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Simple modules for some Cartan-type Lie superalgebras

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Abstract

The modules are induced from the finite-dimensional Cartan-type modular Lie superalgebras W, S, H and K over a field of prime characteristic, respectively. We give the Cartan subalgebras of these modular Lie superalgebras. Using certain properties of the positive root vectors, we discuss the simplicity of these modules.

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1. Introduction

The development of representation theories of Lie algebras and Lie superalgebras over a field of characteristic 0 has shown a remarkable evolution. We know that the representation theory of Lie algebras plays a central role in the classification of simple Lie algebras of characteristic 0. Kac (see [5]) classified finite-dimensional simple Lie superalgebras over the field $\mathbb{C}$, proposing three Cartan series the Witt type, special type and Hamiltonian type, with an additional series of the classical type. Until now, the classification of finite-dimensional simple Lie superalgebras over a field of prime characteristic has not been completed. Zhang (see [11]) studied finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic.

Motivated by the connection between the representation theory of Lie algebras and modular Lie algebras, many researchers have investigated the representation theory of modular Lie algebras (see [1]-[4], [6]-[11]). Many results have also been obtained for the representative theory of modular Lie algebras, i.e., Lie algebras over a field of characteristic $p > 0$ (see [1]-[4], [7]-[9]). The modular Lie superalgebra has experienced rather

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vigorous development and further Cartan-type Lie superalgebras in prime characteristic are constructed. For a restricted Cartan-type Lie algebra, the restricted simple modules have been determined in the sense that their isomorphism classes have been parameterized and their dimensions have been computed. Concrete constructions and dimensions of these modules were obtained by Shen and Holmes (see [1]-[4], [7]-[9]). Shen (see [7]-[9]) constructed the graded modules for the Witt, special and Hamiltonian Lie algebras. Shen determined the simple modules having fundamental dominant weights, except the contact algebra. Holmes (see [1]) solved the remaining problem regarding the contact algebra. With a few exceptional weights, he showed that the simple restricted modules were induced from the restricted universal enveloping algebra for the homogeneous component of degree zero extended trivially to positive components.

However, there are few results for the representation theory of Lie superalgebras over a field of characteristic $p > 0$, i.e., modular Lie superalgebras. Liu (see [6]) solved the dimension formula of induced modules and obtained the properties of induced modules. The structure of Cartan-type Lie superalgebras is not as symmetric as that of classical type Lie superalgebras. The representation theory of Cartan-type Lie superalgebras seems to be more difficult than that of classical type Lie superalgebras. Motivated by the ideas of Holmes (see [1]-[4], [7]-[9]), in this paper we construct modules of Lie superalgebras $W(m,n,\underline{1})$, $S(m,n,\underline{1})$, $H(m,n,\underline{1})$ and $K(m,n,\underline{1})$, induced from the homogeneous components of their restricted universal enveloping superalgebras. We show that the generator $1 \otimes m$ of these constructed modules belongs to their nonzero submodules, before showing that the sufficient conditions of these modules are simple modules.

2. Preliminaries

In this paper, let $\mathbb{F}$ always denote an algebraically closed field of characteristic $p > 0$.

First, we recall the definitions of $W(m,n,\underline{t}), S(m,n,\underline{t}), H(m,n,\underline{t})$ and $K(m,n,\underline{t})$.

Let $\mathbb{N}$, and $\mathbb{N}_0$ be the set of positive integers, and the set of nonnegative integers, respectively. Let $m,n \in \mathbb{N}$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m$, $\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{N}_0^m$. Then we define

(1) $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_m + \beta_m)$. $0 = (0, \cdots, 0)$. $1 = (1, \cdots, 1)$.

(2) $\binom{\alpha}{\beta} = \prod\limits_{i=1}^{m} \binom{\alpha_i}{\beta_i}$, where $\binom{\alpha_i}{\beta_i}$ denotes the binomial coefficient.

(3) $\alpha \leqslant \beta :\Leftrightarrow \alpha_i \leqslant \beta_i$, $i = 1, \cdots, m$.

(4) $\varepsilon_i := (0, \cdots, 1, \cdots, 0)$, where 1 occurs at the ith place.

(5) For every element $(x_1, \cdots, x_m)$, we put $x^{(\alpha)} := \prod\limits_{i=1}^{m} x_i^{\alpha_i}$, noting that (5) implies that $x^0 = 1$.

Let $\mathcal{U}(m)$ denote the $\mathbb{F}$-algebra of divided power series in the variables $x_1, \ldots, x_m$ and define the multiplication by $x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}$, where $\alpha, \beta \in \mathbb{N}_0^m$.

Let $\Lambda(n)$ be an exterior algebra over $\mathbb{F}$, generated by $x_{m+1}, \cdots, x_s$, where $s = m+n$. Put $\Lambda(m,n) := \mathcal{U}(m) \otimes \Lambda(n)$. We write fg for $f \otimes g$ in the following, where $f \in \mathcal{U}(m)$, $g \in \Lambda(n)$. The following identities hold in $\Lambda(m,n)$:

(1) $x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}$.

(2) $x_i x_j = -x_j x_i$, $i,j = m+1, \cdots, s$.

(3) $x^{(\alpha)}x_j = x_j x^{(\alpha)}$, $\forall\ \alpha \in \mathbb{N}_0^m$, $j = m+1, \cdots, s$.

For $k = 1, \cdots, n$, define $B_k := \{(i_1, \cdots, i_k) \mid m+1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant s\}$. Let $B(n) = \bigcup\limits_{k=0}^{n} B_k$, where $B_0 = \emptyset$. If $u = (i_1, i_2, \cdots, i_r) \in B_r$, where $m+1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant s$, then we set $x^u := x_{i_1}x_{i_2}\cdots x_{i_r}$, $|u| := r$. Put $x^{\emptyset} = 1$. Note that $\Lambda(m,n)$ is $\mathbb{Z}_2$-graded by $\Lambda(m,n)_{\bar{0}} = \mathcal{U}(m) \otimes \Lambda(n)_{\bar{0}}$, $\Lambda(m,n)_{\bar{1}} = \mathcal{U}(m) \otimes \Lambda(n)_{\bar{1}}$. Note that $\Lambda(m,n)$ is $\mathbb{Z}$-

graded by $\Lambda(m,n)_i = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid \alpha \in \mathbb{N}_0^m, u \in B(n), |\alpha|+|u| = i\}$, where $|\alpha| = \sum\limits_{i=1}^{m} \alpha_i$. Obviously, $\Lambda(m,n)$ is a $\mathbb{Z}$-graded associative superalgebra.

Put $Y_0 := \{1, 2, \cdots, m\}$, $Y_1 := \{m+1, \cdots, s\}$, $Y = Y_0 \cup Y_1$.

Let ∂_i be the special derivation of $\Lambda(n)$ defined by $\partial_i(x_j) = \delta_{ij}$, where $i, j \in Y_1$ and δ_{ij} is the Kronecker delta. For $i \in Y$, we consider $D_i \in \mathrm{Der}_{\mathbb{F}}(\Lambda(m,n))$ given by $D_i(x^{(\alpha)}x^u) := \begin{cases} x^{(\alpha-\varepsilon_i)}x^u, & \forall i \in Y_0 \\ x^{(\alpha)}\partial_i(x^u), & \forall i \in Y_1 \end{cases}$. $D_1, \cdots, D_s$ are called the special derivations of $\Lambda(m,n)$.

We put $\underline{t} := (t_1, \cdots, t_m) \in \mathbb{N}^m$, $\pi_i := p^{t_i} - 1$, $\forall i \in Y_0$. Denote

$$A(m, \underline{t}) := \{\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m \mid 0 \leqslant \alpha_i \leqslant \pi_i, i \in Y_0\}.$$

$$\Lambda(m, n, \underline{t}) := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid \alpha \in A(m, \underline{t}), u \in B(n)\}.$$

$$\Lambda(m, n, \underline{t})_i := \Lambda(m, n)_i \cap \Lambda(m, n, \underline{t}).$$

Then $\Lambda(m, n, \underline{t}) = \bigoplus\limits_{i=0}^{\xi} \Lambda(m, n, \underline{t})_i$ is a $\mathbb{Z}$-graded superalgebra, where $\xi := \sum\limits_{i=1}^{m} \pi_i + n$.

Remark: In this paper, $\mathrm{hg}(L)$ denotes the set of homogeneous elements of Lie superalgebra L, i.e., $\mathrm{hg}(L) = L_{\bar{0}} \oplus L_{\bar{1}}$. If $f \in \mathrm{hg}(L)$, then $\mathrm{d}(f)$ always denotes the $\mathbb{Z}_2$-graded degree of the element f. Write $\tau(i) := \begin{cases} \bar{0}, & \forall i \in Y_0 \\ \bar{1}, & \forall i \in Y_1 \end{cases}$.

In the following, we illustrate the definitions of the graded Cartan-type Lie superalgebras $W(m, n, \underline{t})$, $S(m, n, \underline{t})$, $H(m, n, \underline{t})$ and $K(m, n, \underline{t})$ (see [1]).

$$W(m, n, \underline{t}) := \{\sum_{i=1}^{s} f_i D_i \mid f_i \in \Lambda(m, n, \underline{t}),\ \forall i \in Y\}.$$

$$S(m, n, \underline{t}) := \mathrm{span}_{\mathbb{F}}\{D_{ij}(f) \mid i, j \in Y, f \in \mathrm{hg}(\Lambda(m, n, \underline{t}))\},$$

where $D_{ij}(f) = (-1)^{\tau(i)\tau(j)} D_i(f) D_j - (-1)^{\mathrm{d}(f)(\tau(i)+\tau(j))} D_j(f) D_i$.

Let $m = 2r$ or $m = 2r + 1$. Put

$$i' := \begin{cases} i + r, & 1 \leqslant i \leqslant r \\ i - r, & r < i \leqslant 2r \\ i, & 2r < i \leqslant s \end{cases}.$$

$$\sigma(i) := \begin{cases} 1, & 1 \leqslant i \leqslant r \\ -1, & r < i \leqslant 2r \\ 1, & 2r < i \leqslant s \end{cases}.$$

Write $D_H(f) = \sum\limits_{i=1}^{s} \sigma(i')(-1)^{\tau(i')\mathrm{d}(f)} D_{i'}(f) D_i$, $\forall i \in Y$, $f \in \mathrm{hg}(\Lambda(m, n, \underline{t}))$. Then

$$H(m, n, \underline{t}) := \mathrm{span}_{\mathbb{F}}\{D_H(f) \mid f \in \mathrm{hg}(\Lambda(m, n, \underline{t}))\},$$

where $m = 2r$ is even.

Let $L = \Lambda(m, n, \underline{t})$ and $m = 2r + 1$ be odd. Define a multiplication on L by means of

$$\begin{aligned} [f, g] \quad = \quad & (2f - \sum_{i \in Y \setminus \{m\}} x_i D_i(f)) D_m(g) \\ & -(-1)^{\mathrm{d}(f)\mathrm{d}(g)}(2g - \sum_{i \in Y \setminus \{m\}} x_i D_i(g)) D_m(f) \\ & + \sum_{i \in Y \setminus \{m\}} \sigma(i)(-1)^{\tau(i)\mathrm{d}(f)} D_i(f) D_{i'}(g). \end{aligned}$$

Then L is a Lie superalgebra. $K(m,n,\underline{t}) := [L,L]$see [1] shows that

$$K(m,n,\underline{t}) := \begin{cases} L, \quad \text{if} \quad n-m-3 \not\equiv 0(\text{mod } p) \\ \bigoplus_{i=0}^{\xi-1} \Lambda(m,n,\underline{t})_i, \quad \text{if} \quad n-m-3 \equiv 0(\text{mod } p) \end{cases}.$$

In the following, we simply write $W := W(m,n,\underline{1})$, $S := S(m,n,\underline{1})$, $H := H(m,n,\underline{1})$ and $K := K(m,n,\underline{1})$, respectively. They inherit $\mathbb{Z}$-gradations from $\Lambda(m,n,\underline{t})$ by means of

$$W = \bigoplus_{i=-1}^{\eta-1} W_i,$$

where $W_i = \text{span}_{\mathbb{F}}\{x^{(\alpha)}x^u D_j \mid |\alpha| + |u| = i+1, j \in Y\}$, $\eta = m(p-1)+n$.

$$S = \bigoplus_{i=-1}^{\eta-2} S_i,$$

where $S_i = \text{span}_{\mathbb{F}}\{D_{ij}(x^{(\alpha)}x^u) \mid |\alpha| + |u| = i+2, i,j \in Y\}$.

$$H = \bigoplus_{i=-1}^{\eta-3} H_i,$$

where $H_i = \text{span}_{\mathbb{F}}\{D_H(x^{(\alpha)}x^u) \mid |\alpha| + |u| = i+2\}$.

$$K = \bigoplus_{i=-2}^{\lambda} K_i,$$

where $K_i = \text{span}_{\mathbb{F}}\{(x^{(\alpha)}x^u) \mid |\alpha| + \alpha_m + |u| = i+2\}$. And

$$\lambda = \begin{cases} \eta + \pi_m - 2, & n-m-3 \not\equiv 0(\text{mod } p) \\ \eta + \pi_m - 3, & n-m-3 \equiv 0(\text{mod } p) \end{cases}.$$

We first recall the definition of the restricted universal enveloping superalgebra. Let $(L,[p])$ be a restricted Lie superalgebra. A pair $(u(L),i)$ consisting of an associative $\mathbb{F}$-superalgebra with unity and a restricted homomorphism $i : L \to u(L)^-$, is called a restricted universal enveloping superalgebra if given any associative $\mathbb{F}$- superalgebra A with unity and any restricted homomorphism $f : L \to A^-$, there is a unique homomorphism $\overline{f} : u(L) \to A$ of associative $\mathbb{F}$-superalgebra such that $\overline{f} \circ i = f$. The category of $u(L)$-modules and that of restricted L-modules are equivalent. According to the PBW theorem, the following statements hold: Let $(L,[p])$ be a restricted Lie superalgebra. If $(u(L),i)$ is a restricted universal enveloping superalgebra and $(l_j)_{j\in J_0} \cup (f_j)_{j\in J_1}$ is an ordered basis of L over $\mathbb{F}$, where $l_j \in L_{\bar{0}}$, $f_j \in L_{\bar{1}}$, then the elements $i(l_{j_1})^{s_1} i(l_{j_2})^{s_2} \cdots i(l_{j_n})^{s_n} i(f_{i_1}) i(f_{i_2}) \cdots i(f_{i_m})$, $j_1 < \cdots < j_n$, $0 \leqslant s_k \leqslant p-1$, $1 \leqslant k \leqslant n$, $i_1 < \cdots < i_m$, consist of a basis of $u(L)$ over $\mathbb{F}$. Sometimes, with no confusion, we will identify L with its image $i(L)$ in $u(L)$. Note that $D_i^p = 0$, for $i \in Y_0$; $D_i^2 = 0$, for $i \in Y_1$.

We know that W, S, H and K are restricted Lie superalgebras. They are also simple Lie superalgebras.

2.1. Definition. Let L be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. Suppose that A is a Cartan subalgebra of L_0, where L_0 is the set of the 0th homogenous elements of $\mathbb{Z}$-graded Lie superalgebra L. For $\lambda \in A^*$ and a $u(L_0)$-module V, we set $V_\lambda := \{v \in V \mid h \cdot v = \lambda(h)v, \forall h \in A\}$. If $V_\lambda \neq 0$, then λ is called a weight and a nonzero vector v in V_λ is called a weight vector (of weight λ). A nonzero vector $v \in V_\lambda$ is called a maximal vector (of weight λ) provided $x \cdot v = 0$, where x is any positive root vector of L_0.

Let $L = \Sigma_{i\in\mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. Set $N^+ := \sum_{i>0} L_i$, where L_i denotes the homogeneous component of degree i in the $\mathbb{Z}$-graded Lie superalgebra L. Then $N^+ \lhd N^+ + L_0 := L^+$ and $L^+/N^+ \cong L_0$. In particular, any L_0-module becomes a L^+-module by letting $\mathbb{N}^+$ act trivially. Define $M_L(B) := u(L) \otimes_{u(L^+)} B$, where $u(L)$ and $u(L^+)$ denote the restricted universal enveloping superalgebras of L and L^+, respectively, and B is a simple $u(L_0)$-module. According to the classical theory, for each weight λ, there exists a simple $u(L_0)$-module $B(\lambda)$ which is generated by a maximal vector of weight λ.

In the following, we will discuss the simplicity of $M_L(B(\lambda)) = u(L) \otimes_{u(L^+)} B(\lambda)$, where L denotes one of four classes of Cartan-type Lie superalgebras W, S, H or K.

Remark:

(1) If C is a subset of some linear space, then $\langle C\rangle$ denotes the subspace spanned by the set C over $\mathbb{F}$.

(2) ω always denotes a maximal vector of a Cartan-type Lie superalgebra L.

(3) $\widehat{D_i}$ means that D_i is deleted.

3. The simple module of Lie superalgebra $W(m, n, \underline{1})$

3.1. Lemma. $A = \sum\limits_{i=1}^{s} \mathbb{F}x_iD_i$ is a Cartan subalgebra of W_0. The positive root vectors of W_0 are $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$.

Proof. Let $\varphi_W : W_0 \to gl(\Lambda(m, n, \underline{1})_1)$ be a homomorphism of Lie superalgebras such that $\varphi_W(x_iD_j) = E_{ij}$, where $gl(\Lambda(m, n, \underline{1})_1)$ is the general linear Lie superalgebra and E_{ij} is the $s \times s$-matrix with 1 in the (i, j)-position and zeros elsewhere, $\forall i, j \in Y$. Note that φ_W is an isomorphism. By a straightforward computation, we obtain that the Cartan subalgebra of $gl(\Lambda(m, n, \underline{1})_1)$ is $\langle\{E_{ii} \mid i \in Y\}\rangle$, and the positive root vectors of $gl(\Lambda(m, n, \underline{1})_1)$ are $\{E_{ij} \mid 1 \leqslant i < j \leqslant s\}$. By the isomorphism φ_W, we find that $A = \sum\limits_{i=1}^{s} \mathbb{F}x_iD_i$ is a Cartan subalgebra of W_0 and $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$ are positive root vectors of W_0. □

According to the Definition 2.1 and Lemma 3.1, the following statements hold: If $A = \sum\limits_{i=1}^{s} \mathbb{F}x_iD_i$ is a Cartan subalgebra of W_0, V is a $u(W_0)$-module and $\lambda \in A^*$, then $V_\lambda = \{v \in V \mid (x_iD_i) \cdot v = \lambda(x_iD_i)v, i \in Y\}$. Write $\lambda_i := \lambda(x_iD_i)$. A nonzero element $v \in V_\lambda$ is a maximal vector (of weight λ) provided $x_iD_j \cdot v = 0$, $\forall\ 1 \leqslant i < j \leqslant s$.

3.2. Lemma. Suppose that $M := < u(W)D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega >$, for $i \in Y_0$. If the following each situation holds, respectively,

(1) $\lambda_i \neq -1, \beta_i = p-1$.

(2) $\lambda_i = 0, \beta_i \neq 1$.

(3) $\lambda_i = -1, \beta_i \neq p-1$.

(4) There exists $j \in Y_0$, $1 \leqslant i < j \leqslant m$, such that $\lambda_j \neq -1$, and $\beta_j = p-1$.

(5) There exists $j \in Y_0$, $1 \leqslant j < i \leqslant m$, such that $\lambda_j \neq 0$, and $\beta_j = 0$.

Then $D_i^{\beta_i-1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(6) In addition, if $\lambda_i \neq 0, \beta_i = p-2$, then $D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

Proof. Obviously, M is a W-module. With the equality

$$[fD, gE] = fD(g)E - (-1)^{\mathrm{d}(fD)\mathrm{d}(gE)} gE(f)D + (-1)^{\mathrm{d}(D)\mathrm{d}(g)} fg[D, E],$$

where $f, g \in \mathrm{hg}(\Lambda(m,n,\underline{t}))$, $D, E \in \mathrm{hg}(\mathrm{Der}(\Lambda(m,n,\underline{t})))$, we obtain

$$\begin{aligned}
& (x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & (D_i\cdot(x^{(2\varepsilon_i)}D_i)-x_iD_i)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & (D_i\cdot(x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i-1}-D_i^{\beta_i-1}\cdot(x_iD_i)+(\beta_i-1)D_i^{\beta_i-1})D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & D_i^{\beta_i-1}\cdot(x^{(2\varepsilon_i)}D_i)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_i^{\beta_i-1}\cdot(x_iD_i)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
& +(1+2+\cdots+(\beta_i-1))D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega.
\end{aligned}$$

Since the $\mathbb{Z}$-graded degree of $x^{(2\varepsilon_i)}D_i$ is 1 and ω is a maximal vector of weight λ, the first term vanishes and the second term equals $-\beta_i\lambda_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega$. Then

$$(x^{(2\varepsilon_i)}D_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega=-\frac{\beta_i}{2}(2\lambda_i-\beta_i+1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.$$

Satisfying the situation (1), (2) or (3), respectively, we can conclude that $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(4) According to the situation (4), there exists $j\in Y_0$, such that $\lambda_j\neq-1$ and $\beta_j=p-1$. We know that $D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. Hence, we obtain

$$\begin{aligned}
& (x_iD_j)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & [D_i\cdot(x_iD_j)-D_j]D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
& -\beta_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-1}D_{j+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}\tag{3.1}$$

The maximal vector ω implies that the first term vanishes. Now (3.1) yields the desired result, $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(5) For $1\leqslant j<i\leqslant m$, we have

$$\begin{aligned}
& (x_iD_j)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & (D_i\cdot(x_iD_j)-D_j)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}\tag{3.2}$$

With the equality $[x^{(2\varepsilon_j)}D_j,x_iD_j]=-x_ix_jD_j$ and (3.2) multiplied on the left by $x^{(2\varepsilon_j)}D_j$, we have

$$\begin{aligned}
& (x^{(2\varepsilon_j)}D_j)\cdot(D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega) \\
= \; & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\cdot(x^{(2\varepsilon_j)}D_j)\cdot(x_iD_j)\otimes\omega \\
& -\beta_i(D_j\cdot(x^{(2\varepsilon_j)}D_j)-x_jD_j)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & \beta_i\lambda_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}\tag{3.3}$$

The assertion follows from (3.3).

(6) In the particular case of $\lambda_i\neq0$ and $\beta_i=p-2$, we obtain

$$\begin{aligned}
& (x^{((p-1)\varepsilon_i)}D_i)\cdot D_i^{p-2}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & (D_i\cdot(x^{((p-1)\varepsilon_i)}D_i)-x^{((p-2)\varepsilon_i)}D_i)D_i^{p-3}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega \\
= \; & -\lambda_iD_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}\tag{3.4}$$

Along with (3.4), we get the result. □

3.1. Theorem. $M_W(B(\lambda))$ is simple, if one of the following situations holds:

(1) $(\lambda_1, \cdots, \lambda_m) \neq \sum_{t=k+1}^{m} -\varepsilon_t$, (the empty sum being zero) for $0 \leqslant k \leqslant m$.

(2) There exist $i, j \in Y_1$ and $j < i$ such that $\lambda_i \neq 1$, $\lambda_j \neq 0$.

Proof. Let $M^{'}$ be a nonzero submodule of $M_W(B(\lambda)) = u(W) \otimes_{u(W^+)} B(\lambda)$. $u(W^+) \cdot B(\lambda) = u(W_0) \cdot B(\lambda)$ implies that $u(W) \cdot B(\lambda) = (u(W_0) + u(W_{-1})) \cdot B(\lambda) \subseteq u(W_{-1}) \cdot B(\lambda)$. Then, $M_W(B(\lambda)) = u(W) \otimes_{u(W^+)} B(\lambda) = (u(W_{-1}) + u(W_0)) \otimes_{u(W^+)} B(\lambda) \subseteq u(W_{-1}) \otimes_{u(W^+)} B(\lambda)$, namely,

(3.5) $\quad M_W(B(\lambda)) = u(W_{-1}) \otimes_{u(W^+)} B(\lambda).$

Choose $v \neq 0 \in M^{'}$. (3.5) implies that v can be written by

$$v = \Sigma_{\beta \in I} c(\beta) i(D_1)^{\beta_1} \cdots i(D_s)^{\beta_s} \otimes b_\beta,$$

where $\beta = (\beta_1, \cdots, \beta_s)$, $I := \{(\beta_1, \cdots, \beta_s) \mid 0 \leqslant \beta_i \leqslant p-1, \text{for any } i \in Y_0; \beta_i = 0 \text{ or } 1, \text{for any } i \in Y_1\} \subset \mathbb{Z}^s$, $c(\beta) \in \mathbb{F}$, $b_\beta \in B(\lambda)$. In the following, with no confusion, we usually write D_j to instead $i(D_j)$ in $u(L)$. Then

(3.6) $\quad v = \Sigma_{\beta \in I} c(\beta) D_1^{\beta_1} \cdots D_s^{\beta_s} \otimes b_\beta.$

Define an order of I such that $\beta = (\beta_1, \cdots, \beta_s) < \beta' = (\beta_1', \cdots, \beta_s')$ if and only if there exists $k \in \{1, 2, \cdots, s\}$ such that $\beta_i = \beta_i'$ for all $i > k$ and $\beta_k < \beta_k'$. Let $\mathscr{C} := \{\beta \in I \mid c(\beta) \neq 0\}$, where $c(\beta)$ comes from the right side of the equality (3.6). According to the order of I, we choose the least element $\eta = (\eta_1, \cdots, \eta_s) \in \mathscr{C}$. Obviously, $c(\eta) \neq 0$. Put $y := \prod_{t=1}^{m} D_t^{p-1-\eta_t} \prod_{l=m+1}^{s} D_l^{1-\eta_l}$. Since $[D_i, D_j] = 0$, namely, $D_i D_j = (-1)^{\tau(i)\tau(j)} D_j D_i$ holds in $u(W)$. Then

$$\begin{aligned} yv &= \prod_{t=1}^{m} D_t^{p-1-\eta_t} \prod_{l=m+1}^{s} D_l^{1-\eta_l} (\Sigma_{\beta \in I} c(\beta) D_1^{\beta_1} \cdots D_s^{\beta_s} \otimes b_\beta) \\ &= \alpha c(\eta) \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes b_\beta \in M', \end{aligned}$$

where $\alpha = 1$ or -1. Consequently, $\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes b_\beta \in M'$.

$\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda)$ is a $u(W_0)$-module. In fact, if $k \in Y_0$, a straightforward computation shows that,

$$\begin{aligned} & (x_k D_l) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ = & \prod_{t_1=1}^{k-1} D_{t_1}^{p-1} \cdot (x_k D_l) \cdot D_k^{p-1} \prod_{t_2=k+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ = & \prod_{t_1=1}^{k-1} D_{t_1}^{p-1} (D_k^{p-1} \cdot x_k D_l - (p-1) D_l D_k^{p-2}) \prod_{t_2=k+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\ \subseteq & \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda). \end{aligned}$$

Similarly, if $k \in Y_1$, then we have

$$\begin{aligned}
& (x_k D_l) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\
= & \prod_{t=1}^{m} D_t^{p-1} \cdot (x_k D_l) \cdot \prod_{l=m+1}^{s} D_l \otimes B(\lambda) \\
\subseteq & \prod_{t=1}^{m} D_t^{p-1} \cdot \prod_{l_1=m+1}^{k-1} D_{l_1} \cdot (-1)^{\mathrm{d}(x_k D_l)} (D_k \cdot (x_k D_l) - D_l) \cdot \prod_{l_2=k+1}^{s} D_{l_2} \otimes B(\lambda) \\
\subseteq & \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes B(\lambda).
\end{aligned}$$

As $B(\lambda)$ is a simple $u(W_0)$-module, we see that $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes B(\lambda)$ is a simple $u(W_0)$-module. It can be regarded as a $u(W)$-module. By virtue of $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes b_\beta \in M' \cap \left(\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes B(\lambda) \right)$, we have $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes B(\lambda) \subseteq M'$. Thus, there exists a maximal vector ω of weight λ such that $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

For the case of the situation (1), without loss of generality, we assume $\prod\limits_{t=i}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$, for any $i \in Y_0$. We proceed according to several different situations.
(i) $\lambda_i \neq -1$ and $\lambda_i \neq 0$.

By (1) and (6) in Lemma 3.2, we conclude that $\prod\limits_{t=i+1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) $\lambda_i = -1$.

By $(\lambda_1, \cdots, \lambda_m) \neq \sum\limits_{t=i}^{m} -\varepsilon_t$, there exists $j \in Y_0$, $j > i$ such that $\lambda_j \neq -1$, or $j < i$, $\lambda_j \neq 0$. By (4) or (5) in Lemma 3.2, we have $\prod\limits_{t=i+1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) $\lambda_i = 0$.

By $(\lambda_1, \cdots, \lambda_m) \neq \sum\limits_{t=i+1}^{m} -\varepsilon_t$, there exists $j \in Y_0$, such that $j > i$, $\lambda_j \neq -1$, or $j < i$, $\lambda_j \neq 0$. Also by (4) or (5) in Lemma 3.2, we have $\prod\limits_{t=i+1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

The assumption of arbitrary i implies that $\prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

Furthermore, without loss of the generality, we assume $\prod\limits_{l=j}^{s} D_l \otimes \omega \in M'$. The situation (1) implies that there exists $i \in Y_0$ such that $\lambda_i \neq 0$. Then, $\prod\limits_{l=j}^{s} D_l \otimes \omega \in M'$ multiplied on the left by $x_j x_i D_i$, for $j \in Y_1$, we obtain

$$\begin{aligned}
& (x_j x_i D_i) \cdot D_j \cdots D_s \otimes \omega \\
= & (-D_j \cdot (x_j x_i D_i) + x_i D_i) D_{j+1} \cdots D_s \otimes \omega \\
= & \lambda_i D_{j+1} \cdots D_s \otimes \omega. \qquad (3.7)
\end{aligned}$$

Consequently, $D_{j+1}\cdots D_s\otimes\omega\in M'$. Along with the choice of j, we obtain $1\otimes\omega\in M'$. Since $u(W_0)$-module $M_W(B(\lambda))$ is generated by $1\otimes\omega$, $1\otimes\omega\in M'$ indicates that $M_W(B(\lambda))=M'$. It suffices to demonstrate that $M_W(B(\lambda))$ is simple.

Situation (2) implies that $i\neq m+1\in Y_1$. Obviously, $[x_{m+1}x_iD_i,D_l]=\delta_{m+1,l}x_iD_i-\delta_{il}x_{m+1}D_i$, for $l\in Y_1$. Thus

$$(x_{m+1}x_iD_i)\cdot D_l=-D_l\cdot(x_{m+1}x_iD_i)+\delta_{m+1,l}x_iD_i-\delta_{il}x_{m+1}D_i \tag{3.8}$$

holds in $u(W)$. By virtue of (3.8), we find

$$\begin{aligned}
&(x_{m+1}x_iD_i)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}-D_{m+1}\cdot(x_{m+1}x_iD_i)+x_iD_i)D_{m+2}\cdots D_s\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-1)^{i-1-m}D_{m+1}\cdots D_{i-1}\cdot(x_{m+1}x_iD_i)\cdot D_i\cdots D_s\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_{i-1}\cdot(x_iD_i)\cdot D_i\cdots D_s\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-1)^{n}D_{m+1}\cdots D_s\cdot(x_{m+1}x_iD_i)\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}(-1)^{i-m}D_{m+1}\cdots\widehat{D_i}\cdots D_s\cdot(x_{m+1}D_i)\otimes\omega\\
&+\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_s\cdot(x_iD_i)\otimes\omega\\
&-\prod_{t=1}^{m}D_t^{p-1}D_{m+2}\cdots D_s\otimes\omega.
\end{aligned}$$

As $B(\lambda)$ is a $u(W_0)$-module and ω is a maximal vector of weight λ, the first summation vanishes and so does the second for $m+1<i$. According to the assertion above, we obtain

$$(x_{m+1}x_iD_i)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+1}^{s}D_l\otimes\omega=(\lambda_i-1)\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega.$$

As $\lambda_i\neq1$, this entails $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega\in M'$. If $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=m+2}^{s}D_l\otimes\omega$ multiplied on the left by the elements $x_{m+2}x_iD_i, x_{m+3}x_iD_i,\cdots,x_{i-1}x_iD_i$, successively, yielding $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i}^{s}D_l\otimes\omega\in M'$. By the assumption of the theorem, there exists $m+1\leqslant j<i\leqslant s$ such that $\lambda_j\neq0$. Thus we have

$$\begin{aligned}
&(x_ix_jD_j)\cdot\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i}^{s}D_l\otimes\omega\\
=\;&\prod_{t=1}^{m}D_t^{p-1}(-D_i)\cdot(x_ix_jD_j)\cdot\prod_{l=i+1}^{s}D_l\otimes\omega+\prod_{t=1}^{m}D_t^{p-1}\cdot(x_jD_j)\cdot\prod_{l=i+1}^{s}D_l\otimes\omega.
\end{aligned}\tag{3.9}$$

Using the fact that the $\mathbb{Z}$-graded degree of $x_ix_jD_j$ is 1, we can see that (3.9) coincides with $\lambda_j\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i+1}^{s}D_l\otimes\omega$. The assertion that $\prod_{t=1}^{m}D_t^{p-1}\prod_{l=i+1}^{s}D_l\otimes\omega\in M'$ follows from $\lambda_j \neq 0$. We multiply $\prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \otimes \omega$ by the elements $x_{i+1}x_jD_j, \cdots, x_sx_jD_j$, consecutively. Repeating the process above yields $\prod_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$. The situation (2) ensures that there exists $j \in Y_1$ such that $\lambda_j \neq 0$. Thus, for any $i \in Y_0$, we have

$$\begin{aligned} & (x_ix_jD_j) \cdot \prod_{t=i}^{m} D_t^{p-1} \otimes \omega \\ = \quad & (D_i \cdot (x_ix_jD_j) - x_jD_j) \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \\ = \quad & -\lambda_j \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega. \end{aligned}$$

$\lambda_j \neq 0$ entails $\prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \in M'$. In the general case of i, this implies that $1 \otimes \omega \in M'$.

Hence, $M_W(B(\lambda))$ is simple. □

4. The simple module of Lie superalgebra $S(m, n, \underline{1})$

4.1. Lemma. Let

$A = \langle \{-D_{i,i+1}(x_ix_{i+1}), D_{j,j+1}(x_jx_{j+1}), D_{m,m+1}(x_mx_{m+1}) \mid i, i+1 \in Y_0; j, j+1 \in Y_1\} \rangle$.

Then A is a Cartan subalgebra of S_0. The positive root vectors of S_0 are $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$.

Proof. The homomorphism φ_S is the restriction of the isomorphism $\varphi_W : W_0 \to gl(\Lambda(m, n, \underline{t})_1)$. Note that $S_0 \cong \mathscr{L}$, where $\mathscr{L} := \langle \{ \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right), E_{m,m} + E_{m+1,m+1}, E_{ij} \mid A_1 \in Sl_m(\mathbb{F}), D_1 \in Sl_n(\mathbb{F}); i \in Y_0, j \in Y_1; \text{or } i \in Y_1, j \in Y_0\} \rangle$.
By a straightforward computation, we get the Cartan subalgebra of $\mathscr{L}$ is $\langle \{ \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right), E_{m,m} + E_{m+1,m+1} \mid A_1 \in Sl_m(\mathbb{F}), D_1 \in Sl_n(\mathbb{F})\} \rangle$ and the positive root vectors of $\mathscr{L}$ are $\{E_{ij} \mid 1 \leqslant i < j \leqslant s\}$. By the isomorphism φ_S, we obtain that the Cartan subalgebra of S_0 is

$$A = \langle \{-D_{i,i+1}(x_ix_{i+1}), D_{j,j+1}(x_jx_{j+1}), D_{m,m+1}(x_mx_{m+1}) \mid i, i+1 \in Y_0; j, j+1 \in Y_1\} \rangle$$

and the positive root vectors of S_0 are $\{x_iD_j \mid 1 \leqslant i < j \leqslant s\}$. □

Our preceding results of Lemma 4.1 discuss the weight vectors and a maximal vector. The following facts hold: If A is a Cartan subalgebra of S_0, V is a $u(S_0)$-module and $\lambda \in A^*$. Then

$$V_\lambda = \{v \in V \mid D_{i+1,i}(x_ix_{i+1}) \cdot v = \lambda_i v, \ 1 \leq i \leq m-1; D_{j,j+1}(x_jx_{j+1}) \cdot v = \lambda_j v, m \leqslant j \leqslant s-1\}.$$

A nonzero element $v \in V_\lambda$ is a maximal vector (of weight λ) provided $x_iD_j \cdot v = 0$, whenever $1 \leqslant i < j \leqslant s$.

4.2. Lemma. Suppose that $M := < u(S)D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega >$, for $1 \leqslant i \leqslant m-1$. If the following situations hold, respectively,

(1) $\lambda_i \neq 0, \beta_i = p-1$.
(2) $\lambda_i = 1, \beta_i \neq 1$.
(3) $\lambda_i = 0, \beta_i \neq p-1$.
(4) There exists $j \in Y_0$, $1 \leqslant i < j \leqslant m$, such that $\beta_j = p-1$ and $\lambda_j \neq 0$.
(5) There exists $j \in Y_0$, $2 \leqslant j+1 < i \leqslant m-1$, such that $\beta_j = 0$ and $\lambda_j \neq 0$. In addition, for $i = j+1, \beta_i = p-1$ and $\lambda_j \neq 1$, or $\beta_i \neq p-1$ and $\lambda_j \neq 0$.

Then $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(6) In addition, if $\lambda_i \neq 1, \beta_i = p-2$, then $D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

Proof. By virtue of $[D_k, D_{ij}(f)] = (-1)^{\tau(k)\tau(i)} D_{ij}(D_k(f))$, for all $i, j, k \in Y$, we have

$$\begin{aligned}
& D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& (D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) - D_{i+1,i}(x_i x_{i+1})) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& - D_i^{\beta_i - 1} \cdot D_{i+1,i}(x_i x_{i+1}) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} + (\beta_i - 1) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& D_i \cdot D_{i+1,i}(x^{(2\varepsilon_i)} x_{i+1}) \cdot D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& + (-\lambda_i + \beta_i) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& ((-\lambda_i + \beta_i) + (-\lambda_i + \beta_i - 1) + \cdots + (-\lambda_i + 1)) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
(4.1) = \;& -\frac{\beta_i}{2} (2\lambda_i - \beta_i - 1) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M.
\end{aligned}$$

The foregoing equality (4.1) implies that (1), (2) or (3) can conclude $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$, respectively.

(4) Our assumption of the situation (4) entails that $D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$. Then,

$$\begin{aligned}
& D_{ij}(x^{(2\varepsilon_i)}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& (D_i \cdot (x_i D_j) - D_j) D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_{j-1}^{p-1} D_j^{p-2} D_{j+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
(4.2) \quad & - \beta_i D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_j^{p-1} \cdots D_m^{p-1} \otimes \omega.
\end{aligned}$$

For $i < j$, the first term vanishes. Hence (4.2) implies that $D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \in M$.

(5) For $i > j$, we have

$$\begin{aligned}
& D_{ij}(x^{(2\varepsilon_i)}) \cdot D_i^{\beta_i} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_i D_j D_i^{\beta_i - 1} \cdots D_m^{p-1} \otimes \omega \in M.
\end{aligned}$$

Multiplying this equation by $D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1})$ on the left, we obtain

$$D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (D_i^{\beta_i} \cdot (x_i D_j) \cdot D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_i D_j D_i^{\beta_i - 1} D_{i+1}^{p-1} \cdots D_m^{p-1} \otimes \omega) \in M.$$

If $i = j+1$, then

(4.3)

$$D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (D_{j+1}^{\beta_{j+1}} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega - \beta_{j+1} D_j D_{j+1}^{\beta_{j+1} - 1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega) \in M,$$

where,

$$\begin{aligned}
& D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot D_{j+1}^{\beta_{j+1}} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& (D_{j+1} \cdot D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) - D_{j+1,j}(x^{(2\varepsilon_j)})) D_{j+1}^{\beta_{j+1} - 1} \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& D_{j+1}^{\beta_{j+1}} \cdot D_{j+1,j}(x^{(2\varepsilon_j)} x_{j+1}) \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
& + \beta_{j+1} D_{j+1}^{\beta_{j+1} - 1} \cdot (x_j D_{j+1}) \cdot (x_{j+1} D_j) \cdot D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega \\
= \;& \beta_{j+1} \lambda_j D_{j+1}^{\beta_{j+1} - 1} D_{j+2}^{p-1} \cdots D_m^{p-1} \otimes \omega,
\end{aligned}$$

and

$$\begin{aligned}
&D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1})\cdot(-\beta_{j+1}D_jD_{j+1}^{\beta_{j+1}-1}D_{j+2}^{p-1}\cdots D_m^{p-1}\otimes\omega)\\
=\;&-\beta_{j+1}(D_j\cdot D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1})-D_{j+1,j}(x_jx_{j+1}))D_{j+1}^{\beta_{j+1}-1}D_{j+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&\beta_{j+1}D_{j+1,j}(x_jx_{j+1})\cdot D_{j+1}^{\beta_{j+1}-1}D_{j+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&\beta_{j+1}(\lambda_j+\beta_{j+1}-1)D_{j+1}^{\beta_{j+1}-1}D_{j+2}^{p-1}\cdots D_m^{p-1}\otimes\omega.
\end{aligned}$$

Hence, (4.3) coincides with

$$\beta_{j+1}(2\lambda_j+\beta_{j+1}-1)D_{j+1}^{\beta_{j+1}-1}D_{j+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.$$

If $i>j+1$, then

$$\begin{aligned}
&D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1})\cdot(D_i^{\beta_i}\cdot(x_iD_j)\cdot D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega-\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega)\\
=\;&-\beta_i(D_j\cdot D_{j+1,j}(x^{(2\varepsilon_j)}x_{j+1})-D_{j+1,j}(x_jx_{j+1}))D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&\beta_i\lambda_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.
\end{aligned}$$

We get the desired results.

(6) In the special case, $\beta_i=p-2$, we obtain that

$$\begin{aligned}
&D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1})\cdot D_i^{p-2}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1})\cdot D_{i+1}^{p-1}D_i^{p-2}D_{i+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&(D_{i+1}\cdot D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1})-D_{i+1,i}(x^{((p-1)\varepsilon_i)}))D_{i+1}^{p-2}D_i^{p-2}D_{i+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&(D_{i+1}^{p-1}\cdot D_{i+1,i}(x^{((p-1)\varepsilon_i)}x_{i+1})-x_i^{p-2}D_{i+1}\cdot D_{i+1}^{p-2})D_i^{p-2}D_{i+2}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
=\;&(-\lambda_i+1)D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega.
\end{aligned}$$

From $\lambda_i\neq 1$, we obtain the desired identity, $D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. □

4.3. Lemma. Suppose that $M:=<u(S)D_m^{p-1}D_{m+1}\otimes\omega>$. If the following situations hold, respectively,

(1) $\lambda_m\neq 0,\beta_m=p-1$.

(2) $\lambda_m=1,\beta_m\neq 1$.

(3) $\lambda_m=0,\beta_m\neq p-1$.

(4) There exists $j\in Y_0$, $2\leqslant j+1<m$ such that $\beta_j=0$ and $\lambda_j\neq 1$. In addition, for $m=j+1,\beta_m=p-1,\lambda_j\neq 1$, or $\beta_m\neq p-1,\lambda_j\neq 0$.

Then $D_m^{\beta_m-1}D_{m+1}\otimes\omega\in M$.

(5) In addition, if $\lambda_m\neq 1,\beta_m=p-2$, then $D_{m+1}\otimes\omega\in M$.

Proof. The proof is completely analogous to the one given in Lemma 4.2. □

4.1. Theorem. $M_S(B(\lambda))$ is simple, if one of the following situations holds:

(1) $(\lambda_1,\cdots,\lambda_m)\neq\varepsilon_{i-1}$, for all $1\leqslant i\leqslant m+1$, where $\varepsilon_i:=(0,\cdots,1,\cdots,0)\in\mathbb{N}_0^m$, 1 occurs at the ith place, $\varepsilon_0:=(0,\cdots,0,\cdots,0)$.

(2) There exist $i,j\in Y_1$, such that $\mid j-i\mid>1$ and $\lambda_i\neq 0,\lambda_j\neq 0$.

Proof. Let M' be a nonzero submodule of $M_S(B(\lambda))$. The similar discussion of the Lie superalgebra W applies to the Lie superalgebra S. There exists a maximal vector ω such that $\prod\limits_{t=1}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$.

For the case of the situation (1), in general case, we assume $\prod\limits_{t=i}^{m}D_t^{p-1}\prod\limits_{l=m+1}^{s}D_l\otimes\omega\in M'$, for $1\leqslant i\leqslant m-1$.

(i) $\lambda_i\neq 0$ and $\lambda_i\neq 1$.

By (1) and (6) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) $\lambda_i = 1$.

By (2) in Lemma 4.2, we have $D_i \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. For $(\lambda_1, \cdots, \lambda_m) \neq \varepsilon_i$, then there exists $j \in Y_0$, $j > i$ such that $\lambda_j \neq 0$, or $j < i$ such that $\lambda_j \neq 0$.

If $j > i$, $\lambda_j \neq 0$, by (4) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

If $j + 1 < i$, $\lambda_j \neq 0$, by (5) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

If $j + 1 = i$, $\lambda_j \neq 0$, we have

$$
\begin{aligned}
& D_{j+1,j}(x_j x^{(2\varepsilon_{j+1})}) \cdot D_i \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\
= \; & (D_{j+1} \cdot D_{j+1,j}(x_j x^{(2\varepsilon_{j+1})}) - D_{j+1,j}(x_j x_{j+1})) \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\
= \; & -\lambda_j \prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'.
\end{aligned}
$$

We can conclude that $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) $\lambda_i = 0$.

Since $(\lambda_1, \cdots, \lambda_m) \neq \varepsilon_{i-1}$, there exists $j \in Y_0$, such that $j > i$, $\lambda_j \neq 0$, or $j + 1 < i$, $\lambda_j \neq 0$, or $j + 1 = i$, $\lambda_j \neq 1$. By (4) or (5) in Lemma 4.2, we have $\prod_{t=i+1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

The assumption of arbitrary of i implies that $D_m \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$. By Lemma 4.3, the similar discussion as before, we have $\prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

In general, we set $\prod_{l=j}^{s} D_l \otimes \omega \in M'$, for $j \in Y_1$. The situation (1) implies that there exists $i \in Y_0$, $1 \leqslant i \leqslant m$ such that $\lambda_i \neq 0$. Therefore, $\prod_{l=j}^{s} D_l \otimes \omega$ multiplied on the left by $D_{i+1,i}(x_i x_{i+1} x_j)$, we obtain

$$
\begin{aligned}
& D_{i+1,i}(x_i x_{i+1} x_j) \cdot D_j \cdots D_s \otimes \omega \\
= \; & (-D_j \cdot D_{i,i+1}(x_i x_{i+1} x_j) + D_{i+1,i}(x_i x_{i+1})) D_{j+1} \cdots D_s \otimes \omega \\
= \; & \lambda_i D_{j+1} \cdots D_s \otimes \omega \in M'.
\end{aligned}
$$

Then $D_{j+1} \cdots D_s \otimes \omega \in M'$ follows from $\lambda_i \neq 0$. According to the arbitrary $j \in Y_1$, as well as $1 \otimes \omega \in M'$, we obtain that $M_S(B(\lambda))$ is simple.

For the case of the situation (2), if $i \neq m+1 \in Y_1$, we have

$$
\begin{aligned}
& D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\
= & -\prod_{t=1}^{m} D_t^{p-1} D_{m+1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=m+2}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=m+2}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1}(-1)^{i-m-1} D_{m+1}\cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=i}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2}\cdots D_{i-1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1}(-1)^{i-m} D_{m+1}\cdots D_i \cdot D_{i,i+1}(x_{m+1}x_{m+i}x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1}(-1)^{i+1-m} D_{m+1}\cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2}\cdots D_i \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2}\cdots D_{i-1} \cdot D_{i,i+1}(x_{i+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1}(-1)^{i+1-m} D_{m+1}\cdots D_{i+1} \cdot D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1}(-1)^{i+1-m} D_{m+1}\cdots D_i \cdot D_{i,i+1}(x_{m+1}x_i) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1}(-1)^{i+2-m} D_{m+1}\cdots D_{i-1} \cdot D_{i,i+1}(x_{m+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{m+2}\cdots D_{i+1} \cdot D_{i,i+1}(x_ix_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1}(-1)^{i+1-m} D_{m+1}\cdots \widehat{D_i} D_{i+1} \cdot D_{i,i+1}(x_{m+1}x_{i+1}) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega \\
& +\prod_{t=1}^{m} D_t^{p-1} D_{m+2}\cdots D_i \cdot D_{i,i+1}(x_i) \cdot \prod_{l=i+2}^{s} D_l \otimes \omega - \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+2}^{s} D_l \otimes \omega.
\end{aligned}
$$

Since the $\mathbb{Z}$-graded degree of $D_{i,i+1}(x_{m+1}x_ix_{i+1})$ is 1, it implies that the first term vanishes. The definition of a maximal vector ω implies that the second and the forth terms vanish. Finally, we obtain

$$\begin{aligned}
& D_{i,i+1}(x_{m+1}x_ix_{i+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \\
= & -\prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+2}^{s} D_l \cdot D_{i,i+1}(x_ix_{i+1}) \otimes \omega \\
= & \lambda_i \prod_{t=1}^{m} D_t^{p-1} \prod_{l=m+2}^{s} D_l \otimes \omega.
\end{aligned}$$

Since $\lambda_i \neq 0$, we have $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$.

Multiply $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega$ by $D_{i,i+1}(x_{m+2}x_ix_{i+1}), \cdots, D_{i,i+1}(x_{i-1}x_ix_{i+1})$, in turn. The same calculation as above yields $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i}^{s} D_l \otimes \omega \in M'$, where $m+1 < i < s$.

If $m+1 \leqslant j < i-1 \leqslant s$, then we have

$$\begin{aligned}
& D_{j,j+1}(x_ix_jx_{j+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1} (-1)^{s-i+1} D_i D_{i+1} \cdots D_s \cdot D_{j,j+1}(x_ix_jx_{j+1}) \otimes \omega \\
& -\prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \cdot D_{j,j+1}(x_jx_{j+1}) \otimes \omega.
\end{aligned}$$

It also can be found that the first term vanishes. By the assumption of the Theorem 4.1, we have $\lambda_j \neq 0$. Hence, from the second term we can conclude that $\prod\limits_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \otimes \omega \in M'$. Multiplying $\prod\limits_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_s \otimes \omega$ on the left by $D_{j,j+1}(x_{i+1}x_jx_{j+1}), \cdots, D_{j,j+1}(x_sx_jx_{j+1})$, we obtain that $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

If $m+1 \leqslant i+1 < j \leqslant s$, then we have

$$\begin{aligned}
& D_{j,j+1}(x_ix_jx_{j+1}) \cdot \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i} D_i \cdots D_{j-1} \cdot D_{j,j+1}(x_ix_jx_{j+1}) \cdot \prod_{l=j}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} \cdot D_{j,j+1}(x_jx_{j+1}) \cdot \prod_{l=i+1}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+1} D_i \cdots D_j \cdot D_{j,j+1}(x_ix_jx_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i} D_i \cdots D_{j-1} \cdot D_{j,j+1}(x_ix_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_{j-1} D_j \cdot D_{j,j+1}(x_jx_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} D_{i+1} \cdots D_{j-1} \cdot D_{j,j+1}(x_{j+1}) \cdot \prod_{l=j+1}^{s} D_l \otimes \omega \\
= & \prod_{t=1}^{m} D_t^{p-1} (-1)^{s-i+1} \prod_{l=i}^{s} D_l \cdot D_{j,j+1}(x_ix_jx_{j+1}) \cdot \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+2} D_i \cdots \widehat{D_{j+1}} \cdots D_s \cdot D_{j,j+1}(x_ix_j) \otimes \omega \\
& + \prod_{t=1}^{m} D_t^{p-1} (-1)^{j-i+1} D_i \cdots \widehat{D_j} \cdots D_s \cdot (x_iD_j) \otimes \omega \\
& - \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \cdot D_{j,j+1}(x_jx_{j+1}) \otimes \omega \\
= & \lambda_j \prod_{t=1}^{m} D_t^{p-1} \prod_{l=i+1}^{s} D_l \otimes \omega.
\end{aligned}$$

Obviously, we can obtain $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+1}^{s} D_l \otimes \omega \in M'$, for $\lambda_j \neq 0$. Similarly, considering $D_{j,j+1}(x_{i+1}x_jx_{j+1}) \cdot \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+1}^{s} D_l \otimes \omega$, for $j \neq i+1$, it implies that $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+2}^{s} D_l \otimes \omega \in M'$. Continue to multiply $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=i+2}^{s} D_l \otimes \omega \in M'$ on the left by $D_{i,i+1}(x_{i+2}x_ix_{i+1}), \cdots$, $D_{i,i+1}(x_sx_ix_{i+1})$, consecutively. Finally, we obtain $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

If $i = m+1$, we have $D_{j,j+1}(x_{m+1}x_jx_{j+1}) \cdot \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega = \lambda_j \prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$, furthermore, $\prod\limits_{t=1}^{m} D_t^{p-1} \prod\limits_{l=m+2}^{s} D_l \otimes \omega \in M'$. Imitating the process of calculation above for $i \neq m+1$, we have $\prod\limits_{t=1}^{m} D_t^{p-1} \otimes \omega \in M'$.

Using the fact that there exists $j \in Y_1$ such that $\lambda_j \neq 0$, and we see, for $i \in Y_0$,

$$
\begin{aligned}
& D_{j,j+1}(x_i x_j x_{j+1}) \cdot \prod_{t=i}^{m} D_t^{p-1} \otimes \omega \\
= \quad & (D_i \cdot D_{j,j+1}(x_i x_j x_{j+1}) - D_{j,j+1}(x_j x_{j+1})) D_i^{p-2} \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \\
= \quad & \lambda_j D_i^{p-2} \prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega .
\end{aligned}
$$

$(p-1)$-fold multiplication with $\prod_{t=i}^{m} D_t^{p-1} \otimes \omega$ implies that $\prod_{t=i+1}^{m} D_t^{p-1} \otimes \omega \in M'$. In general case of i, it entails $1 \otimes \omega \in M'$. $M_S(B(\lambda))$ is simple as desired. $\square$

5. The simple module of Lie superalgebra $H(m, n, \underline{1})$

In this section, we consider the Lie superalgebra $H(m, n, \underline{1})$. First, we suppose that $n = 2q$ is an even number.

5.1. Lemma. $A = \langle \{-D_H(x_i x_{i'}), \mu D_H(x_{j^v} x_j) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ is a Cartan subalgebra of H_0, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leqslant j \leqslant m+q \\ j-q, & m+q+1 \leqslant j \leqslant m+2q \end{cases}$.
The positive root vectors of H_0 are $\{D_H(-x_i x_{j'}), 1 \leqslant i < j \leqslant r; D_H(x_i x_j), 1 \leqslant i < j \leqslant r; D_H(x_i^2), 1 \leqslant i \leqslant r; D_H(-\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(-x_i x_j + \mu x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(\frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}), m+1 \leqslant i < j \leqslant m+q; D_H(x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}), m+1 \leqslant i < j \leqslant m+q\}$.

Proof. Let $\varphi : H_0 \to \mathscr{L} := \{\left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \in pl(m,n) \mid A_1^t G + G A_1 = 0, B_1^t G + C_1 = 0, D_1^t + D_1 = 0\}$ be a homomorphism of Lie superalgebras such that $\varphi(D_H(x_i D_j)) = \sigma(j)(-1)^{\tau(j)}(E_{ij'} + \sigma(i)\sigma(j)(-1)^{\tau(i)\tau(j)+\tau(i)+\tau(j)} E_{ji'})$, where $G = \left(\begin{smallmatrix} & I_r \\ -I_r & \end{smallmatrix}\right)$, I_r is the $r \times r$ identity matrix, $pl(m,n) := pl_{\bar{0}}(m,n) \oplus pl_{\bar{1}}(m,n)$, for $pl_{\bar{0}}(m,n) := \{\left(\begin{smallmatrix} A_1 & 0 \\ 0 & D_1 \end{smallmatrix}\right) \mid A_1$ is the $m \times m$ matrix over $\mathbb{F}$, D_1 is the $n \times n$ matrix over $\mathbb{F}\}$, $pl_{\bar{1}}(m,n) := \{\left(\begin{smallmatrix} 0 & B_1 \\ C_1 & 0 \end{smallmatrix}\right) \mid B_1$ is the $m \times n$ matrix over $\mathbb{F}$, C_1 is the $n \times m$ matrix over $\mathbb{F}\}$. It can be checked easily that $H_0 \cong \mathscr{L} \cong \mathscr{L}(P) := \{P^{-1} E P \mid E \in \mathscr{L}\}$, where $P := \left(\begin{smallmatrix} I_m & 0 \\ 0 & P_n \end{smallmatrix}\right)$, $P_n := \left(\begin{smallmatrix} I_q & \frac{1}{2} I_q \\ -\mu I_q & \frac{\mu}{2} I_q \end{smallmatrix}\right)$.
By straightforward computation we find that the Cartan subalgebra of $\mathscr{L}$ is

$$\langle \{\left(\begin{smallmatrix} E_{ii} - E_{i'i'} & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{jj} - E_{j^v, j^v} \end{smallmatrix}\right) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle,$$

and the positive root vectors are
$\{\left(\begin{smallmatrix} E_{ij} - E_{j'i'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i < j \leqslant r$; $\left(\begin{smallmatrix} E_{ij'} + E_{ji'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i < j \leqslant r$;
$\left(\begin{smallmatrix} E_{ii'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r$; $\left(\begin{smallmatrix} 0 & E_{ij} \\ -E_{j^v i'} & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$;
$\left(\begin{smallmatrix} 0 & E_{ij^v} \\ -E_{ji'} & 0 \end{smallmatrix}\right)$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{ij} - E_{j^v i^v} \end{smallmatrix}\right)$, for $m+1 \leqslant i < j \leqslant m+q$;
$\left(\begin{smallmatrix} 0 & 0 \\ 0 & E_{ij^v} - E_{ji^v} \end{smallmatrix}\right)$, for $m+1 \leqslant i < j \leqslant m+q\}$

By the isomorphism φ, we can show that the Cartan subalgebra of H_0 is
$A = \langle \{-D_H(x_i x_{i'}), \mu D_H(x_{j^v} x_j) \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ and the set of positive root vectors of H_0 are $\{D_H(-x_i x_{j'}), 1 \leqslant i < j \leqslant r; D_H(x_i x_j), 1 \leqslant i < j \leqslant r; D_H(x_i^2), 1 \leqslant i \leqslant r; D_H(-\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(-x_i x_j + \mu x_i x_{j^v}), 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; D_H(\frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}), m+1 \leqslant i < j \leqslant m+q; D_H(x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}), m+1 \leqslant i < j \leqslant m+q\}$. $\square$

From the Definition 2.1 and Lemma 5.1, we can get $V_\lambda = \{y \in V \mid D_H(x_i x_{i'}) \cdot y = \lambda_i y, D_H(x_j x_{j^v}) \cdot y = \lambda_j y, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}$, where y is a maximal vector

and satisfies the following statements:
$D_H(x_ix_j)\cdot y=0$, for $1\leqslant i,j\leqslant r$, or $1\leqslant i\leqslant r, i'<j\leqslant 2r$;
$D_H(x_ix_j)\cdot y=0$, for $1\leqslant i\leqslant r, m+1\leqslant j\leqslant s$;
$D_H(x_ix_j+\mu x_jx_{i^v})\cdot y=0$, for $m+1\leqslant i<j\leqslant m+q$;
$D_H(x_ix_j-\mu x_jx_{i^v})\cdot y=0$, for $m+q+1\leqslant i<j\leqslant s$.

5.2. Lemma. Suppose that $M:=<u(H)D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega>$, where $1\leqslant i\leqslant r$. If the following situations hold, respectively,

(1) $\lambda_i\neq 0,\beta_i=p-1$.
(2) $\lambda_i=0,\beta_i\neq p-1$.
(3) $\lambda_i=-1,\beta_i\neq 1$.
(4) There exists $j\in Y_0$, $1\leqslant i<j\leqslant r$, such that $\lambda_j\neq 0$ and $\beta_j=p-1$.
(5) There exists $j\in Y_0$, $1\leqslant j<i\leqslant r$, such that $\lambda_j\neq -1$ and $\beta_j=0$.

Then $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(6) If $\lambda_i\neq -1,\beta_i=p-2$, then $D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

Proof. By virtue of $[D_j,D_H(f)]=D_H(D_j(f))$, for $j\in Y$, we obtain

$$\begin{aligned}
& D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & (D_{i'}\cdot D_H(x^{(2\varepsilon_i)}x_{i'})-D_H(x^{(2\varepsilon_i)}))\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{p-2}D_{i'+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & D_{i'}^{p-1}\cdot D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& +D_{i'}^{p-2}\cdot D_H(x^{(2\varepsilon_i)})\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_{i'}^{p-1}(D_i\cdot D_H(x^{(2\varepsilon_i)}x_{i'})-D_H(x_ix_{i'}))\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& +D_{i'}^{p-2}\cdot(D_i\cdot D_H(x^{(2\varepsilon_i)})-D_H(x_i))\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_{i'}^{p-1}D_i^2\cdot D_H(x^{(2\varepsilon_i)}x_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& -D_{i'}^{p-1}D_i\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& -D_{i'}^{p-1}\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& +D_{i'}^{p-2}D_i^{\beta_i}\cdot D_H(x^{(2\varepsilon_i)})\cdot D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& -\beta_iD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
& D_H(x_ix_{i'})\cdot D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & (D_i\cdot D_H(x_ix_{i'})-D_H(x_{i'}))\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_i\cdot D_H(x_ix_{i'})\cdot D_i^{\beta_i-2}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& +D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
= & D_i^{\beta_i-1}\cdot D_H(x_ix_{i'})\cdot D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega\\
& +(\beta_i-1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots\widehat{D_{i'}^{p-1}}\cdots D_m^{p-1}\otimes\omega. \qquad (5.2)
\end{aligned}$$

By (5.2), we find that (5.1) equals

$$\begin{aligned}
& (-(\lambda_i+\beta_i-1)-(\lambda_i+\beta_i-2)\cdots-(\lambda_i+\beta_i-\beta_i)-\beta_i)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\\
= & -\frac{\beta_i}{2}(2\lambda_i+\beta_i+1)D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega
\end{aligned}$$

With the conditions (1), (2) or (3), respectively, we have $D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$.

(4) In situation (4), we have

$$D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M.$$

Hence,

$$\begin{aligned} & D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & (D_{j'}\cdot D_H(x_{j'}x_i)-D_H(x_i))D_{j'}^{p-2}D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega \\ = & D_{j'}^{p-1}\cdot D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega \\ = & D_{j'}^{p-1}(D_i\cdot D_H(x_{j'}x_i)-D_H(x_{j'}))D_i^{\beta_i-1}\cdots D_{j-1}^{p-1}D_j^{p-2}D_{j+1}^{p-1}\cdots D_{i'}^{p-1}\cdots\widehat{D_{j'}^{p-1}}\cdots D_m^{p-1}\otimes\omega \\ = & \beta_iD_i^{\beta_i-1}\cdots D_j^{p-1}\cdots D_{i'}^{p-1}\cdots D_{j'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned}$$

The assertion follows from the above equation.

(5) By a straightforward calculation, we obtain

$$\begin{aligned} & D_H(x_{j'}x_i)\cdot D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & (D_i\cdot D_H(x_{j'}x_i)-D_H(x_{j'}))D_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'-1}^{p-1}\cdot D_H(x_{j'}x_i)\cdot D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ & +\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'-1}^{p-1}(D_{j'}\cdot D_H(x_{j'}x_i)-D_H(x_i))D_{j'}^{p-2}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ & +\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & D_i^{\beta_i}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\cdot D_H(x_{j'}x_i)\otimes\omega \\ & +\beta_iD_jD_i^{\beta_i-1}D_{i+1}^{p-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M. \end{aligned} \tag{5.3}$$

(5.3) multiplied by $D_H(x^{(2\varepsilon_j)}x_{j'})$, then we have $-(\lambda_j+1)\beta_iD_i^{\beta_i-1}\cdots D_{j'}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. By the situation (5), we have the desired result.

(6) In the particular case, $\beta_i=p-2$, we obtain

$$\begin{aligned} & D_H(x^{((p-1)\varepsilon_i)}x_{i'})\cdot D_i^{p-2}D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega \\ = & -(\lambda_i+1)D_{i+1}^{p-1}\cdots D_{i'}^{p-1}\cdots D_m^{p-1}\otimes\omega. \end{aligned}$$

For $\lambda_i\neq-1$, we obtain the asserted result, i.e., $D_{i+1}^{p-1}\cdots D_m^{p-1}\otimes\omega\in M$. □

5.3. Lemma. Suppose that $M:=<u(H)D_{r+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{\beta_{i'}}\otimes\omega>$, for $1\leqslant i\leqslant r$. If the following situations hold, respectively,

(1) $\lambda_i\neq-1,\beta_{i'}=p-1$.

(2) $\lambda_i=-1,\beta_{i'}\neq p-1$.

(3) $\lambda_i=0,\beta_{i'}\neq1$.

(4) There exists $j'\in Y_0$, $r+1\leqslant j'<i'\leqslant m$ such that $\lambda_j\neq-1$ and $\beta_{j'}=p-1$.

(5) There exists $j'\in Y_0$, $r+1\leqslant i'<j'\leqslant m$ such that $\lambda_j\neq0$ and $\beta_{j'}=0$. Then $D_{r+1}^{p-1}\cdots D_{i'-1}^{p-1}D_{i'}^{\beta_{i'}-1}\otimes\omega\in M$.

(6) If $\lambda_i\neq0,\beta_{i'}=p-2$, then $D_{r+1}^{p-1}\cdots D_{i'+1}^{p-1}\otimes\omega\in M$.

Proof. For $1 \leqslant i \leqslant r$,

$$
\begin{aligned}
& D_H(x_i x^{(2\varepsilon_{i'})}) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\
= \; & (D_{i'} \cdot D_H(x_i x^{(2\varepsilon_{i'})}) - D_H(x_i x_{i'})) D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \\
= \; & \frac{\beta_{i'}}{2}(-2\lambda_i + \beta_{i'} - 1) D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega. \qquad (5.4)
\end{aligned}
$$

From (5.4), with the situations (1), (2) or (3), respectively, we can conclude $D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$.

(4) We see that $D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} D_{j'}^{p-2} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$. Hence, we have

$$
\begin{aligned}
& D_H(x_j x_{i'}) \cdot D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} D_{j'}^{p-2} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\
= \; & D_{j'}^{p-2}(D_{i'} \cdot D_H(x_j x_{i'}) - D_H(x_j)) D_{r+1}^{p-1} \cdots D_{j'-1}^{p-1} \widehat{D_{j'}^{p-2}} D_{j'+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \\
= \; & -\beta_{i'} D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M.
\end{aligned}
$$

(5) For $i < j$, we have

$$
\begin{aligned}
& D_H(x_j x_{i'}) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \otimes \omega \\
(5.5) = \; & D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}} \cdot D_H(x_j x_{i'}) \otimes \omega - \beta_{i'} D_{j'} D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M.
\end{aligned}
$$

(5.5) multiplied by $D_H(x_j x^{(2\varepsilon_{j'})})$, we have $\beta_{i'} \lambda_j D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{\beta_{i'}-1} \otimes \omega \in M$.

(6) In particular, $\beta_i = p-2$, we see that

$$
\begin{aligned}
& D_H(x^{((p-1)\varepsilon_{i'})} x_i) \cdot D_{r+1}^{p-1} \cdots D_{i'-1}^{p-1} D_{i'}^{p-2} \otimes \omega \\
= \; & -\lambda_i D_{r+1}^{p-1} \cdots D_{i'+1}^{p-1} \otimes \omega \in M.
\end{aligned}
$$

□

5.1. Theorem. $M_H(B(\lambda))$ is simple, if $(\lambda_1, \cdots, \lambda_r) \neq \sum_{t=1}^{k-1} -\varepsilon_t$, (the empty sum being zero) for all $1 \leqslant k \leqslant r+1$, where $\varepsilon_t := (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^r$, 1 occurs at the tth place.

Proof. Let M' be a nonzero submodule of $M_H(B(\lambda))$. In the analogous proof for W described above, we can get $\prod_{t_1=1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

First, without loss of generality, we assume $\prod_{t_1=i}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$, for $1 \leqslant i \leqslant r$.

(i) If $\lambda_i \neq 0$ and $\lambda_i \neq -1$. By (1) and (6) in Lemma 5.2, then we conclude that $\prod_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) If $\lambda_i = -1$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod_{k=1}^{i} -\varepsilon_k$, then there exists $j \in Y_0$, such that $1 \leqslant j < i \leqslant r$ and $\lambda_j \neq -1$, or $i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.2, we get $\prod_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) If $\lambda_i = 0$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod_{k=1}^{i-1} -\varepsilon_k$, then there exists $j \in Y_0$, such that $1 \leqslant j \leqslant i-1$, $\lambda_j \neq -1$, or $1 \leqslant i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma

5.2, we get $\prod\limits_{t_1=i+1}^{r} D_{t_1}^{p-1} \prod\limits_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$. Because i is arbitrary, we know $\prod\limits_{t_2=r+1}^{m} D_{t_2}^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

We assume $\prod\limits_{t_2=r+1}^{i'} D_{t_2}^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega$, for $1 \leqslant i \leqslant r$.

(i) If $\lambda_i \neq 0$ and $\lambda_i \neq -1$, by (1) and (6) of Lemma 5.3, we conclude that $\prod\limits_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(ii) If $\lambda_i = -1$, and $(\lambda_1, \cdots, \lambda_r) \neq \prod\limits_{k=1}^{i} -\varepsilon_k$, then there exists j, $1 \leqslant j < i \leqslant r$ such that $\lambda_j \neq -1$, or $i < j \leqslant r$ such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.3, we conclude that $\prod\limits_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$.

(iii) If $\lambda_i = 0$, and $\lambda \neq \prod\limits_{k=1}^{i-1} -\varepsilon_k$, then there exists j, $1 \leqslant j < i \leqslant r$ such that $\lambda_j \neq -1$, or $i < j \leqslant r$, such that $\lambda_j \neq 0$. By (4) or (5) of Lemma 5.3, we conclude that $\prod\limits_{t_2=r+1}^{i'+1} D_{t_2}^{p-1} \prod\limits_{j=m+1}^{s} D_j \otimes \omega \in M'$. Because i is arbitrary, we know $\prod\limits_{l=m+1}^{s} D_l \otimes \omega \in M'$. The condition (1) implies that there exists $i \in Y_0$, such that $\lambda_i \neq 0$. Then, for $j \in Y_1$,

$$\begin{aligned} & D_H(x_i x_{i'} x_j) \cdot D_j \cdots D_s \otimes \omega \\ = \ & (-D_j \cdot D_H(x_i x_{i'} x_j + D_H(x_i x_{i'}))D_{j+1} \cdots D_s \otimes \omega \\ = \ & \lambda_i D_{j+1} \cdots D_s \otimes \omega. \end{aligned}$$

Along with the arbitrary j, it yields $1 \otimes \omega \in M'$. Then $M_H(B(\lambda))$ is simple. □

For the case of $n = 2q+1$, we substitute $P_n := \begin{pmatrix} I_q & \frac{1}{2} I_q \\ -\mu I_q & \frac{\mu}{2} I_q \end{pmatrix}$ in Lemma 5.1 with $P_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_q & \frac{1}{2} I_q \\ 0 & -\mu I_q & \frac{\mu}{2} I_q \end{pmatrix}$. The remaining proof is treated similarly as above.

6. The simple module of Lie superalgebra $K(m, n, \underline{1})$

We first consider the case $n = 2q$.

6.1. Lemma. $A = \langle \{-x_i x_{i'}, x_m, \mu x_{j^v} x_j \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$ is a Cartan subalgebra of K_0, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leq j \leq m+q \\ j-q, & m+q+1 \leq j \leq m+2q \end{cases}$. The positive root vectors of K_0 are $\{-x_i x_{j'}, 1 \leqslant i < j \leqslant r; x_i x_j, 1 \leqslant i < j \leqslant r; x_i^2, 1 \leqslant i \leqslant r; -\frac{1}{2} x_i x_j - \frac{\mu}{2} x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; -x_i x_j + \mu x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; \frac{1}{2} x_j x_i + \frac{\mu}{2} x_{j^v} x_i + \frac{\mu}{2} x_{i^v} x_j + \frac{1}{2} x_{j^v} x_{i^v}, m+1 \leqslant i < j \leqslant m+q; x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}, m+1 \leqslant i < j \leqslant m+q\}$.

Proof. Let $\varphi : K_0 \to \mathscr{L} = \{\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in pl(m-1, n) \mid A_1^t G + G A_1 = 0, B_1^t G + C_1 = 0, D_1^t + D_1 = 0\}$ be a mapping of vector spaces, given by

$$x_i x_j \mapsto \sigma(j)(-1)^{\tau(j)}(E_{ij'} + \sigma(i)\sigma(j)(-1)^{\tau(i)\tau(j)+\tau(i)+\tau(j)} E_{ji'}), (1 \leqslant i < j \leqslant s, i, j \neq m)$$
$$x_m \mapsto 1 \in \mathbb{F},$$

where $G = \begin{pmatrix} & I_r \\ -I_r & \end{pmatrix}$, $pl(m-1, n) := pl_{\bar{0}}(m-1, n) \oplus pl_{\bar{1}}(m-1, n)$, for $pl_{\bar{0}}(m-1, n) := \{\begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \mid A_1$ is the $(m-1) \times (m-1)$ matrix over $\mathbb{F}$, D_1 is the $n \times n$ matrix over $\mathbb{F}\}$, $pl_{\bar{1}}(m-1, n) := \{\begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix} \mid B_1$ is the $(m-1) \times n$ matrix over $\mathbb{F}$, C_1 is the $n \times (m -$

1) matrix over $\mathbb{F}\}$. It is obvious that $K_0 \cong \mathscr{L} \oplus \mathbb{F} \cong \mathscr{L}(P) \oplus \mathbb{F} := \{P^{-1}EP \mid E \in \mathscr{L}\} \oplus \mathbb{F}$, where $P := \begin{pmatrix} I_m & 0 \\ 0 & P_n \end{pmatrix}$, $P_n := \begin{pmatrix} I_q & \frac{1}{2}I_q \\ -\mu I_q & \frac{\mu}{2}I_q \end{pmatrix}$. By a straight computation, we get the Cartan subalgebra of $\mathscr{L}$ as

$$\langle \{ \begin{pmatrix} E_{ii}-E_{i'i'} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & E_{jj}-E_{j^v,j^v} \end{pmatrix} \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle,$$

and the positive root vectors are
$\{ \begin{pmatrix} E_{ij}-E_{j'i'} & 0 \\ 0 & 0 \end{pmatrix}$, for $1 \leqslant i < j \leqslant r$; $\begin{pmatrix} E_{ij'}+E_{ji'} & 0 \\ 0 & 0 \end{pmatrix}$ for $1 \leqslant i < j \leqslant r$; $\begin{pmatrix} E_{ii'} & 0 \\ 0 & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r$; $\begin{pmatrix} 0 & E_{ij} \\ -E_{j^v i'} & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\begin{pmatrix} 0 & E_{ij^v} \\ -E_{ji'} & 0 \end{pmatrix}$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q$; $\begin{pmatrix} 0 & 0 \\ 0 & E_{ij}-E_{j^v i^v} \end{pmatrix}$, for $m+1 \leqslant i < j \leqslant m+q$; $\begin{pmatrix} 0 & 0 \\ 0 & E_{ij^v}-E_{ji^v} \end{pmatrix}$, for $m+1 \leqslant i < j \leqslant m+q\}$

By the isomorphism φ, we can get the Cartan subalgebra of K_0 is $A = \langle \{-x_i x_{i'}, x_m, \mu x_{j^v} x_j \mid 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\} \rangle$, where $\mu^2 = -1$, $j^v = \begin{cases} j+q, & m+1 \leq j \leq m+q \\ j-q, & m+q+1 \leq j \leq m+2q \end{cases}$. Also, the positive root vectors of K_0 are $\{-x_i x_{j'}, 1 \leqslant i < j \leqslant r; x_i x_j, 1 \leqslant i < j \leqslant r); x_i^2, 1 \leqslant i \leqslant r; -\frac{1}{2}x_i x_j - \frac{\mu}{2}x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; -x_i x_j + \mu x_i x_{j^v}, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q; \frac{1}{2}x_j x_i + \frac{\mu}{2}x_{j^v} x_i + \frac{\mu}{2}x_{i^v} x_j + \frac{1}{2}x_{j^v} x_{i^v}, m+1 \leqslant i < j \leqslant m+q; x_j x_i + \mu x_{i^v} x_j + \mu x_i x_{j^v} + x_{i^v} x_{j^v}, m+1 \leqslant i < j \leqslant m+q\}$. □

With respect to the Definition 2.1 and Lemma 6.1, we can get $V_\lambda = \{y \in V \mid (x_i x_{i'}) \cdot y = \lambda_i y, x_m \cdot \omega = \lambda_m y, (x_j x_{j^v}) \cdot y = \lambda_j y, 1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant m+q\}$, where y is a maximal vector and satisfies the following conditions,
$(x_i x_j) \cdot y = 0$, for $1 \leqslant i, j \leqslant r$, or $1 \leqslant i \leqslant r, i' < j \leqslant 2r$;
$(x_i x_j) \cdot y = 0$, for $1 \leqslant i \leqslant r, m+1 \leqslant j \leqslant s$;
$(x_i x_j + \mu x_j x_{i^v}) \cdot y = 0$, for $m+1 \leqslant i < j \leqslant m+q$;
$(x_i x_j - \mu x_j x_{i^v}) \cdot y = 0$, for $m+q+1 \leqslant i < j \leqslant s$.

6.1. Theorem. $M_K(B(\lambda))$ is simple, if $(\lambda_1, \cdots, \lambda_r, \lambda_m) \neq \zeta_k + (\pm k - r - 1)\varepsilon_m$, for $1 \leqslant k \leqslant r+1$, where $\zeta_k = -\sum_{i=1}^{r-k+1} \varepsilon_i$ (the empty sum being zero), $\varepsilon_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^{r+1}$, 1 occurs at the ith place, for $1 \leqslant i \leqslant r+1$.

Proof. Let M' be a nonzero submodule of $M_K(B(\lambda))$. Take $a \in M'$ and $a \neq 0$. We note

$$a = \Sigma_{\beta \in \mathscr{A}} c(\beta) i(x_1)^{\beta_1} \cdots \widehat{i(x_m)^{\beta_m}} \cdots i(x_s)^{\beta_s} x_0^{\beta_0} \otimes b_\beta,$$

where $\beta = (\beta_1, \cdots, \widehat{\beta_m}, \cdots, \beta_s, \beta_0)$, $c(\beta) \in \mathbb{F}$, $\mathscr{A} := \{a = \Sigma_k \beta_k \varepsilon_k \mid 0 \leqslant \beta_k \leqslant p-1$ for $1 \leqslant k \leqslant m-1$; $\beta_k = 0$ or 1 for $m+1 \leqslant k \leqslant s\} \subset \mathbb{Z}^{s-1}$. Write $i(x_j) = x_j, i(1) = x_0$ in $u(K)$.

Put $\alpha_0 = \min\{\beta_0 \mid a = \Sigma_{\beta \in \mathscr{A}} c(\beta) x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{\beta_0} \otimes b_\beta, c(\beta) \neq 0\}$. Since $[1, x^\alpha] = x^{\alpha - \varepsilon_m}$, $x_0 x_\alpha = x_\alpha x_0 + x^{\alpha - \varepsilon_m}$ holds in $u(K)$. We can get
$x_0^{p-1-\alpha_0} \cdot v = \Sigma_{\beta' \in \mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'} \in M'$, where $\beta' = (\beta_1, \cdots, \widehat{\beta_m}, \cdots, \beta_s, \alpha_0)$.
Put $\alpha_1 := \min\{\beta_1 \mid a = \Sigma_{\beta' \in \mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'}, c(\beta') \neq 0\}$. Multiplying $\Sigma_{\beta' \in \mathscr{A}} c(\beta') x_1^{\beta_1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta'}$ by $x_1^{p-1-\alpha_1}$, we can obtain
$\Sigma_{\beta'' \in \mathscr{A}} c(\beta'') x_1^{p-1} \cdots \widehat{x_m^{\beta_m}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta''} \in M'$, where $\beta'' = (\alpha_1, \beta_2, \cdots, \widehat{\beta_m}, \cdots \beta_s, \alpha_0)$.
Eventually, we can conclude that

$$\Sigma_{\beta^{(m)} \in \mathscr{A}} c(\beta^{(m)}) x_1^{p-1} \cdots x_{m-1}^{p-1} \widehat{x_m^{\beta_m}} x_{m+1}^{\beta_{m+1}} \cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta^{(m)}} \in M', \tag{6.1}$$

where $\beta^{(m)} = (\alpha_1, \alpha_2, \cdots, \alpha_{m-1}, \widehat{\beta_m}, \beta_{m+1}, \cdots, \beta_s, \alpha_0)$.

Put α_{m+1} :=min$\{\beta_{m+1} \mid a = \Sigma_{\beta^{(m)}\in\mathscr{A}} c(\beta^{(m)}) x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{\beta_m}} x_{m+1}^{\beta_{m+1}}\cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta^{(m)}},\ c(\beta^{(m)}) \neq 0\}$. Multiplying (6.1) by $x_{m+1}^{1-\alpha_{m+1}}$, we can obtain $\Sigma_{\beta^{(m+1)}\in\mathscr{A}} c(\beta^{(m+1)}) x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{\beta_m}} x_{m+1}\cdots x_s^{\beta_s} x_0^{p-1} \otimes b_{\beta^{(m)}} \in M'$. Finally, we can conclude that there exists η such that $c(\eta)x_1^{p-1}\cdots x_{m-1}^{p-1}\widehat{x_m^{p-1}} x_{m+1}\cdots x_s x_0^{p-1} \otimes b_\eta \in M'$, where $\eta = (\alpha_1, \alpha_2, \cdots, \widehat{\alpha_m}, \alpha_{m+1}, \cdots, \alpha_s, \alpha_0)$, $c(\eta) \neq 0$. Similarly to the discussion for W, there exists a maximal vector ω such that $\prod_{t=0}^{m-1} x_t^{p-1} \prod_{l=m+1}^{s} x_l \otimes \omega \in M'$. By Lemma 2.12 and Lemma 2.13 of [9], we can get $\prod_{l=m+1}^{s} x_l \otimes \omega \in M'$. By our assumption, we know that there exists i, for $1 \leqslant i \leqslant r$ or $i = m$, such that $\lambda_i \neq 0$.

We assume $\prod_{l=j}^{s} x_l \otimes \omega \in M'$, for $j \in Y_1$.

If $\lambda_i \neq 0$, for $1 \leqslant i \leqslant r$, we have

$$\begin{aligned} & (x_i x_{i'} x_j) \cdot \prod_{l=j}^{s} x_l \otimes \omega \\ = \quad & (-x_j \cdot (x_i x_{i'} x_j) - x_i x_{i'}) \prod_{l=j+1}^{s} x_l \otimes \omega \\ = \quad & -\lambda_i \prod_{l=j+1}^{s} x_l \otimes \omega. \end{aligned}$$

With the generality of $j \in Y_1$, we get $1 \otimes \omega \in M'$.

If $\lambda_m \neq 0$, then

$$\begin{aligned} & (x_{m+1}\cdots x_s x_m) \cdot \prod_{l=m+1}^{s} x_l \otimes \omega \\ = \quad & (x_{m+1} \cdot (x_{m+1}\cdots x_s x_m) + (x_{m+2}\cdots x_s x_m)) \prod_{l=m+2}^{s} x_l \otimes \omega \\ = \quad & (x_s x_m) \cdot x_s \otimes \omega \\ = \quad & -x_m \otimes \omega \\ = \quad & -\lambda_m \otimes \omega. \end{aligned}$$

We also can conclude that $1 \otimes \omega \in M'$. In other words, $M_K(B(\lambda))$ is simple. $\square$

For the case of $n = 2q+1$, we substitute $P_n := \begin{pmatrix} I_q & \frac{1}{2}I_q \\ -\mu I_q & \frac{\mu}{2}I_q \end{pmatrix}$ in Lemma 6.1 with $P_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_q & \frac{1}{2}I_q \\ 0 & -\mu I_q & \frac{\mu}{2}I_q \end{pmatrix}$. The remaining proof is treated similarly as above.

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