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A note on the embedding properties of p-subgroups in finite groups

Boru Zhang, Xiuyun Guo*

Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China

Abstract

In this note, we prove that a finite group G is p-supersolvable if and only if there exists a power d of p with $p^2 \leq d < |P|$ such that $H \cap O^p(G_p^*)$ is normal in $O^p(G)$ for all non-cyclic normal subgroups H of P with $|H| = d$, where P is a Sylow p-subgroup of G. Moreover, we also prove that either $l_p(G) \leq 1$ and $r_p(G) \leq 2$ or else $|P \cap O^p(G)| > d$ if there exists a power d of p with $1 \leq d < |P|$ such that $H \cap O^p(G_{p^2}^*)$ is normal in $O^p(G)$ for all non-meta-cyclic normal subgroups H of P with $|H| = d$.

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1. Introduction

All groups considered in this note are finite. We use conventional notions and notation, as in [9].

It is quite interesting to investigate the structure of a group by using some kind of the embedding properties of subgroups and many interesting results have been given (for example, see [1, 6, 8, 13]). Recently, Guo and Isaacs [6] proved the following theorem by using the embedding condition "$H \cap O^p(G) \trianglelefteq O^p(G)$".

Theorem 1.1. ([6, Theorem B]). *Let $P \in Syl_p(G)$, and let d be a power of p such that $1 \leq d < |P|$. Assume that $H \cap O^p(G) \trianglelefteq O^p(G)$ for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p-supersolvable or else $|P \cap O^p(G)| > d$.*

An interesting idea of [6] is that in the hypothesis of the theorem only the normal subgroups of order d are considered, not necessarily the family of all subgroups of order d. Recall that a subgroup H of a group G is said to be S-semipermutable in G (see [12]) if H permutes with all Sylow q-subgroups of G for the primes q not dividing $|H|$. Ballester-Bolinches etc in their paper [1] proved an analogous result, but they only assume that $H \cap O^p(G)$ are S-semipermutable in $O^p(G)$ instead of assuming that these subgroups are normal in $O^p(G)$.

Theorem 1.2. ([1, Theorem 2]). *Let $P \in Syl_p(G)$, and let d be a power of p such that $1 \leq d < |P|$. Assume that $H \cap O_p(G)$ is S-semipermutable in G for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p-supersolvable or else $|P \cap O_p(G)| > d$.*

*Corresponding Author.
Email addresses: brzhang@live.com (B. Zhang), xyguo@staff.shu.edu.cn (X. Guo)
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More recently, Yu in his paper [13] use the subgroup G_p^* of a group G, and consider the embedding condition $O^p(G_p^*) \cap H \trianglelefteq O^p(G)$ to prove the following result, where G_p^* is the unique smallest normal subgroup of a group G for which the corresponding factor group is abelian of exponent dividing $p-1$.

Theorem 1.3. ([13, Theorem 2]). *Let $P \in Syl_p(G)$, and let d be a power of p such that $1 \le d < |P|$. Then G is p-supersolvable if and only if $|P \cap O^p(G_p^*)| \le d$ and $H \cap O^p(G_p^*) \trianglelefteq O^p(G)$ for all subgroups $H \trianglelefteq P$ with $|H| = d$.*

Remark 2.3 and Example 2.4 in [13] show that it is better to use the embedding condition $O^p(G_p^*) \cap H \trianglelefteq O^p(G)$ to investigate the p-supersolvability of groups. On the other hand, in all of the above results all normal subgroups of order d in P are considered. So we wonder whether we may reduce the number of normal subgroups of order d in P?

In fact, we have the following results.

Theorem 1.4. *Let $P \in Syl_p(G)$, and let d be a power of p such that $p^2 \le d < |P|$. Then G is p-supersolvable if and only if $|P \cap O^p(G_p^*)| \le d$ and $H \cap O^p(G_p^*) \trianglelefteq O^p(G)$ for all non-cyclic subgroups $H \trianglelefteq P$ with $|H| = d$.*

Theorem 1.5. *Let p be a prime dividing the order of a group G of odd order, let d be a power of p such that $1 \le d < |P|$ and $P \in Syl_p(G)$ with $|P| > d$. And suppose that $H \cap O^p(G_{p^2}^*) \trianglelefteq O^p(G)$ for all non-meta-cyclic normal subgroups H in P with $\mid H \mid= d$. Then either p-length $l_p(G) \le 1$ and p-rank $r_p(G) \le 2$ or else $|P \cap O^p(G_{p^2}^*)| > d$, where $G_{p^2}^*$ is the unique smallest normal subgroup of the group G for which the corresponding factor group is abelian of exponent diving $p^2 - 1$.*

We should notice that we assume $d \ge p^2$ in Theorem 1.4. In fact, if $p = 2$ and $d = 2$, then the result is still true. Since $|P \cap O^p(G_p^*)| \le 2$, it follows from Burnside Theorem[9, IV, 2.8] that $O^p(G_p^*)$ is 2-nilpotent, and thus G_p^* is 2-nilpotent. Hence G is 2-supersolvable. However, the result is not true if p is odd prime and $d = p$ in Theorem 1.4. In fact, let D be a non-abelian simple group such that Sylow p-subgroups of D are cyclic of order p, and let $G = D \times C$ with a cyclic group C of order p. Clearly, $G_p^* = G$ and $H \cap O^p(G_p^*)$ is normal in $O^p(G)$ for every non-cyclic normal subgroup H of P of order d, where P is a Sylow p-subgroup. But $|P \cap O^p(G_p^*)| = p$ and G is not p-supersolvable.

We also notice that the hypothesis "G is an odd order group" in Theorem 1.5 can not be removed. In fact, let D be a non-abelian simple group such that Sylow p-subgroups of D are cyclic of order $p^m (d \ge p^m \ge 1)$, and let $G = D \times C$ with a cyclic group C of order $p^n (n \ge 1)$. Clearly, $H \cap O^p(G_{p^2}^*)$ is normal in $O^p(G)$ for every non-meta-cyclic normal subgroup H of P of order d, where P is a Sylow p-subgroup of G. But $|P \cap O^p(G_{p^2}^*)| = p^m \le d$ and G is not p-solvable.

Furthermore, the following example tells us that G may not be p-supersolvable if G satisfies the hypotheses in Theorem 1.5. Let p be an odd prime with $p \ne 2^k - 1$ for all positive integer k. Write $P = \langle a \rangle \times \langle b \rangle \simeq C_{p^n}^2$. So $Aut(\Omega_1(P)) \simeq GL(2,p)$ and there exists a cyclic subgroup T of order p^2-1 in $Aut(\Omega_1(P))$. Note that $p+1$ is not a power of 2, then we can choose an automorphism $\overline{\alpha} \in T$ with order q such that $q|p+1$ and $q \ne 2$. Now, considering the surjective homomorphism $\phi : Aut(P) \to Aut(\Omega_1(P))$; we can choose an automorphism α of P such that $\phi(\alpha) = \overline{\alpha}$. Write the semidirect product $G = P \rtimes \langle \alpha \rangle$, it is clear that $H \cap O^p(G_{p^2}^*)$ is normal in $O^p(G)$ for every non-meta-cyclic normal subgroup H of P of order $d = p^m$ $(m < 2n)$. Now we prove that G is not p-supersolvable. If the action of $\overline{\alpha}$ on $\Omega_1(P) = \langle a_1 \rangle \times \langle b_1 \rangle \simeq C_p^2$ is reducible, then it follows from $p \ne q$ and Maschke's Theorem that $\langle a_1 \rangle, \langle b_1 \rangle$ are $\overline{\alpha}$-invariant. It follows from $g.c.d.(p+1, p-1) = 2$ that $q \nmid p-1$, and therefore $\overline{\alpha}$ acts trivially on both $\langle a \rangle$ and $\langle b \rangle$, that is, $\overline{\alpha}$ acts trivially on $\Omega_1(P)$, a contradiction. Hence $\overline{\alpha}$ acts irreducibly on $\Omega_1(P)$, implying that α acts irreducibly on

$\Omega_1(P)$. Then we have $\Omega_1(P) \simeq C_p^2$ is a minimal normal subgroup of G and so $r_p(G) = 2$. It follows that G is not p-supersolvable.

2. Preliminary results

In this section we list some basic and known results which will be used in our proofs.

Definition 2.1. ([7, Definition 1.9]). Let p be prime. A group G is said to be a $CS(p,n)$-group if G is a p-group with a characteristic series

$$1 = G_0 < G_1 < \cdots < G_m = G$$

such that $|G_i/G_{i-1}| \leq p^n$ for all $i \geq 1$.

It is clear that meta-cyclic p-groups and p-groups of maximal class are both $CS(p,2)$-groups.

Lemma 2.2. ([3, Lemma 1.4]). *Let p be a prime, let P be a p-group and let d be a power of p such that $p^2 \leq d < |P|$. Let $N \trianglelefteq P$, where $|N| \geq d$, and suppose that every normal subgroup of P that has order d and is contained in N is cyclic. Then N is cyclic, dihedral, semidihedral or generalized quaternion.*

Lemma 2.3. ([2, Lemma 2.4]). *Let $P \trianglelefteq G$, where P is a p-group. Also, let $N \leq G$ be a p-subgroup with $|N| \leq |P|$ and $N \nleq P$. Then $N < PN$, and every subgroup H with $N < H \leq NP$ is non-cyclic.*

Lemma 2.4. ([8, Lemma 2.5]). *If a group P of order $2^n > 2^3$ has a subgroup M of order 2^{n-1} of maximal class, then either P is of maximal class or $P/P' \simeq C_2^3$, and P' is cyclic.*

Lemma 2.5. ([3, Exercise 1(b), p.114]). *Let P be dihedral, semidihedral or generalized quaternion, then P has the only one normal subgroup N of order 2^a for every $1 < 2^a < |P|/2$ and N is cyclic.*

Lemma 2.6. ([4, Corollary 69.5]). *Let p be an odd prime and d be a power of p such that $d \geq p^3$, and let N be a normal subgroup of a p-group P with $|N| \geq d$. If every normal subgroup of P that has order d and is contained in N is meta-cyclic, then N is a meta-cyclic group or a 3-group of maximal group.*

Lemma 2.7. *Let p be a odd prime, and let P be a meta-cyclic p-group or a 3-group of maximal class. If N is normal in P, then $\Omega_1(N) \lesssim C_p \times C_p$ or N is a 3-group of maximal class.*

Proof. If P is meta-cyclic, then $\Omega_1(N) \lesssim C_p \times C_p$. Now assume that P is a 3-group of maximal class. It follows from [3, Exercise 9.1] that N is a 3-group of maximal class or absolutely regular, where a p-group N is absolutely regular if $|G/\mho_1(G)| < p^p$ (see [3, List of definitions and notations]). If N is absolutely regular, then $|\Omega_1(N)| = |N/\mho_1(N)| \leq 3^2$, and thus $\Omega_1(N) \lesssim C_p \times C_p$ by [3, Lemma 1.4]. □

Lemma 2.8. *Let p be a prime and d be a power of p such that $p^3 \leq d$, and let P be a p-group. Also, let N and P_1 be normal subgroups of P with $N \lesssim C_p \times C_p$ and $N < P_1$. If P_1 contains a non-meta-cyclic normal subgroup of order d of P , then there exists a non-meta-cyclic normal subgroup H of order d of P such that $N < H \leq P_1$.*

Proof. Let H_1 be a non-meta-cyclic normal subgroup of order d of P with $H_1 \leq P_1$. If $N \nleq H_1$, then $|N \cap H_1| = 1$ or p by $N \lesssim C_p \times C_p$, that is, $|N : N \cap H_1| = p^2$ or p. First, we assume that $|N : N \cap H_1| = p$. Since $N \cap H_1$ is normal in P, there exists a maximal subgroup M of H_1 such that $M \trianglelefteq P$ and $N \cap H_1 \leq M$, and so $H = NM$ is normal in P and $|H| = d$. Noting that H_1 is non-meta-cyclic, we have that M is non-cyclic. It follows from $N \nleq M$ and $N \lesssim C_p^2$ that $\Omega_1(H) > \Omega_1(M) \geq p^2$. Thus H is non-meta-cyclic by Lemma 2.7 and $H \leq P_1$. Now assume that $|N : N \cap H_1| = p^2$ and take a subgroup M_1 of H_1

with $|M_1| = d/p^2$ and $M_1 \trianglelefteq P$. Then $H = NM_1$ is a normal subgroup of P with $|H| = d$. Noticing that $N \simeq C_p \times C_p$ and $N \cap M = 1$, we see $|\Omega_1(H)| \geq |\Omega_1(N)||\Omega_1(M)| \geq p^3$. Hence H is non-meta-cyclic by Lemma 2.7, as we wanted. □

Lemma 2.9. ([7, Lemma 2.2]). *Let P be a p-group. If P has a meta-cyclic maximal subgroup and P is not isomorphic to C_p^3 , then P is a $CS(p, 2)$-group.*

Lemma 2.10. ([7, Lemma 3.2]). *Let an odd order group A act on a $CS(p, 2)$-group P. Then P is centralized by $O^p(A^*_{p^2})$.*

Lemma 2.11. *Let G be a group and let p be a prime of $|G|$. If $G^*_{p^2}$ is p-nilpotent, then G is p-solvable with $l_p(G) \leq 1$ and $r_p(G) \leq 2$.*

Proof. Since $G^*_{p^2}$ is p-nilpotent, G is p-solvable of $l_p(G) \leq 1$. We see G has a chief series

$$1 = K_0 < \cdots < H_0 = O^p(G^*_{p^2}) < H_1 < \cdots < H_n = G^*_{p^2} < \cdots < G$$

Noticing that $O^p(G^*_{p^2}) \leq C_G(H_{i+1}/H_i)$ $(0 \leq i \leq n-1)$, we have $A_G(H_{i+1}/H_i) \simeq G/C_G(H_{i+1}/H_i) \in \mathfrak{D}_p\mathfrak{U}_{p^2-1}$, where $\mathfrak{D}_p$ is the formation consisting of all p-groups and $\mathfrak{U}_{\mathfrak{p}^2-1}$ is the formation consisting of all abelian groups with exponent dividing $p^2 - 1$. Since $O_p(A_G(H_{i+1}/H_i)) = 1$ by [5, A, Lemma 13.6], it follows that $A_G(H_{i+1}/H_i) \in \mathfrak{U}_{\mathfrak{p}^2-1}$, and so $A_G(H_{i+1}/H_i)$ is abelian with exponent dividing $p^2 - 1$.

Write $|H_{i+1}/H_i| = p^m$. By the faithful and irreducible action of the abelian group $A_G(H_{i+1}/H_i)$ on H_{i+1}/H_i, we see that $A_G(H_{i+1}/H_i)$ is cyclic and m is the smallest positive integer such that $|A_G(H_{i+1}/H_i)|$ divides $p^m - 1$ by [9, II, Lemma 3.10], and thus $m \leq 2$ since the exponent of $A_G(H_{i+1}/H_i)$ divides $p^2 - 1$. Then $r_p(G) \leq 2$. □

3. Proof of Theorem 1.4

Lemma 3.1. *Let p be a prime, and let $P \in Syl_p(G)$, where G is a group. If P is a cyclic group, then either G is p-supersolvable or else $P \cap O^p(G^*_p) = P$.*

Proof. Without loss of generality, we assume $P \cap O^p(G^*_p) < P$. If $P \cap O^p(G^*_p) = 1$, then G^*_p is a p-nilpotent, and thus G is p-supersolvable. So $1 < P \cap O^p(G^*_p) < P$, then it follows from [11, Theorem 2.1] that G is p-supersolvable. □

Proof of Theorem 1.4. Note that G is p-supersolvable if and only if G^*_p is p-nilpotent, and so we only need to prove the sufficient. Now assume that G is a counterexample of minimal order. Then G is not p-supersolvable. In particular, G^*_p is not p-nilpotent, and therefore $N = P \cap O^p(G^*_p) > 1$. For convenience, we write

$$\mathfrak{H} = \{H \trianglelefteq P \mid H \text{ is a non-cyclic subgroup with } |H| = d\}$$

and

$$\mathfrak{Y} = \{Y < \cdot P \mid N \nleq Y\}.$$

It is easy to see that $H \cap O^p(G^*_p) \trianglelefteq G$ for all $H \in \mathfrak{H}$. We proceed in a number of steps to derive a contradiction.

Step 1. *P is not cyclic, dihedral, semidihedral or generalized quaternion.*

If P is cyclic, then, by Lemma 3.1, G is p-supersolvable, a contradiction. Now assume that P is dihedral, semidihedral or generalized quaternion. If N is a cyclic subgroup, then it follows from Burnside's theorem [9, IV, 2.8] and $p = 2$ that $O^p(G^*_p)$ is 2-nilpotent, and thus G^*_p is 2-nilpotent, a contradiction. Thus, by Lemma 2.5, we may assume that N is a non-cyclic maximal subgroup of P and $|N| = d$. In this case $P = D_{2^n} (n \geq 3)$, $Q_{2^n} (n \geq 4)$ or SD_{2^n}, and thus there exists a non-cyclic maximal subgroup N_1 of P such that $N \nleq N_1$. For convenience, we write $M_1 = N \cap N_1$ and have

$$M_1 = N \cap O^p(G^*_p) \cap N_1 \cap O^p(G^*_p) \trianglelefteq G.$$

Since $|P : M_1| = 2^2$, M_1 is cyclic by Lemma 2.5. It follows that $O^p(G_p^*)$ is 2-supersolvable and therefore $O^p(G_p^*)$ is 2-nilpotent. Hence G_p^* is 2-nilpotent, a contradiction.

Step 2. *$\mathfrak{H} \neq \phi$.*

Suppose not, that is , all normal subgroups of P with order d are cyclic. Now by Lemma 2.2, P is cyclic, dihedral, semidihedral or generalized quaternion, in contradiction to Step 1.

Step 3. *$O_{p'}(G_p^*) = 1$.*

Write $D = O_{p'}(G_p^*)$ and $\overline{G} = G/D$, and note that $O^p(\overline{G_p^*}) = \overline{O^p(G_p^*)}$ by [13, Lemma 2.9]. It follows from Dedekind's lemma that $O^p(G_p^*) \cap DH = D(O^p(G_p^*) \cap H)$ for $H \in \mathfrak{H}$. In addition, both D and $O^p(G_p^*) \cap H$ are normalized by $O^p(G)$, we see that $O^p(G)$ normalizes $O^p(G_p^*) \cap DH$, or equivalently, $\overline{O^p(G)}$ normalizes $\overline{O^p(G_p^*)} \cap \overline{H}$. Since $PD \cap O^p(G_p^*) = D(P \cap U) = DN$ and $|N| \leq d$, we see that $|\overline{P} \cap \overline{O^p(G_p^*)}| \leq d$. Then $\overline{G}$ satisfies the hypotheses, and therefore $\overline{G}$ is p-supersolvable. It is clear that the subgroups of $\overline{G}$ corresponding to the members of $\mathfrak{H}$ are exactly the subgroups $\overline{H}$ for $H \in \mathfrak{H}$. Hence $\overline{G}$ is p-supersolvable. Futhermore, we see that G is p-supersolvable, which is a contradiction. So we conclude that $D = 1$.

Step 4. *N is normal in G. In fact, G is p-solvable and $P \trianglelefteq G$.*

Since $H \cap O^p(G_p^*) \trianglelefteq O^p(G)$ for $H \in \mathfrak{H}$ and $O^p(G_p^*) \leq O^p(G)$, we see that $H \cap O^p(G_p^*) \trianglelefteq O^p(G_p^*)$ for $H \in \mathfrak{H}$. Then G_p^* satisfies the hypotheses of [8, Theorem 3.2], and thus G_p^* is p-supersolvable. Hence G is p-solvable. Noticing that $O_{p'}(G_p^*) = 1$ and G_p^* is p-supersolvable, we have $P \trianglelefteq G_p^*$ by [9, VI, 6.6]. Then it follows from $P \in Syl_p(G_p^*)$ and $G_p^* \trianglelefteq G$ that P is normal in G. So $N = P \cap O^p(G_p^*)$ is normal in G by $O^p(G_p^*) \trianglelefteq G$.

Step 5. *There exists a maximal subgroup $Y \in \mathfrak{Y}$ with $L = N \cap Y$ is not normal in G and L is cyclic.*

If $N \leq \Phi(P)$, then it follows from Tate's theorem [9, IV, 4.7] that $O^p(G_p^*)$ is p-nilpotent, and therefore G_p^* is p-nilpotent, a contradiction. Thus there exists a maximal subgroup Y of P with $N \not\leq Y$.

Next we prove that there exists $Y \in \mathfrak{Y}$ such that $L = N \cap Y$ is not normal in G. If not, then $L = N \cap Y$ is normal in G and $|N : L| = p$ for all $Y \in \mathfrak{Y}$. So $G_p^* \leq C_G(N/L)$. Noticing that N/L is a normal Sylow p-subgroup of $O^p(G_p^*)/L$, we see $N/L \leq Z(O^p(G_p^*)/L)$, and therefore $O^p(G_p^*)/L$ is p-nilpotent by Burnside's theorem [9, IV, 2.6]. Hence $O^p(O^p(G_p^*)) < O^p(G_p^*)$, a contradiction.

Finally, we prove that L is cyclic. If L is non-cyclic, then there exists $H \in \mathfrak{H}$ such that $L < H \leq Y$. So

$$L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G_p^*) = H \cap O^p(G_p^*) \trianglelefteq G,$$

which is a contradiction.

Step 6. *Y is a cyclic, dihedral, semidihedral or generalized quaternion group.*

Let Y and L be as in Step 5. If there exists a subgroup S in Y such that $S \in \mathfrak{H}$, then, since $|L| < |N| \leq d = |S|$, there exists $H \in \mathfrak{H}$ such that $L < H \leq LS \leq Y$ by Lemma 2.3. In this case, we have

$$L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G_p^*) = H \cap O^p(G_p^*) \trianglelefteq G,$$

in contradiction to Step 5. So every normal subgroup of P that has order d and is contained in Y is cyclic. By Lemma 2.2, the Step 6 is true.

Step 7. *The final contradictions.*

If Y is a cyclic maximal subgroup of P, then it follows from [2, Lemma 2.1(b)] and Step 1 that $O^p(G_p^*)$ acts trivially on P, and therefore G_p^* is p-nilpotent, a contradiction. Now assume that Y is a dihedral, semidihedral or generalized quaternion group. If $|Y| = d$, then $Y \in \mathfrak{H}$, and therefore $L = N \cap Y = P \cap O^p(G_p^*) \cap Y$ is normal in G, in contradiction to Step 5. The remaining case is $|Y| > d$. In this case, since Y is of maximal class, we see that $|Y : Y'| = 2^2$. Furthermore, by $|Y| > d \geq |N|$ and $|N : L| = 2$, we see that $L \leq Y'$ by Lemma 2.5. It follows from Lemma 2.4 that P' is cyclic, and therefore $L \leq Y' \leq P'$ is normal in G by $P \trianglelefteq G$, in contradiction to Step 5, which is the final contradiction. So the proof is complete. □

Now we present some application of Theorem 1.4.

Lemma 3.2. *Let $P \in Syl_p(G)$ with $|P| > p^3$. If P has exactly one non-cyclic maximal subgroup M and $M \trianglelefteq G_p^*$, then G is p-supersolvable.*

Proof. It is easy to see that the hypotheses are inherited by $G/O_{p'}(G_p^*)$ and $P^{G_p^*}$, so we can assume that $O_{p'}(G_p^*) = 1$. If $P^{G_p^*} < G$, then $P^{G_p^*}$ is p-supersolvable by induction. It follows from $O_{p'}(G_p^*) = 1$ and [9, VI, 6.6] that P is normal in $P^{G_p^*}$, and thus $P = P^{G_p^*}$. And since $G_p^* \trianglelefteq G$ and $P \in Syl_p(G^*)$, we see that $P \trianglelefteq G$. Noticing that there exists a cyclic maximal subgroup in P, we see, by [2, Lemma 2.1], that $O^p(G_p^*)$ acts trivially on P. Thus G_p^* is p-nilpotent, and therefore G is p-supersolvable. Now we can assume that $P^{G_p^*} = G$, and in particular, $G_p^* = G$. Then it follows from [8, Lemma 4.1] that G is p-supersolvable. □

Lemma 3.3. *Let a Sylow p-subgroup P of G be a non-cyclic subgroup with $|P| > p^3$. If every non-cyclic maximal subgroup of P is normal in G_p^*, then G is p-supersolvable.*

Proof. By Lemma 3.2, we can assume that P has two distinct non-cyclic maximal subgroups. Then P is normal in G_p^*. In addition, G_p^* is normal in G and $P \in Syl_p(G_p^*)$. Thus P is normal in G. Since $|P| > p^3$, we see, by [2, Theorem A], that $O^p(G_p^*)$ acts trivially on P. Then G_p^* is p-nilpotent, and therefore G is p-supersolvable. □

Corollary 3.4. *Let P be a non-cyclic Sylow p-subgroup of G with $|P| > p^3$, and suppose for every non-cyclic maximal subgroup H of P that $H \cap O^p(G_p^*) \trianglelefteq O^p(G)$. Then G is p-supersolvable.*

Proof. Assume that G is not p-supersolvable. Applying Theorem 1.4 with $d = |P|/p$, we deduce that $O^p(G_p^*) = G_p^*$, and thus every non-cyclic maximal subgroup of P is normal in G_p^*. It follows from Lemma 3.3 that G is p-supersolvable, a contradiction. □

Corollary 3.5. *Let p be an odd prime and $P \in Syl_p(G)$, where P is non-cyclic. Let d be a power of p such that $p^2 \leq d < |P|$, and let $\mathfrak{H}$ be the set of all non-cyclic normal subgroups H of P with $|H| = d$. Assume that $H \cap O^p(G_p^*) \trianglelefteq O^p(G)$ for all $H \in \mathfrak{H}$. If $N_G(H)$ is p-supersolvable for all $H \in \mathfrak{H}$, then G is p-supersolvable.*

Proof. If $|P \cap O^p(G_p^*)| \leq d$, then G is p-supersolvable by Theorem 1.4. Now we can assume that $|P \cap O^p(G_p^*)| > d$. In this case, if there exists $H \in \mathfrak{H}$ such that $H \leq O^p(G_p^*)$, then $H \trianglelefteq O^p(G)$, and thus $H \trianglelefteq PO^p(G) = G$. Hence $G = N_G(H)$ is p-supersolvable. Now we may assume that $N = P \cap O^p(G_p^*)$ is cyclic by Lemma 2.2. Let L be a subgroup of N with order d/p. Since P is non-cyclic, there exists $H \in \mathfrak{H}$ such that $L \leq H$ by Lemma 2.2 and [8, Lemma 2.4], and thus $L = N \cap H$. Noticing that $L = N \cap H = H \cap O^p(G_p^*)$ is normal in $O^p(G)$, we have that L is normal in G. It follows from [11, Theorem 2.1] that $O^p(G_p^*)$ is p-supersolvable, and therefore $N \trianglelefteq O^p(G_p^*)$. In addition, $O^p(G_p^*)$ is normal in G and $N \in Syl_p(O^p(G_p^*))$. Then N is normal in G. Hence, by [2, lemma 2.1] , $O^p(G_p^*)$ acts trivially on N. Furthermore, we see that G_p^* is p-nilpotent and G is p-supersolvable. The proof of the corollary is complete. □

4. Proof of Theorem 1.5

Lemma 4.1. *Let G be a group of odd order and P be a Sylow p-subgroup of G. If P is a meta-cyclic group or 3-group of maximal class, then $l_p(G) \leq 1$ and $r_p(G) \leq 2$.*

Proof. we proceed by induction on $|G|$. It is easy to see that the hypotheses are inherited by $G/O_{p'}(G)$. So we assume that $O_{p'}(G) = 1$. Then $O_p(G) \neq 1$ since G is p-solvable. By Lemma 2.7, we see that $O_p(G)$ is a 3-group of maximal class or $\Omega_1(O_p(G)) \lesssim C_p \times C_p$. If $O_p(G)$ is of maximal class, then $O_p(G)$ is a $CS(p,2)$-group. Hence $O^p(G^*_{p^2})$ acts trivially on $O_p(G)$ by Lemma 2.10. It follows from Hall-Higman lemma [10, Theorem 3.21] that $O^p(G^*_{p^2}) \leq C_G(O_p(G)) \leq O_p(G)$, and thus $G^*_{p^2}$ is p-group. Furthermore, by Lemma 2.11, $l_p(G) \leq 1$ and $r_p(G) \leq 2$. Now we assume that $\Omega_1(O_p(G)) \lesssim C_p \times C_p$. It follows from Lemma 2.10 that $O^p(G^*_{p^2})$ act trivially on $\Omega_1(O_p(G))$, and thus $O^p(G^*_{p^2})$ act trivial on $O_p(G)$ by [9, IV, 5.12]. So $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by using the arguments above. □

Proof of Theorem 1.5. Suppose that G is a counterexample of minimal order. Then $|P \cap O^p(G^*_{p^2})| \leq d$ and $l_p(G) \not\leq 1$ or $r_p(G) \not\leq 2$. By Lemma 2.11, we see that $G^*_{p^2}$ is not p-nilpotent, and $N = P \cap O^p(G^*_{p^2}) > 1$. For convenience, we write

$$\mathfrak{H}_1 = \{H \trianglelefteq P \mid H \text{ is a non-meta-cyclic subgroup with } |H| = d\}$$

and

$$\mathfrak{Y} = \{Y < \cdot P \mid N \not\leq Y\}.$$

It is easy to see $H \cap O^p(G^*_{p^2}) \trianglelefteq G$ for all $H \in \mathfrak{H}_1$. We proceed in a number of steps to derive a contradiction.

Step 1. $O_{p'}(G) = 1$.

Write $D = O_{p'}(G)$ and $\overline{G} = G/D$. We argue that $\overline{G}$ satisfies the hypotheses of the theorem. The subgroups of $\overline{G}$ corresponding to the members of $\mathfrak{H}_1$ are exactly the subgroups $\overline{H}$ for $H \in \mathfrak{H}_1$, and since $O^p(\overline{G}) = \overline{O^p(G)}$ and $O^p(\overline{G^*_{p^2}}) = \overline{O^p(G^*_{p^2})}$, we must show that $\overline{O^p(G)}$ normalizes $\overline{O^p(G^*_{p^2})} \cap \overline{H}$. On the other hand, $\overline{O^p(G^*_{p^2})} \cap \overline{H} = (O^p(G^*_{p^2} \cap H)D/D$ by [13, Lemma 2.8]. Then $O^p(G)$ normalizes $O^p(G^*_{p^2}) \cap H$. Since D and $O^p(G^*_{p^2}) \cap H$ are normalized by $O^p(G)$, this shows that $\overline{G}$ satisfies the hypotheses, as claimed.

If $D > 1$, then $l_p(\overline{G}) \not\leq 1$ or $r_p(\overline{G}) \not\leq 2$, and thus $|\overline{P} \cap \overline{O^p(G^*_{p^2})}| > d$ by the minimality of G. Hence $|PD \cap O^p(G^*_{p^2})| > d|D|$. Since $PD \cap U = D(P \cap O^p(G^*_{p^2})) = DN$, we see that $|\ N\ | > d$, which is a contradiction with $|N| \leq d$. So we conclude that $D = 1$.

Step 2. $d \geq p^3$.

If $d \leq p^2$, then $|N| \leq p^2$. Since G is an odd order group, we see that G is p-solvable. Then it follows from [9, VI, 6.6] that $l_p(O^p(G^*_{p^2})) \leq 1$. In addition, $O_{p'}(G^*_{p^2}) = 1$ since $O_{p'}(G) = 1$. Thus $N \trianglelefteq O^p(G^*_{p^2})$, and therefore $N \trianglelefteq G$. It follows that $O^p(G^*_{p^2})$ acts trivially on N by Lemma 2.10, and so $O^p(G^*_{p^2})$ is p-nilpotent by Burnside's theorem [9, IV, 2.6]. Hence $G^*_{p^2}$ is p-nilpotent, a contradiction.

Step 3. $\mathfrak{H}_1 \neq \varnothing$.

Suppose not, that is, all subgroups of P with order d are meta-cyclic. Now by Lemma 2.6, P is a meta-cyclic group or a 3-group of maximal class. Then it follows form Step 2 and Lemma 4.1 that $l_p(G) \leq 1$ and $r_p(G) \leq 2$, a contradiction.

Step 4. *N is non-meta-cyclic and is normal in G.*

Suppose that N is meta-cyclic, that is, N is a cyclic group or a meta-cyclic group with $d(N) = 2$, where $d(N)$ is a minimal number of generators of N. If N is cyclic, and let A be a subgroup of N with order p, then A is normal in P by $N \trianglelefteq P$, and therefore there exists $H \in \mathfrak{H}_1$ such that $A \leq H$ by Lemma 2.8 and $H \cap N \neq 1$. Hence, by $H \cap N = H \cap P \cap O^p(G^*_{p^2}) \trianglelefteq G$ and [11, Theorem 2.1], $O^p(G^*_{p^2})$ is p-supersovable. Furthermore, it follows from $O_{p'}(G) = 1$ and [9, VI, 6.6] that N is normal in $O^p(G^*_{p^2})$. By Lemma 2.9 and 2.10, we see that $O^p(G^*_{p^2})$ centralizes N, and thus $O^p(G^*_{p^2})$ is p-nilpotent and $G^*_{p^2}$ is p-nilpotent, a contradiction. Now we assume that N is a metacyclic subgroup of P with $d(G) = 2$. Then $\Omega_1(N) \simeq C_p \times C_p$, and thus, by Lemma 2.8, there exists $H \in \mathfrak{H}_1$ such that $\Omega_1(N) \subseteq H$ and $H \cap N \neq 1$. Hence $T = H \cap N = H \cap P \cap O^p(G^*_{p^2}) \trianglelefteq G$. Noticing that $\Omega_1(N) = \Omega_1(T)$ *char* T, we have that $\Omega_1(N)$ is normal in G, and therefore $O^p(G^*_{p^2})$ centralizes $\Omega_1(N)$ by Lemma 2.10. Since p is odd, we see that $O^p(G^*_{p^2})$ centralizes N by [9, IV, 5.12]. Then $O^p(G^*_{p^2})$ is p-nilpotent by Burnside's Theorem [9, IV, 2.6], and thus $G^*_{p^2}$ is p-nilpotent, a contradiction.

Hence N is non-meta-cyclic, and thus there exists $H \in \mathfrak{H}_1$ such that $N \subseteq H$. We see

$$N = N \cap H = O^p(G^*_{p^2}) \cap P \cap H \trianglelefteq G.$$

Step 5. *There exists a maximal subgroup* $Y \in \mathfrak{Y}$ *such that* $N \not\leq Y$.

If $N \leq \Phi(P)$, then it follows from Tate's theorem [9, IV, 4.7] that $O^p(G^*_{p^2})$ is p-nilpotent, and therefore $G^*_{p^2}$ is p-nilpotent, a contradiction. Thus there exists a maximal subgroup Y of P with $N \not\leq Y$.

Step 6. *For any* $Y \in \mathfrak{Y}$, $L = N \cap Y$ *is not normal in* G *and* L *is meta-cyclic.*

First, we prove that $L = N \cap Y$ is not normal in G for any $Y \in \mathfrak{Y}$. If not, then there exists $Y \in \mathfrak{Y}$ such that $L = N \cap Y \trianglelefteq G$. Since $|N : L| = p$ for all $Y \in \mathfrak{Y}$, $G^*_{p^2} \leq C_G(N/L)$. In addition, N/L is a normal Sylow p-subgroup of $O^p(G^*_{p^2})/L$, then $N/L \leq Z(O^p(G^*_{p^2})/L)$, and therefore $O^p(G^*_{p^2})/L$ is p-nilpotent by Burnside's theorem [9, IV, 2.6]. Hence $O^p(O^p(G^*_{p^2})) < O^p(G^*_{p^2})$, a contradiction.

Next, we prove that L is meta-cyclic. If L is non-meta-cyclic, then there exists $H \in \mathfrak{H}_1$ such that $L < H \leq Y$. So

$$L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G^*_{p^2}) = H \cap O^p(G^*_{p^2}) \trianglelefteq G,$$

which is a contradiction.

Step 7. $N \simeq C_p \times C_p \times C_p$.

If not, then, since L is a meta-cyclic maximal subgroup of N, we see that N is a $CS(p, 2)$-group by Lemma 2.9, and thus N is centralized by $O^p(G^*_{p^2})$ by Lemma 2.10. Hence $G^*_{p^2}$ is p-nilpotent, a contradiction.

Step 8. *The final contradiction.*

It is easy to see that $G^*_{p^2}/N$ is p-nilpotent. If $N \leq \Phi(G)$, then $G^*_{p^2}$ is p-nilpotent, a contradiction. Hence there exists a maximal subgroup M of G such that $N \not\leq M$. It is easy to see that N is a minimal normal subgroup of G. If not, there is nothing to be proved. Then $G = NM$ and $N \cap M = 1$. It follows that $P = N(P \cap M)$ by Dedekind's lemma. For convenience, write $S = P \cap M$. Noticing that there exists a maximal subgroup P_1 of P such that $S \leq P_1$ and $N \not\leq P_1$. Write $K = N \cap P_1$ is normal in P and $K \simeq C_p \times C_p$ by Step 7. If there exists $H_1 \in \mathfrak{H}_1$ such that $H_1 \leq P_1$, then, by Lemma 2.8, there exists $H \in \mathfrak{H}_1$ such that $K \leq H \leq P_1$, and thus $K = N \cap P_1 \cap H = H \cap O^p(G^*_{p^2}) \trianglelefteq G$, which contradicts Step 6. Then it follows from Lemma 2.6 and Lemma 4.1 that P_1 is a meta-cyclic group of $d(P_1) = 2$ or a 3-group of maximal class. If P_1 is meta-cyclic of $d(P_1) = 2$, then $\Omega_1(P_1) \simeq C_p \times C_p$, and therefore $\Omega_1(S) \leq \Omega_1(P_1) = K \leq N$. In addition, we know that $\Omega_1(S) \leq S \leq M$ and $N \cap M = 1$. Then $\Omega_1(S) = 1$, and thus $S = 1$. Hence $N = P$, which is a contradiction with $|N| \leq d < |P|$. Now we assume that P_1 is a 3-group of maximal class. Since $p^3 = |N| \leq d < |P|$, we see that $|P_1| \geq p^3$. If $|P_1| \geq 3^4$, then $K \leq \Phi(P_1)$ by [3, Exercise 9.1.]. It follows from Dedekind's lemma that $P_1 = (P_1 \cap N)S$ and $P_1 = S$, which is a contradiction with $P = NS > P$. Now we assume that $|P_1| = 3^3$ and $|P| = 3^4$. Then it follows from $p^3 = |N| \leq d < |P| = p^4$ that $d = p^3$. Hence $P_1 \in \mathfrak{H}_1$. Furthermore, we see that $K = N \cap P_1 = P_1 \cap O^p(G^*_{p^2}) \trianglelefteq G$, which is a contradiction with Step 6. This final contradiction completes the proof. □

Now we may present some applications of Theorem 1.5.

Lemma 4.2. *Let* G *be a group of odd order and* $P \in Syl_p(G)$ *with* $|P| > p^4$. *If* P *has exactly one non-meta-cyclic maximal subgroup* M *and* $M \trianglelefteq G$, *then* $l_p(G) \leq 1$ *and* $r_p(G) \leq 2$.

Proof. We proceed by induction on $|G|$. It is clear that the hypotheses are inherited by $G/O_{p'}(G)$ and P^G, so we can assume that $O_{p'}(G) = 1$. If $P^G < G$, then $l_p(P^G) = 1$ by induction. In addition, $O_{p'}(G) = 1$, then P is normal in P^G. Since $P^G \trianglelefteq G$ and $P \in Syl_p(P^G)$, we see $P = P^G \trianglelefteq G$. Notice that P has a meta-cyclic maximal subgroup and $|P| > p^4$. Then P is a $CS(p, 2)$-group by Lemma 2.11, and thus P is centralized by $O^p(G^*_{p^2})$ by Lemma 2.10. Then it follows from Burnside's theorem [9, IV, 2.6] that $O^p(G^*_{p^2})$ is p-nilpotent. Hence $G^*_{p^2}$ is p-nilpotent, and therefore $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by Lemma 2.11.

Now we can assume that $P^G = G$, and in particular, $G^*_{p^2} = G$. Applying Theorem 1.5, we may assume that $d = |P|/p$ and $|P \cap O^p(G^*_{p^2})| > d$, and therefore $O^p(G^*_{p^2}) = G^*_{p^2}$. Since M is the unique non-meta-cyclic maximal subgroup of P, we see that M has a meta-cyclic maximal subgroup by [7, Lemma 2.3]. On the other hand, M is a $CS(p, 2)$-group since $|M| > p^3$. Then, by Lemma 2.10, $O^p(G^*_{p^2}) = G$ acts trivially on M. Thus P is abelian and $P \simeq C_{p^m} \times C_p \times C_p (m \geq 3)$. Let $N = N_G(P)$. We see that N/P acts on the P and centralizes M. It follows from Fitting's lemma[10, Lemma 4.28] and $P \simeq C_{p^m} \times C_p \times C_p$ that the action of N/P on P is trivial, and therefore $P \leq Z(N)$. So G is p-nilpotent by Burnside's theorem [9, IV, 2.6], and thus $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by Lemma 2.11. □

Lemma 4.3. *Let G be a group of odd order, and let P be a Sylow p-subgroup of G with $|P| > p^4$. If every non-meta-cyclic maximal subgroup of P is normal in G, then $l_p(G) \leq 1$ and $r_p(G) \leq 2$.*

Proof. We proceed by induction on $|G|$. It is easy to see that the hypotheses are inherited by $G/O_{p'}(G)$. so we can assume that $O_{p'}(G) = 1$. It follows from Lemma 2.6 and 4.1 that P has a non-meta-cyclic maximal subgroup. By Lemma 4.2, we can assume that P has two distinct non-meta-cyclic maximal subgroups, and therefore P is normal in G. Since $|P| > p^4$, $O^p(G^*_{p^2})$ acts trivially on P by [7, Theorem A]. Hence $G^*_{p^2}$ is p-nilpotent, and thus $l_p(G) \leq 1$ and $r_p(G) \leq 2$. □

Corollary 4.4. *Let G be an odd order group and P be a Sylow p-subgroup of G with $|P| > p^4$, and suppose for every non-cyclic maximal subgroup H of P that $H \cap U \trianglelefteq U$, where $U = O^p(G)$. Then $l_p(G) \leq 1$ and $r_p(G) \leq 2$.*

Proof. Applying Theorem 1.5 with $d = |P|/p$, we deduce that $O^p(G^*_{p^2}) = G^*_{p^2}$, and thus every non-cyclic maximal subgroup of P is normal in G. It follows from Lemma 4.3 that $l_p(G) \leq 1$ and $r_p(G) \leq 2$, a contradiction. □

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