Hacettepe Journal of Mathematics and Statistics
Volume 44 (4) (2015), 887 – 892

Skew-commuting mappings on semiprime and prime rings

Abbas Najati * and M. Mohammadi Saem †

Abstract

In this paper we study some maps which are skew-commuting on rings. Also we present some results concerning derivations in generalized case.

2000 AMS Classification: 16N60, 16W25.

Keywords: skew-commuting map, additive map, generalized derivation, semiprime ring, prime ring.

Received 09/02/2014 : Accepted 30/04/2014 Doi : 10.15672/HJMS.2015449652

1. Introductions and Preliminaries

Throughout this paper, R be a ring with center $Z(R)$. For an integer $n > 1$, a ring R is called *n-torsion free* if $nx = 0$, $(x \in R)$ implies $x = 0$. As usual we write $[x, y]$ for $xy - yx$ and use the identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for $x, y, z \in R$. Recall that a ring R is *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$ and is *semiprime* if $xRx = \{0\}$ implies $x = 0$. An additive mapping d from R into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f : R \to R$ is said to be a *generalized derivation* if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. A mapping $f : R \to R$ is called *skew-commuting* on R if $f(x)x + xf(x) = 0$ and is called *commuting* on R if $[f(x), x] = 0$ for all $x \in R$.

*Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran.
Email: a.najati@uma.ac.ir, a.nejati@yahoo.com
†Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran.
Email:m.mohammadisaem@yahoo.com

2. Main Results

We shall make use of the following results.

2.1. Theorem. [2] *Let R be a 2-torsion free semiprime ring. If an additive mapping $f : R \to R$ is skew-commuting on R, then $f = 0$.*

2.2. Theorem. [3] *Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation D. If $F(xy) = F(x)F(y)$ for all $x, y \in R$, then $D(I) = 0$.*

2.3. Lemma. [5] *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case, $(a + c)xb = 0$ is satisfied for all $x \in R$.*

2.4. Lemma. [6] *Let R be a semiprime ring and D be a derivation on R. If $D(x)D(y) = 0$ for all $x, y \in R$, then $D = 0$.*

2.5. Theorem. *Let R be a 2-torsion free semiprime ring, D be a derivation and α be a homomorphism on R. Suppose that the mapping $x \mapsto (D(x) + (\alpha(x) - x))$ is skew-commuting on R. In this case $\alpha = I$ and $D = 0$.*

Proof. Put $G(x) = \alpha(x) - x$. By Theorem 2.1, we have $D = -G$. Therefore $D(x)D(y) = 0$ for all $x, y \in R$. Hence $D = 0$ by Lemma 2.4. So we get $\alpha = I$. □

2.6. Theorem. *Let R be a nonzero 2-torsion free semiprime ring, D be a derivation and α be a homomorphism on R. Suppose that the mapping $x \mapsto D(x) + \alpha(x)$ is skew-commuting on R. In this case $\alpha = D = 0$.*

Proof. The result follows by Theorems 2.1 and 2.2. □

In [4] Vukman proved that on a 2-torsion free semiprime ring R, if the mapping $x \mapsto D(x)x + x\alpha(x)$ is commuting on R, then D and $\alpha - I$ map R into $Z(R)$. The next theorem is a version of this result in case of skew-commuting map. We will use the following lemmas in the proofs of next theorems.

2.7. Lemma. [4] *Let R be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.*

2.8. Lemma. [4] *Let R be a 2-torsion free semiprime ring and $\alpha : R \to R$ be an automorphism such that $x[\alpha(x), x] = 0$ or $[\alpha(x), x]x = 0$ for all $x \in R$. Then $\alpha - I$ maps R into $Z(R)$.*

2.9. Lemma. [1] *Let R be a prime ring and let $F : R \to R$ be an additive map. If there exists a positive integer n such that $F(x)x^n = 0$ for all $x \in R$, then $F = 0$.*

2.10. Theorem. *Let R ba a 2 and 3-torsion free semiprime ring. Suppose that D is a derivation and $\alpha : R \to R$ is an onto homomorphism such that the mapping $x \mapsto D(x)x + x\alpha(x)$ is skew-commuting on R. In this case $\alpha - I$ maps R into $Z(R)$.*

Proof. The assumption of the theorem can be written in the form

$$(D(x)x + x\alpha(x))x + x(D(x)x + x\alpha(x)) = 0, \quad x \in R. \tag{2.1}$$

Using the linearization of (2.1), a routine calculation gives

$$A(x)y + D(x)yx + D(y)x^2 + x\alpha(y)x + y\alpha(x)x + x\alpha(x)y + xD(y)x$$
$$+ x^2\alpha(y) + xy\alpha(x) + yD(x)x + yx\alpha(x) = 0, \quad x, y \in R,$$

where $A(x) = D(x)x + xD(x)$. Replacing y by yx in the above relation, we get

$$A(x)yx + D(x)yx^2 + D(y)x^3 + yD(x)x^2 + x\alpha(y)\alpha(x)x + yx\alpha(x)x + x\alpha(x)yx + xD(y)x^2 + xyD(x)x + x^2\alpha(y)\alpha(x) + xyx\alpha(x) + yxD(x)x + yx^2\alpha(x) = 0.$$

It follows from the above relations that

$$(xy + yx)D(x)x + x^2\alpha(y)G(x) + x\alpha(y)G(x)x + xy[x, \alpha(x)] + yx[x, \alpha(x)] + y[x, \alpha(x)]x = 0, \quad x, y \in R, \tag{2.2}$$

where $G(x) = \alpha(x) - x$. Replacing y by xy in (2.2), we get

$$x(xy + yx)D(x)x + x^2\alpha(x)\alpha(y)G(x) + x\alpha(x)\alpha(y)G(x)x + x^2y[x, \alpha(x)] + xyx[x, \alpha(x)] + xy[x, \alpha(x)]x = 0. \tag{2.3}$$

Multiplying the left side of (2.2) by x and then subtracting the obtained relation from (2.3), we obtain

$$x^2G(x)\alpha(y)G(x) + xG(x)\alpha(y)G(x)x = 0, \quad x, y \in R.$$

Since α is onto, therefore

$$x^2G(x)yG(x) + xG(x)yG(x)x = 0, \quad x, y \in R.$$

Hence

$$x^2G(x)yxG(x) + xG(x)yxG(x)x = 0, \quad x, y \in R.$$

By Lemma 2.3, we have

$$(x^2G(x) + xG(x)x)yxG(x) = 0, \quad x, y \in R. \tag{2.4}$$

Replacing y by yx in (2.4), we get

$$(x^2G(x) + xG(x)x)yx^2G(x) = 0, \quad x, y \in R. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$(x^2G(x) + xG(x)x)y(x^2G(x) + xG(x)x) = 0, \quad x, y \in R.$$

Since R is semiprime, this implies

$$x(xG(x) + G(x)x) = 0, \quad x \in R. \tag{2.6}$$

Using the linearization of (2.6), a routine calculation gives

$$x(xG(y) + yG(x) + G(x)y + G(y)x) + y(xG(x) + G(x)x) = 0. \tag{2.7}$$

Replacing y by xy in (2.7), we obtain

$$x(xG(xy) + xyG(x) + G(x)xy + G(xy)x) + xy(xG(x) + G(x)x) = 0, \quad x, y \in R. \tag{2.8}$$

Multiplying the left side of (2.7) by x and then subtracting the obtained relation from (2.8), we get

$$x[\alpha(x), x]y + x^2G(x)\alpha(y) + xG(x)\alpha(y)x = 0, \quad x, y \in R.$$

Using (2.6), we get

$$x[\alpha(x), x]y + xG(x)[\alpha(y), x] = 0, \quad x, y \in R. \tag{2.9}$$

Replacing y by yz in (2.9), we obtain

$$\begin{aligned} 0 &= x[\alpha(x), x]yz + xG(x)\alpha(y)[\alpha(z), x] + xG(x)[\alpha(y), x]\alpha(z) \\ &= -xG(x)[\alpha(y), x]z + xG(x)\alpha(y)[\alpha(z), x] + xG(x)[\alpha(y), x]\alpha(z) \\ &= xG(x)[\alpha(y), x]G(z) + xG(x)\alpha(y)[\alpha(z), x], \quad x, y, z \in R. \end{aligned}$$

Since α is onto, we have

$$xG(x)[y,x]G(z) + xG(x)y[\alpha(z),x] = 0, \quad x,y,z \in R.$$

Putting $y = x$ in the above relation, we infer $xG(x)x[z,x] = 0$ for all $x,z \in R$. If we replace z by zy, then $xG(x)xz[y,x] = 0$ for all $x,y,z \in R$. Putting $y = G(x)$, we have

$$xG(x)xz[G(x),x] = 0, \quad x,z \in R. \tag{2.10}$$

Replacing z by xz in (2.10), we get $xG(x)x^2z[G(x),x] = 0$ for all $x,z \in R$. On the other hand (2.10) gives $x^2G(x)xz[G(x),x] = 0$. Subtracting these two recent relations, we obtain

$$x[G(x),x]xz[G(x),x] = 0, \quad x,z \in R.$$

Hence $x[G(x),x]x = 0$ by semiprimeness of R. According to (2.6), we get $x^2G(x)x = 0$ for all $x \in R$. Therefore $x^2[G(x),x] = 0$ for all $x \in R$. The linearization with a simple calculation leads to

$$x^2[G(y),y] + (xy+yx)([G(x),y] + [G(y),x]) + y^2[G(x),x] = 0, \quad x,y \in R.$$

Replacing y by $x+y$ in the above relation, we get

$$x^2([G(x),y] + [G(y),x]) + (xy+yx)[G(x),x] = 0, \quad x,y \in R. \tag{2.11}$$

Left multiplication (2.11) by $x[G(x),x]$ and using $x[G(x),x]x = 0$, we get

$$x[G(x),x]yx[G(x),x] = 0, \quad x,y \in R.$$

Since R is semiprime, $x[G(x),x] = 0$ for all $x \in R$. So $x[\alpha(x),x] = 0$ for all $x \in R$. Therefore $\alpha - I$ maps R into $Z(R)$ by Lemma 2.8. □

2.11. Theorem. *Let R ba a 2 and 3-torsion free prime ring. Suppose that D is a derivation and $\alpha : R \to R$ is an onto homomorphism such that the mapping $x \mapsto D(x)x + x\alpha(x)$ is skew-commuting on R. The only case for R is $R = \{0\}$.*

Proof. By Theorem 2.10 we obtain that $\alpha - I$ maps R into $Z(R)$. So relation (2.6) gives us $G(x)x^2 = 0$. Hence $\alpha = I$ by Lemma 2.9. Therefore (2.1) and (2.2) give

$$D(x)x^2 + xD(x)x = -2x^3, \quad x^2D(x)x = 0, \quad x \in R. \tag{2.12}$$

Hence we have

$$D(x^3) - x^2D(x) = -2x^3, \quad x \in R. \tag{2.13}$$

It follows from (2.12) that

$$xD(x)x^2 = -2x^4, \quad x \in R. \tag{2.14}$$

Right multiplication of (2.12) by x and then using (2.14), we get $D(x)x^3 = 0$. Hence $D = 0$ by Lemma 2.9. So (2.14) implies $x^4 = 0$ for all $x \in R$. So we get $R = \{0\}$ by Lemma 2.9. □

Vukman [4] proved the result below.

2.12. Theorem. [4] *Let R be a 2-torsion free semiprime ring and $D : R \to R$ be a derivation such that $x[D(x),x] = 0$ or $[D(x),x]x = 0$ for all $x \in R$. Then D maps R into $Z(R)$.*

In the following theorem we generalize this result.

2.13. Theorem. *Let R be a 2-torsion free semiprime ring and F be a generalized derivation associated with a derivation D on R. Also let $[F(x),x]x = 0$ for all $x \in R$. In this case D maps R into $Z(R)$.*

Proof. The linearization of $[F(x), x]x = 0$ gives

$$(2.15) \quad [F(x), y]x + [F(y), x]x + [F(x), x]y = 0, \quad x, y \in R.$$

Replacing y by yx in (2.15), we get

$$[F(x), y]x^2 + [F(y), x]x^2 + y[D(x), x]x + [y, x]D(x)x + [F(x), x]yx = 0.$$

Right multiplication of (2.15) by x and subtracting the obtained relation from the above relation, gives

$$(2.16) \quad y[D(x), x]x + [y, x]D(x)x = 0, \quad x, y \in R.$$

Replacing y by $D(x)y$ in (2.16), we have

$$D(x)y[D(x), x]x + D(x)[y, x]D(x)x + [D(x), x]yD(x)x = 0, \quad x, y \in R.$$

Using (2.16), we infer $[D(x), x]yD(x)x = 0$. Hence (2.16) implies that $[D(x), x]xy[D(x), x]x = 0$ for all $x, y \in R$. Since R is semiprime, $[D(x), x]x = 0$ for all $x \in R$. Therefore D maps R into $Z(R)$ by Theorem 2.12. □

2.14. Theorem. *Let R be a 2-torsion free semiprime ring and let D and G be two derivations on R. Suppose that $(D(x)x + xG(x))x = 0$ for all $x \in R$. In this case D and G map R into $Z(R)$.*

Proof. A routine calculation shows that

$$(2.17) \quad D(x)yx + D(y)x^2 + xG(y)x + yG(x)x + D(x)xy + xG(x)y = 0.$$

Let y be yx in (2.17). Then

$$\begin{aligned} &D(x)yx^2 + D(y)x^3 + yD(x)x^2 + xG(y)x^2 \\ &+ xyG(x)x + yxG(x)x + D(x)xyx + xG(x)yx = 0, \quad x, y \in R. \end{aligned}$$

Multiplying (2.17) from the right by x and then subtracting the obtained relation from the above relation, we get

$$y(D(x)x^2 + xG(x)x) + xyG(x)x - yG(x)x^2 = 0, \quad x, y \in R.$$

Hence by the assumption, we get

$$xyG(x)x - yG(x)x^2 = 0, \quad x, y \in R.$$

Replacing y by $G(x)xy$, we get

$$xG(x)xyG(x)x + G(x)xy(-G(x)x^2) = 0, \quad x, y \in R.$$

By Lemma 2.3 we get

$$(2.18) \quad [G(x), x]xyG(x)x = 0, \quad x, y \in R.$$

If we replace y by yx in (2.18), then

$$[G(x), x]xyxG(x)x = 0, \quad x, y \in R.$$

Multiplying (2.18) from the right by x and subtracting the obtained relation from the above relation, we obtain

$$[G(x), x]xy[G(x), x]x = 0, \quad x, y \in R.$$

Since R is semiprime, $[G(x), x]x = 0$ for all $x \in R$. Hence G maps R into $Z(R)$ by Theorem 2.12. Also using same argument shows that D maps R into $Z(R)$. □

2.15. Theorem. *Let R be a 2-torsion free prime ring and let D and G be two derivations on R. Suppose that $(D(x)x + xG(x))x = 0$ for all $x \in R$. In this case $D = -G$ and R is commutative, unless $D = G = 0$.*

Proof. By Theorem 2.14 we get that G maps R into $Z(R)$. So by the assumption, we obtain $(D+G)(x)x^2 = 0$ for all $x \in R$. Therefore $D+G = 0$ by Lemma 2.9, and we conclude that D maps R into $Z(R)$. □

Our last result generalizes a result of [4].

2.16. Theorem. *Let R be a semiprime ring, F be a generalized derivation associated with a derivation D on R and $\alpha : R \to R$ be an onto homomorphism. If $F(x)x+x(\alpha(x)-x) = 0$ holds for all $x \in R$, then $\alpha = I$ and $F = D = 0$.*

Proof. The linearization of $F(x)x + x(\alpha(x) - x) = 0$ gives

(2.19) $\quad F(x)y + F(y)x + xG(y) + yG(x) = 0,$

where $G(x) = \alpha(x) - x$. Substituting yx for y in (2.19), we have

$$\begin{aligned} 0 &= F(x)yx + F(y)x^2 + yD(x)x + x\alpha(y)\alpha(x) - xyx + yxG(x) \\ &= (F(x)y + F(y)x - xy)x + yD(x)x + x\alpha(y)\alpha(x) + yxG(x) \\ &= -x\alpha(y)x - yG(x)x + yD(x)x + x\alpha(y)\alpha(x) + yxG(x). \end{aligned}$$

Therefore

(2.20) $\quad x\alpha(y)G(x) + y[x, G(x)] + yD(x)x = 0, \quad x, y \in R.$

Replacing y by xy in (2.20), we get

$$x\alpha(x)\alpha(y)G(x) + xy[x, G(x)] + xyD(x)x = 0, \quad x, y \in R.$$

Multiplying (2.20) from the left by x and subtracting the obtained relation from the above relation, we get

$$xG(x)\alpha(y)G(x) = 0, \quad x, y \in R.$$

Since α is onto, $xG(x)yG(x) = 0$ for all $x, y \in R$. Hence $xG(x)yxG(x) = 0$ for all $x, y \in R$. Since R is semiprime, $xG(x) = 0$ for all $x \in R$. By Lemma 2.7 we get $G = 0$, which implies that $\alpha = I$. Now by (2.19) we have $F(x)x = 0$ for all $x \in R$. Hence $F = 0$ by Lemma 2.7. On the other hand, we have $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$. So $xD(y) = 0$ for all $x, y \in R$. Since R is semiprime, we infer $D = 0$. □

Acknowledgments

The authors would like to thank the referee for his (her) useful comments.

References

[1] D. Benkovič and D. Eremita, *Characterizing left centralizers by their action on a polynomial,* Publ. Math. Debercen **64** (2004), 343–351.

[2] M. Brešar, *On Skew-Commuting mappings of rings,* Bull. Austral. Math. Soc. **47** (1993), 291–296.

[3] B. Dhara, *Generalized derivations acting as a homomorphism or anti-homomorphism in semiprime ring,* Beitr. Algebra Geom. **53** (2012), 203–209.

[4] J. Vukman, *Identities with derivations and automorphisms on semiprime rings,* Int. J. Math. Math. Sci. **7** (2005), 1031–1038.

[5] J. Vukman, *Centralizers on semiprime rings,* Comment. Math. Univ. Carolin. **42** (2001), 237–245.

[6] B. Zalar, *On centralizers of semiprime rings,* Comment. Math. Univ. Carolin. **32** (1991), 609–614.