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Exponentiated generalized geometric distribution: A new discrete distribution

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Abstract

In this paper, a new three-parameter extension of the generalized geometric distribution of [6] is introduced. The new discrete distribution belongs to the resilience parameter family and handles a decreasing, increasing, upside-down and bathtub-shaped hazard rate function. The new distributions can also be considered as discrete analogs of some recent continuous distributions belonging to the known Marshall-Olkin family. Here, some basic statistical and mathematical properties of the new distribution are studied. In addition, estimation of the unknown parameters, a simulated example and an application of the new model are illustrated.

Keywords: Geometric distribution, Generalized geometric distribution, Generalized exponential geometric distribution, Marshall-Olkin family, Resilience parameter family.

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1. Introduction

Discretizing continuous distributions has recently received much attention in the literature. Let $F(x) = P(X \leq x)$ be the cumulative distribution function (cdf) of the absolutely continuous random variable X. The corresponding probability mass function (pmf) of X can be obtained by

$$(1.1) \qquad P(X = x) = p_x = F(x+1) - F(x), \quad x \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

In recent years, several new discrete distributions have been appeared in the literature by Eq. (1.1). For example, we can address the works of [12], [15], [9] and [10] which are the discrete versions of Weibull, Rayleigh, half-normal and Burr distributions, respectively. A discrete version of Lindley distribution is introduced by [7] and [4]. [13] obtained a new discrete distribution by discretizing generalized exponential distribution of [8]. Discrete modified Weibull distributions, which are discrete versions of some known modified Weibull distributions, are introduced by [14] and [3]. In addition, [5] introduced the discrete additive Weibull distribution as a discrete version of the additive Weibull distribution of [19].

[11] introduced an extended family of distributions generated by the cdf

$$(1.2) \qquad F(x; \alpha) = \frac{G(x)}{1 - \overline{\alpha}\overline{G}(x)}; \quad x \in \mathbb{R}_X, \quad \alpha > 0,$$

where $\overline{\alpha} = 1 - \alpha$ and $G(x)$ is the cdf of an absolutely continuous distribution. Several new continuous distributions have been obtained in the literature by inserting an arbitrary $G(x)$ into Eq. (1.2). For example, inserting the cdf of the exponential distribution into Eq. (1.2) yields a new distribution, called exponential Marshall-Olkin distribution, with cdf

$$(1.3) \qquad F(x; \alpha, \beta) = \frac{1 - e^{-\beta x}}{1 - \overline{\alpha} e^{-\beta x}}; \quad x > 0, \quad \beta > 0, \alpha > 0,$$

(see [11]). For $0 < \alpha < 1$, (1.3) coincides with the cdf of the exponential-geometric (EG) distribution of [2].

[6] obtained the generalized geometric (GG) distribution by discretizing the exponential Marshall-Olkin distribution using Eq. (1.1). It is evident that when $0 < \alpha < 1$, the GG distribution corresponds to a discrete analogue of the EG distribution.

In this paper, we will introduce the exponentiated generalized geometric (EGG) distribution which is indeed an extension of the GG distribution. This new distribution can also be considered as a discrete version of the generalized exponential-geometric (GEG) distribution of [17].

The paper is organized as follows: In Section 2, we introduce the new distribution and investigate some of its statistical properties. We also derive expressions for the probability generating function, moment generating function and factorial moments. In Section 3, we will show that the proposed distributions are not infinitely divisible in general. The order statistics are discussed in Section 4. Estimation, Fisher information matrix and a kind of simulated example are discussed in Section 5. An application of the new model is illustrated in Section 6. Finally, Section 7 involves some concluding remarks.

2. Three-parameter EGG distribution

Consider the $GG(\alpha, \theta)$ distribution of [6] with the cdf

$$(2.1) \qquad F_{GG}(x; \alpha, \theta) = \frac{1 - \theta^{x+1}}{1 - \overline{\alpha}\theta^{x+1}}, \quad x \geq 0,$$

where $\alpha > 0$ and $0 < \theta < 1$ are the model parameters. By inserting (2.1) into the *resilience parameter family* of distributions, the cdf of the resulting discrete distribution is given by

$$F(x;\alpha,\theta,\gamma) = [F_{GG}(x;\alpha,\theta)]^{\gamma} = \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}, \quad x \geq 0, \tag{2.2}$$

in which $\gamma > 0$ is the *resilience parameter*.

We call such a random variable X, with cdf (2.2), an *exponentiated generalized geometric distribution* with parameters $\alpha > 0$, $0 < \theta < 1$ and $\gamma > 0$ and denote it by $EGG(\alpha, \theta, \gamma)$.

It is evident that when $\gamma > 0$ is an integer value, the cdf given by (2.2) agrees with the cdf of the maximum of γ independent and identical $GG(\alpha, \theta)$ random variables.

The corresponding pmf of a random variable X following an $EGG(\alpha, \theta, \gamma)$ distribution for $x \in \mathbb{N}_0$ is given by

$$f(x;\alpha,\theta,\gamma) = P(X = x) = \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma} - \left[\frac{1-\theta^{x}}{1-\overline{\alpha}\theta^{x}}\right]^{\gamma}. \tag{2.3}$$

[17] introduced the continuous generalized exponential-geometric (GEG) distribution with cdf

$$F(x;\alpha,\theta,\gamma) = \left[\frac{1-e^{-\beta x}}{1-\overline{\alpha}e^{-\beta x}}\right]^{\gamma}, \tag{2.4}$$

where $0 < \alpha < 1$, $\beta > 0$ and $\gamma > 0$ are the model parameters. The above cdf is indeed a kind of exponentiated distribution which contains the EG distribution of [2] as a special case, when $0 < \alpha < 1$. It is interesting to note that for $0 < \alpha < 1$ and $0 < e^{-\beta} = \theta < 1$, the EGG distribution can be viewed as a discrete version of the GEG distribution. In addition, the EGG distribution reduces to the GG distribution when $\gamma = 1$. Several properties of the $GG(\alpha, \theta)$ distribution are obtained for the case $0 < \alpha < 1$; see [6]. We will study several properties of the $EGG(\alpha, \theta, \gamma)$ distribution in this case. Figure 1 plots the pmfs of the $EGG(\alpha, \theta, \gamma)$ distribution for some parameters values.

The survival and hazard rate functions of the $EGG(\alpha, \theta, \gamma)$ distribution are given by

$$S(x;\alpha,\theta,\gamma) = 1 - \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}, \quad x \geq 0$$

and

$$h(x;\alpha,\theta,\gamma) = \frac{\left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma} - \left[\frac{1-\theta^{x}}{1-\overline{\alpha}\theta^{x}}\right]^{\gamma}}{1 - \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}}, \quad x \in \mathbb{N}_0,$$

respectively. As we see from Figure 2, the hazard rate function of the new distribution can be decreasing, increasing, upside-down bathtub and bathtub-shaped, depending on its parameters values, and hence presents a very flexible behavior.

Now, let $b > 1$ and $k > 0$ be real non-integers. If $|z| < 1$, we have the series representations

$$(1-z)^{b} = \sum_{i=0}^{\infty} \frac{\Gamma(b+1)}{\Gamma(b+1-i)i!}(-1)^{i} z^{i} \tag{2.5}$$

and

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^{j}.$$

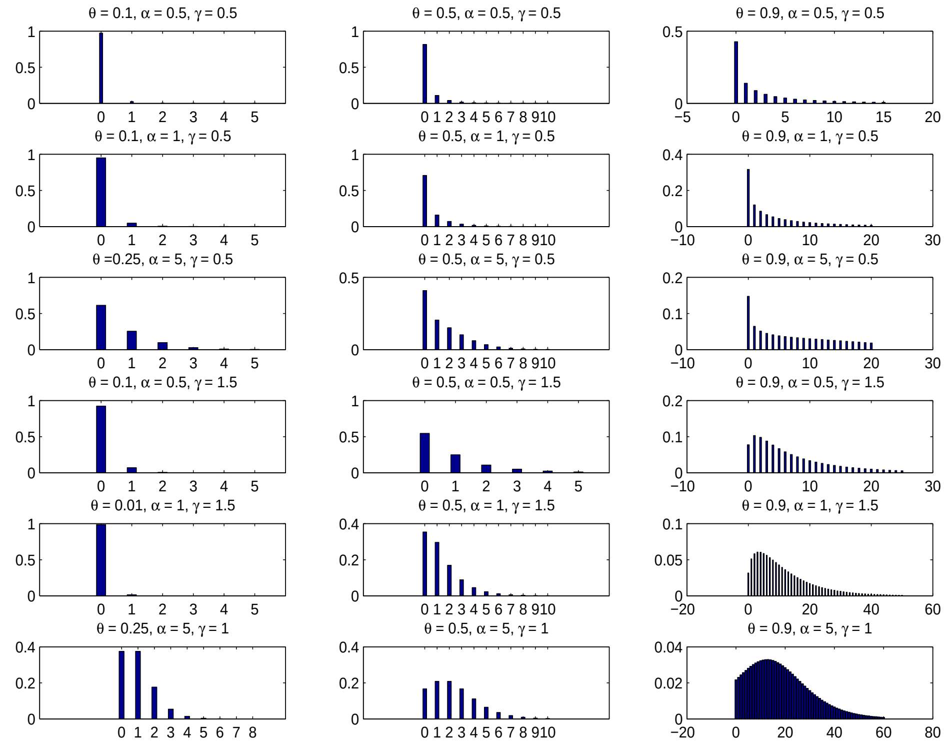

Figure 1. Pmfs of the $EGG(\alpha, \theta, \gamma)$ distribution for some parameter values.

The sum in Eq (2.5) stops at b for integer values of $b > 1$. Using the above series representations, Eq. (2.3), for $0 < \alpha < 1$, can be written as

$$(2.6)\quad f(x; \alpha, \theta, \gamma) = P(X = x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{i,j}(\alpha, \gamma)(1 - \theta^{i+j})\theta^{(i+j)x}, \quad x \in \mathbb{N}_0,$$

where

$$\omega_{i,j}(\alpha, \gamma) = \frac{\Gamma(\gamma + j)\gamma\overline{\alpha}^{j}(-1)^{i+1}}{i!j!\Gamma(\gamma + 1 - i)}.$$

It is clear that the pmf (2.6) is a linear combination of the geometric distributions

$$p_x = (1 - \theta^{i+j})\theta^{(i+j)x}.$$

Hence, several properties of the $EGG(\alpha, \theta, \gamma)$ distribution can be obtained from those of the geometric distribution. For example, the moment and probability generating functions of the proposed distribution are given, respectively, by

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{i,j}(\alpha, \gamma)\frac{1 - \theta^{i+j}}{1 - \theta^{i+j}e^t}$$

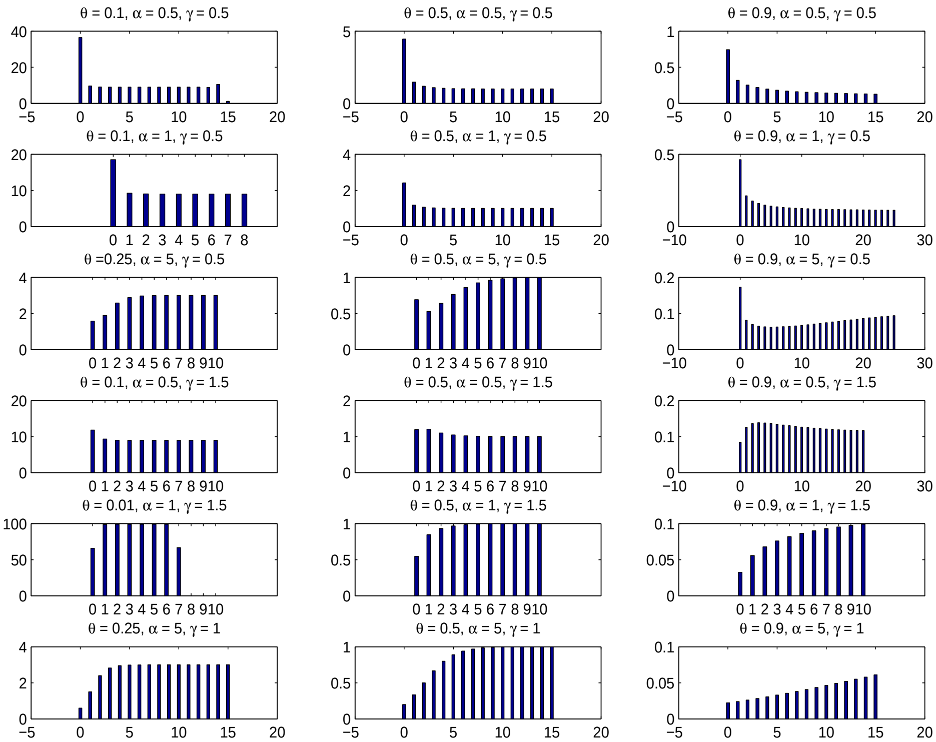

Figure 2. Hazard rate functions of the $EGG(\alpha,\theta,\gamma)$ distribution for some parameter values.

and

$$G_X(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\frac{1-\theta^{i+j}}{1-\theta^{i+j}z}.$$

Moreover, the factorial moments are given by

$$E\left\{X(X-1)...(X-r+1)\right\} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)^{r},$$

for $r = 1, 2, \ldots$. In particular, the mean and variance of the $EGG(\alpha,\theta,\gamma)$ distribution can be obtained by

$$E(X) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right) \tag{2.7}$$

and

$$\begin{aligned} Var(X) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)\left(1+\frac{\theta^{i+j}}{1-\theta^{i+j}}\right) \\ &- \left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)\right\}^{2}. \end{aligned}$$

In addition, the median of the EGG model is given by

$$m = [\frac{1}{\log\theta}\left(\log\frac{1-(1/2)^{1/\gamma}}{1-\overline{\alpha}(1/2)^{1/\gamma}}\right) - 1],$$

where [.] denotes the integer part notation.

The mean and variance of the $EGG(\alpha, \theta, \gamma)$ distribution are calculated in Table 1 for different values of its parameters. It appears that the mean and variance increase, when α, θ, and γ increase. In addition, depending on the values of the parameters, the mean of the distribution can be smaller or greater than its variance.

Table 1 Mean (Variance) of $EGG(\alpha, \theta, \gamma)$ for different values of parameters.

γ	0.5					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0061 (0.0073)	0.0062 (0.0075)	0.0764 (0.1830)	0.0764 (0.1860)	2.4679 (47.6280)	2.4659 (48.6180)
0.5	0.0296 (0.0348)	0.0295 (0.0358)	0.3204 (0.7595)	0.3237 (0.7555)	7.6300 (169.2129)	7.6270 (170.2129)
1.0	0.0570 (0.0606)	0.0569 (0.0670)	0.5546 (1.2778)	0.5555 (1.3000)	11.5117 (268.8771)	11.5116 (269.8771)
2.0	0.1301 (0.1431)	0.1291 (0.1400)	1.0201 (2.2839)	1.0195 (2.29152)	16.0022 (403.8554)	16.6436 (413.8572)
γ	1.0					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0122 (0.0145)	0.0121 (0.0145)	0.1505 (0.3518)	0.1502 (0.3520)	4.5300 (84.5673)	4.5297 (85.5601)
0.5	0.0582 (0.0672)	0.0582 (0.0672)	0.6067 (1.3000)	0.6067 (1.2099)	13.0220 (259.8401)	13.0220 (259.8401)
1.0	0.1111 (0.1235)	0.1111 (0.1235)	1.0000 (2.0000)	1.0000 (2.0000)	19.0000 (380.0000)	19.0000 (380.0000)
2.0	0.2038 (0.1235)	0.2038 (0.1240)	1.5290 (2.8139)	1.5290 (2.8139)	26.5290 (519.9431)	26.5288 (519.9507)
γ	2.0					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0255 (0.0285)	0.0241 (0.0285)	0.2909 (0.6513)	0.2909 (0.6513)	7.8655 (140.2428)	7.8655 (140.2428)
0.5	0.1136 (0.1255)	0.1136 (0.1255)	1.0761 (1.9730)	1.0761 (1.9730)	20.5445 (358.1039)	20.5445 (358.1039)
1.0	0.2121 (0.2163)	0.2121 (0.2163)	1.6667 (2.6667)	1.6667 (2.6667)	28.7436 (475.1874)	28.7436 (475.1874)
2.0	0.3742 (0.3313)	0.3742 (0.3313)	2.3854 (3.2973)	2.3854 (3.2973)	38.4915 (587.3795)	38.4907 (587.4363)

Remark 2.1 Remember that a random variable X with cdf G is stochastically smaller than Y with cdf F, denoted by $X \leq_{st} Y$, if for all x, $G(x) \geq F(x)$. This is the most basic and oldest stochastic order in Probability and Statistics. In this case, if G is simpler than F, $G(x)$ may provide a useful lower bound for $F(x)$ (see, e.g., [16] for more details). Now, let G and F denote the cdfs of the GG and EGG distributions which are defined via Eq.'s (2.1) and (2.2), respectively. It is obvious that for $\gamma > 1$, we have $X \leq_{st} Y$ because $[G(x)]^{\gamma} \leq G(x)$ and if $0 < \gamma < 1$, it follows that $X \geq_{st} Y$. Hence, For $\gamma \geq 1$ it follows that $E(X) \leq E(Y)$ and corresponding result holds if X is stochastically larger than Y. One can consider the results of Table 1 again.

3. Infinite divisibility

The researchers may also here make the following note in regards to the famous structural property of infinite divisibility of the distribution in question. Such a characteristic has a close relation to the Central Limit Theorem and waiting time distributions. Thus, it is a desirable question in modeling to know whether a given distribution is infinitely divisible or not. To settle this question, we recall that according to [18], (pp. 56), if p_x, $x \in \mathbb{N}_0$, is infinitely divisible, then $p_x \leq e^{-1}$ for all $x \in \mathbb{N}$. However, e.g., in an $EGG(0.65, 0.40, 3.80)$ distribution we see that $p_1 = 0.387 > e^{-1} = 0.367$. Therefore, in general, $EGG(\alpha, \theta, \gamma)$ distributions are not infinitely divisible. In addition, since the classes of self-decomposable and stable distributions, in their discrete concepts, are subclasses of infinitely divisible distributions, we conclude that an EDW distribution can be neither self-decomposable nor stable in general.

4. Order statistics

Let $F_i(x;\alpha,\theta,\gamma)$ be the cdf of the i-th order statistic of a random sample $X_1, X_2, ..., X_n$ from $EGG(\alpha,\theta,\gamma)$ distribution. Then, we have

$$F_i(x;\alpha,\theta,\gamma)=\sum_{k=i}^{n}\binom{n}{k}[F(x;\alpha,\theta,\gamma)]^k[1-F(x;\alpha,\theta,\gamma)]^{n-k}.$$

Now, using the binomial expansion for $[1-F(x;\alpha,\theta,\gamma)]^{n-k}$, the expression

$$\begin{aligned}F_i(x;\alpha,\theta,\gamma) &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j[F(x;\alpha,\theta,\gamma)]^{k+j}\\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j[G(x;\alpha,\theta)]^{\gamma(k+j)}\\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j F_{EGG}(x;\alpha,\theta,(k+j)\gamma),\end{aligned}$$

is obtained for the cdf of the i-th order statistic. The corresponding pmf of the i-th order statistic, $f_i(x;\alpha,\theta,\gamma)=F_i(x;\alpha,\theta,\gamma)-F_i(x-1;\alpha,\theta,\gamma)$ for an integer value of x, then is given by

$$f_i(x;\alpha,\theta,\gamma)=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j f_{EGG}(x;\alpha,\theta,(k+j)\gamma),$$

where f_{EGG} denotes the pmf of an EGG distribution.

Remark 2.2 In view of the fact that $f_i(x;\alpha,\theta,\gamma)$ is a linear combination of a finite number of $EGG(\alpha,\theta,\gamma(k+j))$ distributions, we may obtain some properties of order statistics, such as their moments, from the corresponding EGG distribution (see [13]). For example, the mean of the i-th order statistic is given by

$$\mu_{i:n}=\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}(-1)^j\binom{n}{k}\binom{n-k}{j}\omega_{i,l}(\alpha,(k+j)\gamma)\frac{\theta^{i+l}}{1-\theta^{i+l}}.$$

5. Estimation

Let $X_1,\ldots,X_n$ be a random sample of size n from the $EGG(\alpha,\theta,\gamma)$ distribution and $\mathbf{\Theta}=(\alpha,\theta,\gamma)$ be the unknown parameters vector. The log-likelihood function is given by

$$l(\mathbf{\Theta}) = \sum_{i=1}^{n}\log\left[\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\,\theta^{x_i+1}}\right)^{\gamma}-\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\,\theta^{x_i}}\right)^{\gamma}\right].$$

The maximum likelihood estimation (MLE) of $\mathbf{\Theta}$ is obtained by solving the nonlinear equations, $U(\mathbf{\Theta})=(U_\alpha(\mathbf{\Theta}),U_\theta(\mathbf{\Theta}),U_\gamma(\mathbf{\Theta}))^T=\mathbf{0}$, where

$$U_\alpha(\mathbf{\Theta})=\frac{\partial l(\mathbf{\Theta})}{\partial\alpha}=\sum_{i=1}^{n}\frac{-\frac{\left(1-\theta^{x_i+1}\right)^{\gamma}\gamma\,\theta^{x_i+1}}{\left(1-(1-\alpha)\theta^{x_i+1}\right)^{\gamma-1}}+\frac{(1-\theta^{x_i})^{\gamma}\gamma\,\theta^{x_i}}{(1-(1-\alpha)\theta^{x_i})^{\gamma-1}}}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma}-\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma}},$$

$$U_\theta(\mathbf{\Theta})=\frac{\partial l(\mathbf{\Theta})}{\partial\theta}=\sum_{i=1}^{n}\frac{-\alpha\,\gamma\,(x_i+1)\,\theta^{x_i}\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma-1}+\alpha\,x_i\theta^{x_i-1}\gamma\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma-1}}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma}-\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma}},$$

$$U_\gamma(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial \gamma} = \sum_{i=1}^{n} \frac{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^\gamma \ln\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right) - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^\gamma \ln\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^\gamma - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^\gamma}.$$

We need the observed information matrix for interval estimation and hypotheses tests on the model parameters. The 3×3 Fisher information matrix, $J = J_n(\mathbf{\Theta})$, is given by

$$J = -\left[\begin{array}{ccc} J_{\alpha\alpha} & J_{\alpha\theta} & J_{\alpha\gamma} \\ J_{\theta\alpha} & J_{\theta\theta} & J_{\theta\gamma} \\ J_{\gamma\alpha} & J_{\gamma\theta} & J_{\gamma\gamma} \end{array}\right], \tag{5.1}$$

whose elements are given in Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, asymptotically

$$\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \sim N_3(0, I(\mathbf{\Theta})^{-1}),$$

where $I(\mathbf{\Theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\mathbf{\Theta})$ replaced by $J_n(\hat{\mathbf{\Theta}})$, i.e., the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$.

5.1. A simulated example. Let X be a random variable that follows a GEG distribution given by Eq. (2.4). Then, $[X]$ has an $EGG(\alpha, \theta, \gamma)$ distribution. Therefore, we can simulate an $EGG(\alpha, \theta, \gamma)$ random variable from the corresponding continuous GEG distribution. Table 2 below presents the maximum likelihood estimates of $\mathbf{\Theta} = (\alpha, \theta, \gamma)^T$ from an $EGG(\alpha, \theta, \gamma)$ distribution and also contains their standard errors for different values of n as a kind of simulated example. Standard errors are attained by means of the asymptotic covariance matrix of the MLEs of $EGG(\alpha, \theta, \gamma)$ parameters when the Newton-Raphson procedure converges in, e.g., MATLAB software.

Table 2 MLEs of the $EGG(\alpha,\theta,\gamma)$ parameters for different values of n.

		$\alpha=0.50$, $\theta=0.25$, $\gamma=0.75$			$\alpha=0.75$, $\theta=0.75$, $\gamma=0.50$	
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	0.492(0.477)	0.213(0.527)	0.0.984(2.764)	0.812(0.396)	0.753(0.297)	0.442(0.393)
200	0.511(0.323)	0.288(0.410)	0.788(0.1.347)	0.730(0.268)	0.719(0.237)	0.551(0.379)
500	0.501(0.242)	0.217(0.264)	0.792(0.1.099)	0745(0.158)	0.751(0.129)	0.526(0.204)
1000	0.568(0.175)	0.257(0.185)	0.799(0.675)	0.743(0.108)	0.745(0.090)	0.534(0.144)
		$\alpha=2.0$, $\theta=0.5$, $\gamma=3.0$			$\alpha=3.0$, $\theta=0.9$, $\gamma=2.0$	
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	2.077(0.951)	0.564(0.349)	2.656(2.652)	2.912(1.197)	0.897(0.156)	1.872(1.542)
200	1.904(0.663)	0.494(0.289)	2.941(2.352)	2.937(0.818)	0.888(0.113)	2.022(1.163)
500	1.915(0.465)	0.462(0.187)	3.290(1.880)	3.153(0.605)	0.914(0.065)	1.980(0.781)
1000	2.004(0.321)	0.511(0.124)	2.950(1.068)	2.918(0.306)	0.895(0.041)	1.981(0.427)
		$\alpha=1.00$, $\theta=0.50$, $\gamma=1.00$			$\alpha=1.50$, $\theta=0.95$, $\gamma=0.50$	
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	1.278(0.723)	0.601(0.416)	0.864(1.005)	1.257(0.436)	0.867(0.150)	0.808(0.494)
200	0.933(0.363)	0.488(0.293)	0.974(0.850)	1.443(0.393)	0.947(0.060)	0.471(0.198)
500	0.982(0.230)	0.484(0.177)	1.043(0.553)	1.521(0.233)	0.957(0.023)	0.522(0.125)
1000	1.058(0.172)	0.542(0.122)	0.909(0.318)	1.507(0.177)	0.955(0.012)	0.481(0.087)

6. Application

In this section, the EGG model will be examined for a real data set. The data are integer parts of the lifetimes of fifty devices given by [1] and have also been analyzed by [14] and [3]. The data are: 0, 0, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

First, we obtain the MLE of the $EGG(\alpha,\theta,\gamma)$ parameters using the Newton-Raphson procedure. Then, we compare the EGG model with the discrete modified weibull (DMW) distribution of [14] as a rival model. In addition, a four-parameter discrete model, i.e., the discrete additive Weibull (DAddW) distribution of [5], is compared. A summery of computations which consists of the MLEs, Akaike information criterion (AIC) and the values of log-likelihood functions is given in Table 3.

Table 3 MLEs, maximized log-likelihoods and AIC values of the fitted models.

Model	Estimated parameters	$\ell(\hat{\boldsymbol{\theta}})$	AIC
EGG	$\hat{\alpha}$=36011.39, $\hat{\theta}$=0.8845, $\hat{\gamma}=0.1783$	-226.5	459.0
DMW	$\hat{q}$=0.9403, $\hat{c}$=1.0241, $\hat{\theta}$=0.3450	-229.1	464.2
DAddW	$\hat{q}$=0.9216, $\hat{b}$=0.000060091, $\hat{\theta}$=0.4541, $\hat{\gamma}=2.8387$	-228.2	464.4
GG	$\hat{\alpha}$=2.7934, $\hat{\theta}$=0.9674	-239.9	483.7

According to the results of Table 3, it seems that the EGG distribution gives a better fit than the GG (as a sub-model), the DMW (as a three-parameter rival model) and the four-parameter DAddW distributions.

7. Concluding remarks

We proposed the exponentiated generalized geometric (EGG) distribution belonging to the *resilience parameter family*. This new discrete distribution contains the generalized geometric (GG) distribution of [6] as a special case. Moreover, the EGG distribution coincides with the discrete counterpart of the generalized exponential-geometric distribution of [17]. We investigated the basic statistical and mathematical properties of the new model and illustrated that the hazard rate function of the new model can be increasing, decreasing, upside-down bathtub and bathtub-shaped. In addition, fitting the EGG model to a real data set indicated the capacity of the proposed distribution in data modeling.

Appendix

The elements of the 3×3 information matrix in Eq. (5.1) are given by

$$\begin{aligned}
J_{\alpha\alpha} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha^2}\\
&= \sum_{i=1}^{n}\frac{d_i^{\gamma}\left(\theta^{x_i+1}\right)^2\gamma\,(\gamma-1)-\omega_i^{\gamma}\theta^{2x_i}\gamma(\gamma-1)}{d_i^{\gamma}-{\omega_i}^{\gamma}}\\
&- \left[\frac{-d_i^{\gamma}\left(1-(1-\alpha)\theta^{x_i+1}\right)\gamma\,\theta^{x_i+1}+\omega_i^{\gamma}(1-(1-\alpha)\theta^{x_i})\gamma\theta^{x_i}}{d_i^{\gamma}-\omega_i^{\gamma}}\right]^2,
\end{aligned}$$

$$\begin{aligned}
J_{\alpha\theta} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha\partial\theta}\\
&= \sum_{i=1}^{n}\left\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}}\right.\\
&\times \left[d_i^{\gamma-1}\gamma\,\theta^{x}\,(x_i+1)\left(\gamma\,\theta^{x_i+1}-1+\theta^{x_i+1}\right)-d_i^{\gamma}\gamma\,\theta^{2\,x_i+1}\,(\gamma-1)\,(1-\alpha)\,(x_i+1)\right.\\
&- \left.\omega_i^{\gamma-1}\gamma\,\theta^{x_i-1}x_i\,(\gamma\,\theta^{x_i}-1+\theta^{x_i})+\omega_i^{\gamma}\gamma\,\left(\theta\,(\gamma-1)\right)^{2\,x_i-1}(1-\alpha)\,x_i\right]\\
&+ \frac{d_i^{\gamma}(1-(1-\alpha)\theta^{x_i})\gamma\,\theta^{x_i+1}-\omega_i^{\gamma}\gamma\,\theta^{x_i}(1-(1-\alpha)\,\theta^{x_i})}{(d_i^{\gamma}-\omega_i^{\gamma})^2}\\
&\times \left[d_i^{\gamma-1}\gamma(1-(1-\alpha)\theta^{x_i+1})^{-1}\theta^{-1}(x_i+1)\theta^{x_i+1}\,(-1+d_i\,(1-\alpha))\right.\\
&- \left.\left.\omega_i^{\gamma-1}\gamma(1-(1-\alpha)\theta^{x_i})^{-1}\theta^{-1}x_i\theta^{x_i}\,(-1+\omega_i\,(1-\alpha))\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
J_{\alpha\gamma} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha\partial\gamma} \\
&= \sum_{i=1}^{n}\Bigg\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}} \\
&\times \big[d_i^{\gamma}\big(\theta\left(\gamma\,\ln\left(1-(1-\alpha)\,\theta^{x_i+1}\right)-\gamma\,\ln\left(1-\theta^{x_i+1}\right)-1\right)\big)^{x_i+1}\left(1-(1-\alpha)\,\theta^{x_i+1}\right) \\
&+ \omega_i^{\gamma}(1-\theta^{x_i})^{\gamma}\left(\theta\left(\ln\left(1-\theta^{x_i}\right)\gamma+1-\ln\left(1-(1-\alpha)\,\theta^{x_i}\right)\gamma\right)\right)^{x_i}\left(1-(1-\alpha)\,\theta^{x_i}\right)\big] \\
&+ \frac{\gamma\,\theta^{x_i}\left(\theta\,\left(1-\theta^{x_i+1}\right)^{\gamma}-\left(1-\theta^{x_i}\right)^{\gamma}\right)}{\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{\gamma-1}} \\
&\times \big[\frac{d_i^{\gamma}\ln(d_i)-\omega_i^{\gamma}\ln(\omega_i)}{(d_i^{\gamma}-\omega_i^{\gamma})^2}\big]\Bigg\}, \\
J_{\theta\theta} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\theta^2} \\
&= \sum_{i=1}^{n}\Bigg\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}}\Big[d_i^{\gamma-2}\gamma^2 \\
&\times \left(-\frac{\theta^{x_i+1}\,(x_i+1)}{\theta\,\left(1-(1-\alpha)\,\theta^{x_i+1}\right)}+d_i(1-\alpha)\,\theta^{x_i+1}\,(x_i+1)\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{-1}\theta^{-1}\right)^2 \\
&+ {d_i}^{\gamma-1}\gamma\big[\frac{\theta^{x_i+1}\,(x_i+1)\,(-x_i)}{\theta^2\,\left(1-(1-\alpha)\,\theta^{x_i+1}\right)} \\
&+ 2\,d_i\left(1-\theta^{x_i+1}\right)(1-\alpha)^2\left(\theta^{x_i+1}\right)^2(x_i+1)^2\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{-2}\theta^{-2} \\
&+ \frac{(-1+\alpha)\,\theta^{x_i+1}\,(x_i+1)\left(3\,\theta^{x_i+1}x+2\,\theta^{x_i+1}-x\right)}{\theta^2\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^2}\big] \\
&- \alpha\,(x_i+1)^2\left(\theta^{x_i+1}\right)^2\gamma\left(-\frac{-1+\theta^{x_i+1}}{\theta^{x_i+1}\alpha-\theta^{x_i+1}+1}\right)^{\gamma} \\
&\times \left(-1+\theta^{x_i+1}\right)^{-2}\theta^{-2}\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^{-1} \\
&+ \left(-\frac{-1+\theta^{x_i+1}}{\theta^{x_i+1}\alpha-\theta^{x_i+1}+1}\right)^{\gamma}\gamma\left(\theta^{x_i+1}\right)^2(x_i+1)^2 \\
&\times \alpha\,(-1+\alpha)\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^{-2}\theta^{-2} \\
&\times \left(-1+\theta^{x_i+1}\right)^{-1}-\alpha^2{x_i}^2\left(\theta^{x_i}\right)^2\gamma^2\left(-\frac{-1+\theta^{x_i}}{\theta^{x_i}\alpha-\theta^{x_i}+1}\right)^{\gamma} \\
&\times (-1+\theta^{x_i})^{-2}\theta^{-2}\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^{-2}-\omega_i^{\gamma-1}\gamma\big[-\frac{\theta^{x_i}x_i\,(x_i-1)}{\theta^2\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)} \\
&+ 2\,\frac{\left(1-\theta^{x_i}\right)(1-\alpha)^2\left(\theta^{x_i}\right)^2{x_i}^2}{\left(1-(1-\alpha)\,\theta^{x_i}\right)^3\theta^2}+\frac{\theta^{x_i-2}x_i\,(-1+\alpha)\left(3\,\theta^{x_i}x_i-\theta^{x_i}-x_i+1\right)}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\big] \\
&+ \omega_i^{\gamma-1}(1-\theta^{x_i})\big[\frac{\gamma\,\theta^{x_i}{x_i}^2\alpha\,\theta^{x_i-2}}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\big]+\frac{\left(1-\theta^{x_i}\right)^{\gamma-1}\gamma\,\theta^{2\,x_i-2}{x_i}^2\alpha\,(-1+\alpha)}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^{\gamma-2}}\Big] \\
&- \frac{1}{(d_i^{\gamma}-\omega_i^{\gamma})^2} \\
&\times \big[d_i^{\gamma-1}\gamma\big(-\frac{\theta^{x_i+1}\,(x_i+1)\,\alpha}{\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^2\theta}\big)+\frac{\omega_i^{\gamma-1}\gamma\,\theta^{x_i-1}x\alpha}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\big]^2\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\theta\gamma} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\theta\partial\gamma} \\
&= \sum_{i=1}^{n}\left\{\frac{1}{d_i^\gamma-\omega_i^\gamma}\left[d_i^{\gamma-1}\ln(d_i)\gamma(-\frac{\theta^{x_i}\,(x_i+1)\,\alpha}{(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1)^2})\right.\right. \\
&- d_i^{\gamma-1}\left(\frac{\theta^{x_i}\,(x_i+1)\,\alpha}{(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1)^2}\right) \\
&- \left.\omega_i^{\gamma-1}\ln(\omega_i)\gamma\left(-\frac{\theta^{x_i-1}x_i\alpha}{(\theta^{x_i}\alpha-\theta^{x_i}+1)^2}\right)+\omega_i^{\gamma-1}\left(\frac{\theta^{x_i}x_i\alpha}{(\theta^{x_i}\alpha-\theta^{x_i}+1)^2\,\theta}\right)\right] \\
&- \frac{1}{(d_i^\gamma-\omega_i^\gamma)^2}\{[d_i^{\gamma-1}\left(-\frac{\gamma\,\theta^{x_i}\,(x+1)\,\alpha}{(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1)^2}\right)+\omega_i^{\gamma-1}\left(\frac{\gamma\,\theta^{x_i-1}x_i\alpha}{(\theta^{x_i}\alpha-\theta^{x_i}+1)^2}\right)] \\
&\times \left.[d_i^\gamma\ln(d_i)-\omega_i^\gamma\ln(\omega_i)]\}\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\gamma\gamma} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\gamma^2} \\
&= \sum_{i=1}^{n}\{\frac{d_i^\gamma\,(\ln(d_i))^2-\omega_i^\gamma\,(\ln(\omega_i))^2}{d_i^\gamma-\omega_i^\gamma}-\frac{(d_i^\gamma\ln(d_i)-\omega_i^\gamma\ln(\omega_i))^2}{(d_i^\gamma-\omega_i^\gamma)^2}\},
\end{aligned}
$$

where $d_i=\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}$ and $\omega_i=\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}$.

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