and is presented in this paper. When we defined a crossed module of hypergroups we thought normal subgroup condition $gN = Ng$ since hypergroup does not have inverse element. The importance of this application comes from this point of view. Polygroups and hypergroups studies can give a new direction to the different studies such as equivalent categories of simplicial polygroups and cat^1-polygroups. Therefore, properties of crossed module of hypergroups are given very cletailed in this paper.

Hypergroup theory was born in 1934, when Marty [22] gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and relational fractions. Nowadays the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, ethnology, etc. (see [10, 11]).

An outline of the paper is as follows. After the introduction, in Section 2, we give the very well known definition of crossed module and its examples. Definition, properties and examples of hypergroups are presented in Section 3. To define crossed module of hypergroups we need hypergroup action and a strong homomorphism. Two important needs are presented. Specially, hypergroup action and its examples are given in Section 4 due to [24] and [21]. Crossed module of hypergroups and its components such as examples and properties are given in Section 5.

2. Crossed modules

In this section we recall the definition of crossed module.

2.1. Definition. Let G be a group and X be a non-empty set. A *(left) group action* is a binary operator $\tau : G \times X \to X$ that satisfies the following two axioms:

(1) $\tau(gh, x) = \tau(g, \tau(h, x))$, for all $g, h \in G$ and $x \in X$,
(2) $\tau(e, x) = x$, for all $x \in X$.

For $x \in X$ and $g \in G$, we write ${}^{g}x := \tau(g, x)$.

2.2. Definition. A *crossed module* $X = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G \times M \to M$ on M, satisfying the conditions:

(1) $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$ and $g \in G$,
(2) ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

The standard examples of crossed modules are inclusion $M \hookrightarrow G$ of a normal subgroup M of G, the zero homomorphism $M \to G$ when M is a G-module, and any surjection $M \to G$ with central kernel, i.e., the kernel is a subset of center. There is also an important topological example: if $F \to E \to B$ is a fibration sequence of pointed spaces, then the induced homomorphism $\pi_1 F \to \pi_1 E$ of fundamental groups is naturally a crossed module [7].

In the next sections of the paper we present a very powerful application of crossed module due to [25]. The importance of this application comes from the fact that hypergroups do not have inverse element. From this reason we have to pay more attention to define hypergroup action and crossed module of hypergroup.

3. Hypergroups

Let H be a non-empty set and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \star)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \star B = \bigcup_{\substack{a\in A\\ b\in B}} a \star b, \ A \star x = A \star \{x\} \text{ and } x \star B = \{x\} \star B.$$

A hypergroupoid $(H, \star)$ is called a *semihypergroup* if for all a, b, c of H we have $(a\star b)\star c = a \star (b \star c)$, which means that

$$\bigcup_{u\in a\star b} u \star c = \bigcup_{v\in b\star c} a \star v.$$

A hypergroupoid $(H, \star)$ is called a *quasihypergroup* if for all a of H we have $a \star H = H \star a = H$. This condition is also called the *reproduction axiom*.

3.1. Definition. A hypergroupoid $(H, \star)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

3.2. Remark. Every group is a hypergroup.

In a hypergroup $(H, \star)$, an element $e \in H$ is called a *scalar identity element* if $e \star x = x \star e = \{x\} := x$, for all $x \in H$.

There exist many examples of hypergroups in [9, 11]. Here, we present two examples of hypergroups.

3.3. Example. (1) [9, 11] Let $(G, \cdot)$ be a group and H be a non-normal subgroup of it. If we denote $G/H = \{xH \mid x \in G\}$, then $(G/H, \star)$ is a hypergroup, where for all xH, yH of G/H, we have $xH \star yH = \{zH \mid z \in xHy\}$.

(2) [14] Let $H = \{1, 2, 3, 4\}$ with the hyperoperation defined in the following table:

$\star$	1	2	3	4
1	1	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 4\}$
2	$\{1, 2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3, 4\}$
3	$\{1, 2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3, 4\}$
4	$\{1, 4\}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	4

Then, $(H, \star)$ is a hypergroup.

3.4. Definition. Let $(C, \star)$ and $(H, \circ)$ be two hypergroups. Let ∂ be a map from C into H. Then, ∂ is called

(1) an *inclusion homomorphism* if

$$\partial(x \star y) \subseteq \partial(x) \circ \partial(y), \text{ for all } x, y \in C;$$

(2) a *strong homomorphism* or a *good homomorphism* if

$$\partial(x \star y) = \partial(x) \circ \partial(y), \text{ for all } x, y \in C.$$

3.5. Example. In Example 3.3(1), suppose that G is the symmetric group of degree 3, $H = \langle(1\ 2)\rangle$ and $C = \langle(2\ 3)\rangle$. Then, we have

$$\begin{aligned}&H = (1\ 2)H = \{e,\ (1\ 2)\}\\&(1\ 3)H = (1\ 3\ 2)H = \{(1\ 3),\ (1\ 3\ 2)\}\\&(2\ 3)H = (1\ 2\ 3)H = \{(2\ 3),\ (1\ 2\ 3)\}\end{aligned}$$

Hence, $G/H = \{H,\ (1\ 3)H,\ (2\ 3)H\}$. By easy calculation we obtain the following multiplication table on G/H.

$\circ$	H	$(1\ 3)H$	$(2\ 3)H$
H	H	$\{(1\ 3)H,\ (2\ 3)H\}$	$\{(1\ 3)H,\ (2\ 3)H\}$
$(1\ 3)H$	$(1\ 3)H$	$\{H,\ (2\ 3)H\}$	$\{H,\ (2\ 3)H\}$
$(2\ 3)H$	$(2\ 3)H$	$\{H,\ (1\ 3)H\}$	$\{H,\ (1\ 3)H\}$

Similarly, we have

$$\begin{aligned}&C = (2\ 3)C = \{e,\ (2\ 3)\}\\&(1\ 2)C = (1\ 3\ 2)C = \{(1\ 2),\ (1\ 3\ 2\}\\&(1\ 3)C = (1\ 2\ 3)C = \{(1\ 3),\ (1\ 2\ 3\}\end{aligned}$$

Hence, $G/C = \{C,\ (1\ 2)C,\ (1\ 3)C\}$. Again, by easy calculation we obtain the following multiplication table on G/C.

$\star$	C	$(1\ 2)C$	$(1\ 3)C$
C	C	$\{(1\ 2)C,\ (1\ 3)C\}$	$\{(1\ 2)C,\ (1\ 3)C\}$
$(1\ 2)C$	$(1\ 3)C$	$\{C,\ (1\ 3)C\}$	$\{C,\ (1\ 3)C\}$
$(1\ 3)C$	$(2\ 3)C$	$\{C,\ (1\ 2)C\}$	$\{C,\ (1\ 2)C\}$

Now, we define the map $\partial : G/C \to G/H$ by $\partial(C) = H$, $\partial((1\ 2)C) = (1\ 3)H$ and $\partial((1\ 3)C) = (2\ 3)H$. It is straightforward to that ∂ is a strong homomorphism.

4. Hypergroup action

According to [17, 24], we can consider a *generalized permutation* on a non-empty set X as a map $f : X \to \mathcal{P}^*(X)$ such that the *reproductive axiom* holds, i.e.,

$$\bigcup_{x\in X} f(x) = f(X) = X.$$

We denote the set of all generalized permutations by M_X. A generalized permutation f is said to satisfy the condition θ if $x \in X$ and $z \in f(x)$, then $f(z) = f(x)$ [24]. We denote the set of all generalized permutations that satisfies the condition θ by M_θ.

4.1. Proposition. [24] *Let $f \in M_\theta$ and $M_f = \{g \in M_X \mid g \subseteq f\}$. Then, M_f is a hypergroup with respect to the hyperoperation $\star$ defined by $f_1 \star f_2 = \{p \in M_X \mid p \subseteq f_1 \circ f_2\}$, where $f_1 \circ f_2$ is defined by $f_1 \circ f_2 = \bigcup\limits_{y\in f_2(x)} f_1(y)$.*

Several mathematicians considered actions of algebraic hyperstructures, for example see [21, 12, 26]. In [21], Madanshekaf and Ashrafi considered a generalized action of a hypergroup H on a non-empty set X and obtained some results in this respect. For the definition of crossed modules of hypergroups, we need the notion of hypergroup action. So, we recall the following definition from [21].

4.2. Definition. Let $(H, \star)$ be a hypergroup and X be a non-empty set. A map $\alpha : H \times X \to \mathcal{P}^*(X)$ is called a *generalized action* of H on X, if the following axiom hold:

(1) $\alpha(g \star h, x) \subseteq \alpha(g, \alpha(h, x))$, for all $g, h \in H$ and $x \in X$, where

$$\alpha(g \star h, x) = \bigcup_{k\in g\star h} \alpha(k, x).$$

(2) For all $h \in H$, $\alpha(h, X) = X$, where

$$\alpha(h, X) = \bigcup_{x \in X} \alpha(h, x).$$

If the equality holds in axiom (1) of Definition 4.2, the action is called *strong generalized action*. Moreover, if H has the scalar identity element e, then the following condition must holds too,

(3) $\alpha(e, x) = \{x\} := x$, for all $x \in X$.

4.3. Example. [21]

(1) For any hypergroup $(H, \star)$ and any non-empty set X, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, given by $\alpha(h, x) = X$ is a strong generalized action of H on X. If we define $\alpha(h, x) = \{x\}$, then this map is also a strong generalized action of H on X.
(2) Let $(H, \star)$ be a hypergroup. Then, the map $\alpha : H \times H \to \mathcal{P}^*(H)$, given by $\alpha(h, x) = h \star x$ is a strong generalized action of H on H.

4.4. Example. [21] Let X be a non-empty set, $f \in M_\theta$ and $H = M_f$. Then, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, defined by $\alpha(h, x) = h(x)$ is a strong generalized action of H on X.

For $x \in X$, we put ${}^{h}x := \alpha(h, x)$. Then, for a strong generalized action, we have

(1) ${}^{g}(\ {}^{h}x) ={}^{g\star h} x$, for all $g, h \in H$ and $x \in X$.
(2) $\bigcup_{x \in X} {}^{h}x = X$, for all $h \in H$.

4.5. Example. Consider Example 3.3(1). We define the map $\alpha : G/H \times G \to \mathcal{P}^*(G)$ by ${}^{yH}x := yHx$. Then, α is a strong generalized action.

4.6. Example. Suppose that G/H and G/C are the hypergroups defined in Example 3.5 and ∂ is the homomorphism between them. We define $\alpha : G/H \times G/C \to \mathcal{P}^*(G/C)$ by

$${}^{gH}xC := \{zC \mid z \in gHx\}.$$

We show that α is a strong generalized action.

(1) For all $g_1H, g_2H \in G/H$ and $xC \in G/C$ we have

$$\begin{aligned} {}^{g_2H}\left(\ {}^{g_1H}xC\right) &= {}^{g_2H}\left(\{zC \mid z \in g_1Hx\}\right) \\ &= \{aC \mid a \in g_2Hz,\ z \in g_1Hx\} \\ &= \{aC \mid a \in g_2Hg_1Hx\}, \end{aligned}$$

$$\begin{aligned} {}^{g_2H \circ g_1H}xC &= {}^{\{zH \mid z \in g_2Hg_1\}}xC \\ &= \{bC \mid b \in zHx,\ z \in g_2Hg_1\} \\ &= \{bC \mid b \in g_2Hg_1Hx\}. \end{aligned}$$

Thus, ${}^{g_2H}\left(\ {}^{g_1H}xC\right) ={}^{g_2H \circ g_1H} xC$.

(2) Clearly, for all $gH \in G/H$ we have $\bigcup_{xC \in G/C} {}^{gH}xC = G/C$.

5. Crossed module of hypergroups

Now, in this section, we give the notion of crossed module of hypergroups. To define a crossed module of hypergroups, we need the notion of hypergroup action and boundary strong homomorphism.

5.1. Definition. A *crossed module of hypergroups* $\mathcal{X} = (C, H, \partial, \alpha)$ consists of hypergroups $(C, \star)$ and $(H, \circ)$ together with a strong homomorphism $\partial : C \to H$ and a strong generalized action $\alpha : H \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $h \circ \partial(c) \subseteq \partial({}^{h}c) \circ h$, for all $c \in C$ and $h \in H$.
(2) $c \star c' \subseteq {}^{\partial(c)}c' \star c$, for all $c, c' \in C$.

5.2. Example. Suppose that H is a non-empty set. We define the hyperoperation $\circ$ on H by

$$h_1 \circ h_2 = \{h_1,\ h_2\},\ \text{for all } h_1, h_2 \in H.$$

Then, $(H, \circ)$ is a hypergroup. Suppose that C is a subhypergroup of H and $\partial : C \to H$ is the identity map. The map $\alpha : H \times C \to \mathcal{P}^*(C)$ is defined by ${}^{h}c := C$ is a strong generalized action. Moreover,

(1) For all $c \in C$ and $h \in H$, we have

$$h \circ \partial(c) = h \circ c = \{h,\ c\} \subseteq C \cup \{h\} = C \circ h = \partial(C) \circ h = \partial({}^{h}c) \circ h.$$

(2) For all $c, c' \in C$, we have

$$c \circ c' = \{c,\ c'\} \subseteq C = C \circ c = {}^{c}c' \circ c = {}^{\partial(c)}c' \circ c.$$

Therefore, $\mathcal{X} = (C, H, \partial, \alpha)$ is a crossed module of hypergroups.

5.3. Example. Suppose that G is an abelian group and P a non-empty subset of G. We consider the P-hyperoperation $\star_P$ on G as follows:

$$x \star_P y = xyP,\ \text{for all } x, y \in G.$$

Then, $(G, \star_P)$ is a hypergroup. Suppose that $\partial : G \to G$ is the identity map. The map $\alpha : G \times G \to \mathcal{P}^*(G)$ is defined by ${}^{g}x := \{x\}$ is a strong generalized action. Moreover,

(1) For all $x, y \in G$, we have

$$g \star_P \partial(x) = g \star_P x = gxP = xgP = x \star_P g = \partial(x) \star_P g = \partial({}^{g}x) \star_P g$$

(2) For all $x, y \in G$, we have

$$x \star_P y = xyP = yxP = y \star_P x = {}^{x}y \star_P x = {}^{\partial(x)}y \star_P x.$$

Therefore, $\mathcal{X} = ((G, \star_P), (G, \star_P), \partial, \alpha)$ is a crossed module of hypergroups.

5.4. Example. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules of hypergroups has source $C_1 \times C_2$, range $H_1 \times H_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $H_1 \times H_2$ acting obviously on $C_1 \times C_2$.

5.5. Theorem. *Every crossed module is a crossed module of hypergroups.*

Proof. By using Remark 3.2, the proof is straightforward. □

Let $(H\circ)$ be a hypergroup. We define the relation β^*_H as the smallest equivalence relation on H such that the quotient H/β^*_H, the set of all equivalence classes, is a group. In this case β^*_H is called the *fundamental equivalence relation* on H and H/β^*_H is called the *fundamental group*. The product $\odot$ in H/β^*_H is defined as follows: $\beta^*_H(x) \odot \beta^*_H(y) = \beta^*_H(z),$ for all $z \in \beta^*_H(x) \circ \beta^*_H(y)$. This relation is introduced by Koskas [18] and studied mainly by Corsini [9], Leoreanu-Fotea [19] and Freni [15] concerning hypergroups, Vougiouklis [24] concerning H_v-groups, Davvaz concerning polygroups [11], and many others. We consider the relation β_H as follows:

$$x\ \beta_H\ y \ \Leftrightarrow\ \text{there exist } z_1, \ldots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni proved that for hypergroups $\beta = \beta^*$ i in [15]. The kernel of the *canonical map* $\varphi_H : H \longrightarrow H/\beta^*_H$ is called the *heart* of H and is denoted by ω_H. Here we also denote by ω_H the unit of H/β^*_H. The heart of a hypergroup H is the intersection of all subhypergroups of H, which are complete parts.

5.6. Lemma. [9] *ω_P is a subhypergroup of H.*

Throughout the paper, we denote the binary operations of the fundamental groups H/β_H^* and C/β_C^* by $\odot$ and $\otimes$, respectively.

Now, we consider the notion of kernel of a strong homomorphism of hypergroups.

5.7. Definition. Let $(H, \circ)$ and $(C, \star)$ be two hypergroups and $\partial : C \to H$ be a strong homomorphism. The *core-kernel* of ∂ is defined by

$$ker^*\partial = \{x \in C \mid \partial(x) \in \omega_H\}.$$

5.8. Theorem. *$ker^*\partial$ is a subhypergroup of C.*

Proof. Suppose that $x, y \in ker^*\partial$ are arbitrary. Then, $\partial(x), \partial(y) \in \omega_H$ and so

$$\beta_H^*((\partial(x \star y))) = \beta_H^*(\partial(x) \circ \partial(y)) = \beta_H^*(\partial(x)) \otimes \beta_H^*(\partial(x)) = \omega_H \otimes \omega_H = \omega_H.$$

Therefore, $\partial(x \star y) \subseteq \omega_H$. This implies that $x \star y \subseteq ker^*\partial$. Now, we show that $x \star ker^*\partial = ker^*\partial \star x = ker^*\partial$, for all $x \in ker^*\partial$. Clearly, according to the above proof, we have $x \star ker^*\partial \subseteq ker^*\partial$. So, we show that $ker^*\partial \subseteq x \star ker^*\partial$. Suppose that $x, y \in ker^*\partial$. Then, there exists $z \in C$ such that $y \in x \star z$. Hence,

$$\partial(y) \in \partial(x \star z) = \partial(x) \circ \partial(z).$$

This implies that

$$\beta_H^*(\partial(y)) = \beta_H^*(\partial(x) \circ \partial(z)) = \beta_H^*(\partial(x)) \odot \beta_H^*(\partial(z))$$

and so we obtain $\omega_H = \omega_H \odot \beta_H^*(\partial(z))$. Hence, $z \in ker^*\partial$. Thus, $y \in x \star ker^*\partial$. Similarly, we can show that $ker^*\partial \star x = ker^*\partial$. □

5.9. Definition. We say that $(A, B, \partial', \alpha')$ is a *subcrossed module* of the crossed module of hypergroups (C, H, ∂, α) if

(1) A is a subhypergroup of C, and B is a subhypergroup of H,
(2) ∂' is the restriction of ∂ to A,
(3) the action of B on A is induced by the action of H on C.

5.10. Definition. Let $\mathcal{X} = (C, P, \partial, \alpha)$ and $\mathcal{X}' = (C', P', \partial', \alpha')$ be two crossed modules of hypergroups. A *crossed module of hypergroups morphisms*

$$< \theta, \phi >: (C, H, \partial, \alpha) \to (C', H', \partial', \alpha')$$

is a commutative diagram of strong homomorphisms of hypergroups

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[d, "\partial"'] & C' \arrow[d, "\partial'"] \\ H \arrow[r, "\phi"'] & H' \end{tikzcd}$$

such that for all $h \in H$ and $c \in C$, we have

$$\theta({}^{h}c) = {}^{\phi(h)}\theta(c).$$

We say that $< \theta, \phi >$ is an *isomorphism* if θ and ϕ are both isomorphisms. Similarly, we can define *monomorphism*, *epimorphism* and *automorphism* of crossed modules of hypergroups.

5.11. Proposition. *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\partial : C \to H$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta_C^* \to H/\beta_H^*$ by setting*

$$\mathcal{D}(\beta_C^*(c)) = \beta_H^*(\partial(c)), \text{ for all } c \in C.$$

Proof. First, we prove that $\mathcal{D}$ is well defined. Suppose that $\beta^*_C(c_1) = \beta^*_C(c_2)$. Then, there exist $a_1, \ldots, a_n$ such that $\{c_1, c_2\} \subseteq \star \prod_{i=1}^{n} a_i$. So,

$$\{\partial(c_1), \partial(c_2)\} \subseteq \partial\left(\star \prod_{i=1}^{n} a_i\right) = \circ \prod_{i=1}^{n} \partial(a_i).$$

Hence, $\partial(c_1)\ \beta^*_H\ \partial(c_2)$, which implies that $\mathcal{D}\left(\beta^*_C(c_1)\right) = \mathcal{D}\left(\beta^*_C(c_2)\right)$. Now, we have

$$\begin{aligned}
\mathcal{D}(\beta^*_C(c_1) \otimes \beta^*_C(c_2)) &= \mathcal{D}(\beta^*_C(c_1 \star c_2)) = \beta^*_H(\partial(c_1 \star c_2)) \\
&= \beta^*_H(\partial(c_1) \circ \partial(c_2)) = \beta^*_H(\partial(c_1)) \odot \beta^*_H(\partial(c_2)) \\
&= \mathcal{D}(\beta^*_C(c_1)) \odot \mathcal{D}(\beta^*_C(c_2)).
\end{aligned}$$

□

We say the action of H on C is *productive*, if for all $c \in C$ and $h \in H$ there exist $c_1, \ldots, c_n$ in C such that ${}^{h}c = c_1 \star \ldots \star c_n$.

Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\alpha : H \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : H/\beta^*_H \times H/\beta^*_C \to \mathcal{P}^*(H/\beta^*_C)$ as usual manner:

$$\psi(\beta^*_H(h), \beta^*_C(c)) = \{\beta^*_C(x) \mid x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_H(h)}} {}^{z}y\}.$$

By the definition of β^*_C, since the action of H on C is productive, we conclude that $\psi(\beta^*_H(h), \beta^*_C(c))$ is singleton, i.e., we have

$$\psi : H/\beta^*_H \times H/\beta^*_C \to H/\beta^*_C,$$
$$\psi(\beta^*_H(h), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_H(h)}} {}^{z}y.$$

We denote $\psi(\beta^*_H(h), \beta^*_C(c)) = {}^{[\beta^*_H(h)]}[\beta^*_C(c)]$.

5.12. Proposition. *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\alpha : H \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group H/β^*_H on the group C/β^*_C.*

Proof. Suppose that $g, h \in H$ and $c \in C$. Then, we have

$$\psi(\beta^*_H(h) \odot \beta^*_H(g), \beta^*_C(c)) = \psi(\beta^*_H(h \circ g), \beta^*_C(c)) = {}^{[\beta^*_H(h \circ g)]}[\beta^*_C(c)],$$

and

$$\psi(\beta^*_H(h), \psi(\beta^*_H(g), \beta^*_C(c))) = \psi\left(\beta^*_H(h), {}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right) = {}^{[\beta^*_H(h)]}\left({}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right).$$

By the condition (1) of Definition 4.2, we have ${}^{h}({}^{g}c) = {}^{h \circ g}c$. Now, it is not difficult to see that

$${}^{[\beta^*_H(h \circ g)]}[\beta^*_C(c)] = {}^{[\beta^*_H(h)]}\left({}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right).$$

□

5.13. Theorem. *Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crosed module of hypergroups such that the action of H on C is productive. Then, $X = (C/\beta^*_C, H/\beta^*_H, \mathcal{D}, \psi)$ is a crossed module.*

Proof. By Propositions 5.11 and 5.12, it is enough to show that the conditions of Definition 2.2 hold. Suppose that $c \in C$ and $h \in H$ are arbitrary. Then, we have

$$\begin{aligned}\mathcal{D}\left({}^{[\beta^*_H(h)]}([\beta^*_C(c)])\right) \odot \beta^*_H(h) &= \mathcal{D}\left([\beta^*_C(z]\right) \odot \beta^*_H(h), \text{ for all } z \in {}^{h}c\\ &= \beta^*_H(\partial(z)) \odot \beta^*_H(h), \text{ for all } z \in {}^{h}c\\ &= \beta^*_H(\partial({}^{h}c)) \circ h))\\ &= \beta^*_H(h \circ \partial(c))\\ &= \beta^*_H(h) \odot \beta^*_H(\partial(c))\\ &= \beta^*_H(h) \odot \mathcal{D}(\beta^*_C(c)),\end{aligned}$$

which implies that $\mathcal{D}\left({}^{[\beta^*_H(h)]}([\beta^*_C(c)])\right) = \beta^*_H(h) \odot \mathcal{D}(\beta^*_C(c)) \odot \beta^*_H(h)^{-1}$. So, the first condition of Definition 2.2 holds. For the second condition, suppose that $c, c' \in C$ are arbitrary. Then, we have

$$\begin{aligned}{}^{[\mathcal{D}(\beta^*_C(c))]}[\beta^*_C(c')] \otimes \beta^*_C(c) &= {}^{[\beta^*_P(\partial(c))]}[\beta^*_C(c')] \otimes \beta^*_C(c)\\ &= \beta^*_C(z) \otimes \beta^*_C(c), \text{ for all } z \in {}^{\partial(c)}c'\\ &= \beta^*_C\left({}^{\partial(c)}c' \star c\right)\\ &= \beta^*_C(z), \text{ for all } z \in c \star c'\\ &= \beta^*_C(c \star c')\\ &= \beta^*_C(c) \otimes \beta^*_C(c'),\end{aligned}$$

which implies that ${}^{[\mathcal{D}(\beta^*_C(c))]}[\beta^*_C(c')] = \beta^*_C(c) \otimes \beta^*_C(c') \otimes \beta^*_C(c)^{-1}$. □

5.14. Theorem. *Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crossed module of hypergroups, φ_C and φ_P be canonical maps. Then, $< \varphi_C, \varphi_H >$ is a crossed modules of hypergroups morphisms.*

Proof. Note that according to Theorem 5.13, we can consider $(C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ as a crossed module of hypergroups. We show that the following diagram is commutative.

$$\begin{tikzcd} C \arrow[r, "\varphi_C"] \arrow[d, "\partial"'] & C/\beta^*_C \arrow[d, "\mathcal{D}"] \\ H \arrow[r, "\varphi_H"'] & H/\beta^*_H \end{tikzcd}$$

Indeed, we have $\mathcal{D}\varphi_C(c) = \mathcal{D}\left(\beta^*_C(c)\right) = \beta^*_H(\partial(c)) = \varphi_H\partial(c)$, for all $c \in C$. Moreover,

$$\varphi_C({}^{h}c) = \beta^*_C({}^{h}c) = {}^{[\beta^*_H(h)]}[\beta^*_C(c)] = {}^{\varphi_H(h)}\varphi_C(c),$$

for all $c \in C$ and $h \in H$. Therefore, $< \varphi_C, \varphi_H >$ is a crossed module of hypergroup morphism. □

The following example give us another crossed module structure on the fundamental groups.

5.15. Example. Suppose that $(H, \circ)$ is a hypergroup. Then, H/β_H^* is a group. Suppose that $Aut\,(H/\beta_H^*)$ its group of automorphisms. There is an obvious action α of $Aut\,(H/\beta_H^*)$ on H/β_H^*, and a group homomorphism $\partial : H/\beta_H^* \to Aut\,(H/\beta_H^*)$ sending each $\beta_H^*(h) \in P/\beta_P^*$ to the inner automorphism of conjugation by $\beta_P^*(p)$. These together form a crossed module $(H/\beta_H^*, Aut\,(H/\beta_H^*)\,, \partial, \alpha)$.

References

[1] M. Alp, *Enumeration of 1-truncated simplicial groups of low order*, Indian J. Pure Appl. Math., 35 (3) (2004), 333-345.

[2] M. Alp, *Pullbacks of Crossed Modules and Cat1- Commutative Algebras*, Turkish J. Math., 30 (2006), 237-246.

[3] M. Alp, *Pullback Crossed modules of Algebroids*, Iranian J. Sci. & Tech., Transaction A, 32(A3) (2008), 175-181.

[4] M. Alp and B. Davvaz, *Crossed polymodules and fundamental relations*, U.P.B. Sci. Bull., Series A, 77(2) (2015), 129-140

[5] M. Alp and C. D. Wensley, *Enumeration of Cat1-groups of low order*, Internat. J. Algebra Comput., 10(4) (2000), 407-424.

[6] M. Alp and C. D. Wensley, *XMOD, Crossed modules and cat1-groups in GAP, version 2.26*, (2013), 1-49.

[7] R. Brown and N.D. Gilbert, *Algebraic models of 3-types and authomorphism structures for crossed modules*, Proc. London Math. Soc., 59(3) (1989), 51-73.

[8] R. Brown, and P. J. Higgins, *On the connection between the second relative homotopy group some related space*, Proc. London. Math. Soc., 36 (1978), 193-212.

[9] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviain editore, 1993.

[10] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[11] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[12] B. Davvaz, *On polygroups and permutation polygroups*, Math. Balkanica (N.S.), 14(1-2) (2000), 41-58.

[13] B. Davvaz and M. Alp, *Cat^1-Polygroup and Pullback cat^1-Poly group*, Bull. Iranian Math. Soc., 40(3) (2014), 721-735.

[14] M. Farshi, B. Davvaz and S. Mirvakili, *Hypergraphs and hypergroups based on a special relation*, Comm. Algebra, 42 (2014), 3395-3406.

[15] D. Freni, *A note on the core of a hypergroup and the transitive closure β^* of β*, Riv. Mat. Pura Appl., 8 (1991), 153-156.

[16] GAP4 The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.5.5*; 2012, (`http://www.gap-system.org`).

[17] M. Gionfriddo, *Hypergroups associated with multihomomorphisms between generalized graphs*, Convegno su sistemi binari e loro applicazioni, Edited by P. Corsini. Taormina, 1978, pp. 161-174.

[18] M. Koskas, *Groupoids, demi-groupes et hypergroupes*, J. Math. Pures Appl., 49 (1970), 155-192.

[19] V. Leoreanu, *The heart of some important classes of hypergroups*, Pure Math. Appl., 9 (1998), 351-360.

[20] J. L. Loday, *Spaces with finitely many non-trivial homotopy groups*, J. App. Algebra, 24 (1982), 179-202.

[21] A. Madanshekaf and A. R. Ashrafi, *Generalized action of a hypergroup on a set*, Ital. J. Pure Appl. Math., 3 (1998), 127-135.

[22] F. Marty, *Sur une generalization de la notion de groupe*, 8^th Congres Math. Scandinaves, Stockholm (1934), 45-49.

[23] K. J. Norrie, *Actions and automorphisms of crossed modules*, Bull. Soc. Math. France, 118 (1990), 129-146.

[24] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Inc., 1994.

[25] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc., 55 (1949), 453-496.

[26] J. Zhan, S. Sh. Mousavi and M. Jafarpour, *On hyperactions of hypergroups*, U.P.B. Sci. Bull., Series A,, 73(1) (2011), 117-128.