Analysis of ruin measures for two classes of risk processes with stochastic income

Wuyuan Jiang[* † ‡] and Chaoqun Ma[§]

Abstract

In this paper, we consider the ruin measures for two classes of risk processes. We assume that the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. Historically, it has been assumed that the premium size is a constant. In this contribution, the premium income arrival process is a Poisson process. In this framework, both the integro-differential equation and the Laplace transform for the expected discounted penalty function are established. Explicit expressions for the expected discounted penalty function are derived when the claim amount distributions belong to the rational family. Finally, Numerical examples are considered.

1. Introduction

In the actuarial literature, many researchers studied the ruin measures for a risk model involving two independent classes of risks. Among them, [9] considered the expected discounted penalty functions for two classes of risk processes by assuming that the two claim number processes are independent Poisson and generalized Erlang(2) processes, respectively. A system of integro-differential equations for the expected discounted penalty functions were derived and explicit results when the claim sizes are exponentially distributed were obtained. [13] extended the model of [9], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival

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*School of Business Administration, Hunan University, Changsha 410082, China,

†Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China Email: `csujw@163.com`

‡Corresponding Author.

§School of Business Administration, Hunan University, Changsha 410082, China, Email: `cqma1998@hnu.edu.cn`

times. The authors derived an integro-differential equation system for the expected discounted penalty functions, and obtained their Laplace transforms when the corresponding Lundberg equation has distinct roots. [5] investigated the risk model with two classes of renewal risk processes by assuming that both of the two claim number processes have phase-type inter-claim times. A system of integro-differential equations for the expected discounted penalty function was derived and solved. For more related references on two classes of risk processes problem, the reader may consult the following publications and references therein, [12], [8], [3], etc.

Under the above risk models, premiums are assumed to be received by insurance companies at a constant rate over time. In fact, the insurance company may have lump sums of income. For example, insurances of traveling art collections or ship and plane insurances might be expected to have a significant impact on the premium income. [2] first considered the risk model with stochastic premium income by adding a compound Poisson process with positive jumps to the classical risk model. Subsequently, [1] and [10] studied the ruin probabilities for the risk models with stochastic premiums. Recently, [6] considered a risk model with stochastic premium income, where both premiums and claims follow compound Poisson processes. Both a defective renewal equation and an integral equation satisfied by the expected discounted penalty function are established. [14] extended the model in [6] by assuming that there exists a dependence structure among the claim sizes, inter-claim times and premium sizes. [11] studied a risk model with a dependence setting where there exists a specific structure among the time between two claim occurrences, premium sizes and claim sizes. Given that the premium size is exponentially distributed, both the Laplace transforms and defective renewal equations for the expected discounted penalty functions are obtained.

To the best of our knowledge, there is less work in the literature on two classes of risk models with stochastic premiums. Henceforth, the purpose of this paper is to investigate the expected discounted penalty functions in a risk model involving two independent classes of risks and the premium income arrival process is a Poisson process, in which the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. The structure of the paper is as follows. Section 2 describes two classes of risk processes with stochastic income. In Section 3, we derive the system of integro-differential equations for the expected discounted penalty functions. Then Section 4 presents the Laplace solutions of the expected discounted penalty functions and provides closed forms for rational family claim-size distribution. Numerical examples are considered in Section 5. Last, Section 6 concludes.

2. Model and assumptions

The surplus process $R(t)$ is given by

$$(2.1) \qquad R(t) = u + \sum_{i=1}^{M(t)} X_i - S(t), \quad t \geq 0,$$

where $u \geq 0$ is the initial surplus, $M(t)$ denotes the number of insurer's premium income up to time t and follows a Poisson process with intensity $\mu > 0$. $\{X_1, X_2, \cdots\}$ are independent and identically distributed (i.i.d.) positive random variables (r.v.'s) representing the individual premium amounts with common distribution P, probability density function (p.d.f.) p and Laplace transform (LT) $\tilde{p}(s) = \int_0^\infty e^{-sx} p(x) dx$. The aggregate-claim process $\{S(t) : t \geq 0\}$ is defined by

$$S(t) = \sum_{i=1}^{N_1(t)} Y_i + \sum_{i=1}^{N_2(t)} Z_i, \quad t \geq 0,$$

where $\{Y_1, Y_2, \cdots\}$ are i.i.d. positive r.v.'s representing the successive individual claim amounts from the first class. These r.v.'s are assumed to have common cumulative distribution function $F(x), x \geq 0$, with p.d.f. $f(x) = F'(x)$, of which the LT is $\tilde{f}(s) = \int_0^\infty e^{-sx} f(x) dx$, while $\{Z_1, Z_2, \cdots\}$ are i.i.d. positive r.v.'s representing the claim amounts from the second class with common cumulative distribution function $G(x), x \geq 0$ and p.d.f. $g(x) = G'(x)$, of which the LT is $\tilde{g}(s) = \int_0^\infty e^{-sx} g(x) dx$.

The counting process $\{N_1(t); t \geq 0\}$ is assumed to be a Poisson process with parameter λ, representing the number of claims from the first class up to time t. While the counting process $\{N_2(t); t \geq 0\}$, representing the number of claims from the second class up to time t, is defined as follows. $N_2(t) = \sup\{n : W_1 + W_2 + \cdots + W_n \leq t\}$, where $\{W_1, W_2, \cdots\}$ are the i.i.d. positive r.v.'s representing the second class inter-claim times. In this paper, we suppose that $W_i's$ are generalized Erlang(n) distributed with n possibly different parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then W_i can be expressed as $W_i = W_{i1} + W_{i2} + \cdots + W_{in}$, where W_{ij} is exponentially distributed with parameter $\frac{1}{\lambda_i}$.

In addition, we assume that $\{X_1, X_2, \cdots\}$, $\{Y_1, Y_2, \cdots\}$, $\{Z_1, Z_2, \cdots\}$, $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are mutually independent, and $\mu E(X_1) > \lambda E(Y_1) + \frac{E(Z_1)}{\sum_{i=1}^n \frac{1}{\lambda_i}}$, providing a positive safety loading factor.

The time of (ultimate) ruin is $T = \inf\{t | R(t) < 0\}$, where $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The probability of ruin is $\psi(u) = Pr(T < \infty)$.

For $x_1, x_2 \geq 0, k = 1, 2$, let $w_k(x_1, x_2)$ be two possibly distinct non-negative value functions. For $\delta \geq 0$, the expected discounted penalty function at ruin if the ruin is caused by a claim from class k is defined by

$$m_k(u) = E[e^{-\delta T} w_k(R(T-), |R(T)|) I(T < \infty, J = k) | R(0) = u], \quad u \geq 0,$$

where J is defined to be the cause-of-ruin random variable, and $J = k$ if the ruin is caused by a claim of class $k, k = 1, 2$. $R(T-)$ is the surplus immediately before ruin, $|R(T)|$ is the deficit at ruin, $I(\cdot)$ is an indicator function.

When $\delta = 0$ and $w_k(R(T-), |R(T)|) = 1$, let

$$\psi_k(u) = E[I(T < \infty, J = k) | R(0) = u], \quad u \geq 0, k = 1, 2,$$

is the ruin probability due to a claim from class k. The probability of ruin $\psi(u)$ can be decomposed as $\psi(u) = \psi_1(u) + \psi_2(u)$.

3. System of integro-differential equations

In this section, we derive the integro-differential equations for the expected discounted penalty function. Since every inter-claim time with generalized Erlang(n) distribution can be decomposed into the independent sum of n exponential r.v.'s with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, each causing a sub-claim of size 0 and at the time of the nth sub-claim an actual claim with distribution function G occurs. This can be realized by considering n states of the risk process (2.1) for the second class claim. Starting at time 0 in state 1, every sub-claim causes a transition to the next state and at the time of the occurrence of the nth sub-claim, an actual claim with distribution function G occurs and the risk process jumps into state 1 again. We define the corresponding expected discounted penalty function by $m_{kj}, j = 1, 2, \ldots, n$, when ruin is caused by a claim from class $k, k = 1, 2$ and the risk process is in state j. Obviously, $m_{k1}(u) = m_k(u)$.

Considering an infinitesimal time interval $(0, dt)$, there are five possible events regarding to the occurrence of the premium and claim and change of the state: (1) no premium and claim arrival and no change of state; (2) a premium arrival but no claim arrival and no change of state; (3) a claim arrival but no premium arrival and no change of state; (4) a change of state but no claim and premium arrival; (5) two or more events occur.

By conditioning on the above five events in $(0, dt)$ when $j = 1, 2, \ldots, n - 1$, we have

$$
\begin{aligned}
m_{1j}(u) &= (1 - \mu dt)(1 - \lambda dt)(1 - \lambda_j dt)e^{-\delta dt}m_{1j}(u) \\
&+ \mu dt(1 - \lambda dt)(1 - \lambda_j dt)e^{-\delta dt}\int_0^\infty m_{1j}(u + x)p(x)dx \\
&+ (1 - \mu dt)\lambda dt(1 - \lambda_j dt)e^{-\delta dt}\times \\
&\quad \left[\int_0^u m_{1j}(u - x)f(x)dx + \int_u^\infty w_1(u, x - u)f(x)dx\right] \\
&+ (1 - \mu dt)(1 - \lambda dt)\lambda_j dt e^{-\delta dt}m_{1,j+1}(u) + o(dt).
\end{aligned}
$$
(3.1)

From (3.1) it follows that

$$
\begin{aligned}
m_{1j}(u) &= \frac{\mu}{\lambda_j^* + \delta}\int_0^\infty m_{1j}(u + x)p(x)dx \\
&+ \frac{\lambda}{\lambda_j^* + \delta}\left[\int_0^u m_{1j}(u - x)f(x)dx + \zeta_1(u)\right] \\
&+ \frac{\lambda_j}{\lambda_j^* + \delta}m_{1,j+1}(u),
\end{aligned}
$$
(3.2)

where $\lambda_j^* = \mu + \lambda + \lambda_j$, $\zeta_1(u) = \int_u^\infty w_1(u, x - u)f(x)dx$.

When $j = n$, we obtain

$$
\begin{aligned}
m_{1n}(u) &= (1 - \mu dt)(1 - \lambda dt)(1 - \lambda_n dt)e^{-\delta dt}m_{1n}(u) \\
&+ \mu dt(1 - \lambda dt)(1 - \lambda_n dt)e^{-\delta dt}\int_0^\infty m_{1n}(u + x)p(x)dx \\
&+ (1 - \mu dt)\lambda dt(1 - \lambda_n dt)e^{-\delta dt}\times \\
&\quad \left[\int_0^u m_{1n}(u - x)f(x)dx + \int_u^\infty w_1(u, x - u)f(x)dx\right] \\
&+ (1 - \mu dt)(1 - \lambda dt)\lambda_n dt e^{-\delta dt}\int_0^u m_1(u - x)g(x)dx + o(dt).
\end{aligned}
$$
(3.3)

Which results in

$$
\begin{aligned}
m_{1n}(u) &= \frac{\mu}{\lambda_n^* + \delta}\int_0^\infty m_{1n}(u + x)p(x)dx \\
&+ \frac{\lambda}{\lambda_n^* + \delta}\left[\int_0^u m_{1n}(u - x)f(x)dx + \zeta_1(u)\right] \\
&+ \frac{\lambda_n}{\lambda_n^* + \delta}\int_0^u m_1(u - x)g(x)dx,
\end{aligned}
$$
(3.4)

where $\lambda_n^* = \mu + \lambda + \lambda_n$.

By similar arguments, we get

$$
\begin{aligned}
m_{2j}(u) &= \frac{\mu}{\lambda_j^* + \delta}\int_0^\infty m_{2j}(u + x)p(x)dx \\
&+ \frac{\lambda}{\lambda_j^* + \delta}\int_0^u m_{2j}(u - x)f(x)dx \\
&+ \frac{\lambda_j}{\lambda_j^* + \delta}m_{2,j+1}(u), \quad j = 1, 2, \ldots, n - 1.
\end{aligned}
$$
(3.5)

and

$$
\begin{aligned}
m_{2n}(u) &= \frac{\mu}{\lambda_n^* + \delta}\int_0^\infty m_{2n}(u + x)p(x)dx \\
&+ \frac{\lambda}{\lambda_n^* + \delta}\int_0^u m_{2n}(u - x)f(x)dx \\
&+ \frac{\lambda_n}{\lambda_n^* + \delta}\left[\int_0^u m_2(u - x)g(x)dx + \zeta_2(u)\right],
\end{aligned}
$$
(3.6)

where $\zeta_2(u) = \int_u^\infty w_2(u, x - u)g(x)dx$.

4. Analysis of the integro-differential equations with exponential premiums

In this section, we assume that the premium sizes are exponentially distributed with p.d.f. $p(x) = \beta e^{-\beta x}, \beta > 0, x \geq 0$. Throughout this paper, we will use a hat $\sim$ to designate the Laplace transform of a function f, namely, $\tilde{f}(s) = \int_0^\infty e^{-sx}f(x)dx$. Now, we introduce a complex operator T_r of an integrable real-valued function f which will be necessary in order to obtain the main results. T_r is defined as

$$
T_r f(x) = \int_x^\infty e^{-r(u-x)}f(u)du, \quad r \in \mathbb{C}, x \geq 0,
$$

where r has a non-negative real part, $\Re(r) \geq 0$. [7] provide a list of properties of the operator T_r and we recall two of them that will be used in the following:

(1) $T_r f(0) = \int_0^\infty e^{-ru} f(u)du = \tilde{f}(r), r \in \mathbb{C}$, is the Laplace transform of f.

(2) $T_r T_s f(x) = T_s T_r f(x) = \frac{T_s f(x) - T_r f(x)}{r-s}, s \neq r \in \mathbb{C}, x \geq 0$.

4.1. Laplace transform. In the following, for notational convenience, Let $H_{kj}(u) = \int_0^\infty m_{kj}(u+x)p(x)dx, k = 1, 2, j = 1, 2, \ldots, n$. Taking Laplace transforms on both sides of (3.2) and (3.4) yields

$$(4.1) \quad \tilde{m}_{1j}(s) = \frac{\mu}{\lambda_j^* + \delta}\tilde{H}_{1j}(s) + \frac{\lambda}{\lambda_j^* + \delta}\tilde{m}_{1j}(s)\tilde{f}(s) + \frac{\lambda}{\lambda_j^* + \delta}\tilde{\zeta}_1(s) + \frac{\lambda_j}{\lambda_j^* + \delta}\tilde{m}_{1,j+1}(s),$$

and

$$(4.2) \quad \tilde{m}_{1n}(s) = \frac{\mu}{\lambda_n^* + \delta}\tilde{H}_{1n}(s) + \frac{\lambda}{\lambda_n^* + \delta}\tilde{m}_{1n}(s)\tilde{f}(s) + \frac{\lambda}{\lambda_n^* + \delta}\tilde{\zeta}_1(s) + \frac{\lambda_n}{\lambda_n^* + \delta}\tilde{m}_1(s)\tilde{g}(s).$$

Since, for $j = 1, 2, \ldots, n, s \neq \beta$,

$$(4.3) \quad \begin{aligned} \tilde{H}_{1j}(s) &= \int_0^\infty e^{-su}\int_0^\infty m_{1j}(u+x)\beta e^{-\beta x}dxdu \\ &= \int_0^\infty \{\int_0^\infty e^{-su}m_{1j}(u+x)du\}\beta e^{-\beta x}dx \\ &= \int_0^\infty T_s m_{1j}(x)\beta e^{-\beta x}dx = \beta T_\beta T_s m_{1j}(0) \\ &= \beta\frac{\tilde{m}_{1j}(s) - \tilde{m}_{1j}(\beta)}{\beta - s}. \end{aligned}$$

Substituting (4.3) into (4.1) and (4.2), respectively, we have

$$(4.4) \quad \left[\frac{\mu\beta}{\beta-s} - \lambda_j^* - \delta + \lambda\tilde{f}(s)\right]\tilde{m}_{1j}(s) + \lambda_j\tilde{m}_{1,j+1}(s) = \frac{\mu\beta}{\beta-s}\tilde{m}_{1j}(\beta) - \lambda\tilde{\zeta}_1(s),$$

and

$$(4.5) \quad \left[\frac{\mu\beta}{\beta-s} - \lambda_n^* - \delta + \lambda\tilde{f}(s)\right]\tilde{m}_{1n}(s) + \lambda_n\tilde{g}(s)\tilde{m}_1(s) = \frac{\mu\beta}{\beta-s}\tilde{m}_{1n}(\beta) - \lambda\tilde{\zeta}_1(s).$$

Let $\tilde{\mathbf{m}}_k(s) = (\tilde{m}_{k1}(s), \tilde{m}_{k2}(s), \ldots, \tilde{m}_{kn}(s))^\top$, $\tilde{\mathbf{m}}_k(\beta) = (\tilde{m}_{k1}(\beta), \tilde{m}_{k2}(\beta), \ldots, \tilde{m}_{kn}(\beta))^\top$, $k = 1, 2$, $\mathbf{m}^\top$ denotes the transpose of $\mathbf{m}$, and

$$\mathbf{A}_\delta(s) = \begin{pmatrix} \frac{\mu\beta}{\beta-s} - \lambda_1^* - \delta + \lambda\tilde{f}(s) & \lambda_1 & 0 & \cdots & 0 \\ 0 & \frac{\mu\beta}{\beta-s} - \lambda_2^* - \delta + \lambda\tilde{f}(s) & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \\ \lambda_n\tilde{g}(s) & 0 & 0 & \cdots & \frac{\mu\beta}{\beta-s} - \lambda_n^* - \delta + \lambda\tilde{f}(s) \end{pmatrix}.$$

Then (4.4) and (4.5) can be rewritten as the following matrix form

$$(4.6) \quad \mathbf{A}_\delta(s)\tilde{\mathbf{m}}_1(s) = \frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1,$$

where $\mathbf{e}_1$ denotes a column vector of length n with all elements being one.

Similarly, from (3.5) and (3.6) we can obtain the following matrix form for $\tilde{\mathbf{m}}_2(s)$

$$(4.7) \quad \mathbf{A}_\delta(s)\tilde{\mathbf{m}}_2(s) = \frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta) - \lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2,$$

where $\mathbf{e}_2 = (0, 0, \ldots, 0, 1)^\top$ denotes a $n \times 1$ column vector.

When $det[\mathbf{A}_\delta(s)] \neq 0$, solving the linear systems (4.6) and (4.7), we obtain

$$(4.8) \quad \tilde{\mathbf{m}}_1(s) = \frac{\mathbf{A}_\delta^\star(s)\left[\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]}{det[\mathbf{A}_\delta(s)]},$$

and

(4.9) $\quad \tilde{\mathbf{m}}_2(s) = \dfrac{\mathbf{A}_\delta^\star(s) \left[\frac{\mu\beta}{\beta-s} \tilde{\mathbf{m}}_2(\beta) - \lambda_n \tilde{\zeta}_2(s) \mathbf{e}_2 \right]}{det[\mathbf{A}_\delta(s)]},$

where $\mathbf{A}_\delta^\star(s)$ is the adjoint matrix of $\mathbf{A}_\delta(s)$.

4.1. Theorem. *For $\delta > 0$, the generalized Lundberg's fundamental equation $det[\mathbf{A}_\delta(s)] = 0$ has exactly n roots, say $\rho_1, \rho_2, \ldots, \rho_n$ with $\Re(\rho_i) > 0$.*

Proof. $det[\mathbf{A}_\delta(s)] = 0$ can be rewritten as

$$\frac{1}{(\beta-s)^n} \left\{ \prod_{i=1}^{n} \{\mu\beta - [\lambda_i^* + \delta - \lambda\tilde{f}(s)](\beta - s)\} - (\prod_{i=1}^{n} \lambda_i)\tilde{g}(s)(\beta - s)^n \right\} = 0.$$

Thus, it is only needed to prove

(4.10) $\quad \displaystyle\prod_{i=1}^{n} \{\mu\beta - [\lambda_i^* + \delta - \lambda\tilde{f}(s)](\beta - s)\} - (\prod_{i=1}^{n} \lambda_i)\tilde{g}(s)(\beta - s)^n = 0$

has exactly n roots in the right half complex plane. Let $z = (\beta - s)/\beta$, then (4.10) may be expressed as

(4.11) $\quad \displaystyle\prod_{i=1}^{n} \{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z\} - (\prod_{i=1}^{n} \lambda_i)\tilde{g}(\beta(1 - z))z^n = 0.$

When $\delta > 0$, choose $r \in (0, 1)$ such that $(\mu + \delta)r > \mu$, and denote $C_z = \{z \in \mathbb{C} | |z| = r\}$. Obviously, $\prod_{i=1}^{n} \{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z\}$ and $(\prod_{i=1}^{n} \lambda_i)\tilde{g}(\beta(1 - z))z^n$ are analytic on and inside the contour C_z.

We first prove that each of equations $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z = 0$, $i = 1, \cdots, n$ has exactly one root in the interior of C_z. For any $z \in C_z$, we have

$$|(\lambda_i^* + \delta)z - \mu| \geq |(\lambda_i^* + \delta)z| - \mu > (\lambda_i + \lambda)|z| > \lambda|z| \geq |\lambda\tilde{f}(\beta(1 - z))]z|.$$

By virtue of Rouché's theorem, $(\lambda_i^* + \delta)z - \mu = 0$ and $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z = 0$ have the same number of roots inside C_z. Thus $\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z = 0$ has exactly one root inside C_z. It implies that

(4.12) $\quad \displaystyle\prod_{i=1}^{n} \{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z\} = 0$

has exactly n roots inside C_z.

Furthermore, for any $z \in C_z$,

$$\begin{aligned}
& | \prod_{i=1}^{n} \{\mu - [\lambda_i^* + \delta - \lambda\tilde{f}(\beta(1 - z))]z\}| \\
= {}& \prod_{i=1}^{n} |(\lambda_i^* + \delta)z - \mu - \lambda\tilde{f}(\beta(1 - z))z| \\
\geq {}& \prod_{i=1}^{n} \{|(\lambda_i^* + \delta)z - \mu| - |\lambda\tilde{f}(\beta(1 - z))z|\} \\
\geq {}& \prod_{i=1}^{n} \{|(\lambda_i^* + \delta)z - \mu| - |\lambda z|\} \\
= {}& \prod_{i=1}^{n} \{|(\lambda_i + \lambda)z + (\mu + \delta)z - \mu| - |\lambda z|\} \\
> {}& \prod_{i=1}^{n} |\lambda_i z| \geq |(\prod_{i=1}^{n} \lambda_i)\tilde{g}(\beta(1 - z))z^n|.
\end{aligned}$$

In the last second step, we use $z \in C_z = \{z \in \mathbb{C} | |z| = r\}$ and $r \in (\mu/(\mu + \delta), 1)$.

By Rouché's theorem, both Eq. (4.12) and Eq. (4.11) have the same number of roots inside C_z. Then, we conclude that the equation Eq. (4.11) has exactly n roots inside C_z. That is to say, Lundberg's equation $det[\mathbf{A}_\delta(s)] = 0$ has exactly n roots in $C_s = \{s \in \mathbb{C} | |\beta - s| = r\beta\}$. From $r \in (\mu/(\mu + \mathbb{C}), 1)$, the interior of C_s is entirely contained in the right half complex plane. This completes the proof. $\qquad\square$

4.2. Remark. If $\delta \to 0+$ then $\rho_i(\delta) \to \rho_i(0)$ for $i = 1, \cdots, n$, and we have that $s = 0$ is one of the roots from Lundberg's equation $det[\mathbf{A}_\delta(s)] = 0$.

In what follows, we assume that $\rho_1, \rho_2, \ldots, \rho_n$ are distinct.

Divided difference plays an important role in the present paper. Now we recall divided differences of a matrix $\mathbf{L}(s)$ with respect to distinct numbers $r_1, r_2, \cdots$, which are defined recursively as follows:

$$\mathbf{L}[r_1, s] = \frac{\mathbf{L}(s) - \mathbf{L}(r_1)}{s - r_1}, \quad \mathbf{L}[r_1, r_2, s] = \frac{\mathbf{L}[r_1, s] - \mathbf{L}[r_1, r_2]}{s - r_2},$$

and so on.

4.3. Theorem. $\tilde{\mathbf{m}}_1(\beta)$ and $\tilde{\mathbf{m}}_2(\beta)$ are given by

(4.13)

$$\tilde{\mathbf{m}}_1(\beta) = \frac{\lambda}{\mu\beta} \left(\sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod\limits_{l=i}^{n}(\beta - \rho_l)} \right)^{-1} \left(\sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]\tilde{\zeta}_1[\rho_i, \cdots, \rho_n] \right) \mathbf{e}_1,$$

(4.14)

$$\tilde{\mathbf{m}}_2(\beta) = \frac{\lambda_n}{\mu\beta} \left(\sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod\limits_{l=i}^{n}(\beta - \rho_l)} \right)^{-1} \left(\sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]\tilde{\zeta}_2[\rho_i, \cdots, \rho_n] \right) \mathbf{e}_2.$$

Proof. Since $\tilde{m}_{kj}(s)$ is finite for $k = 1, 2, j = 1, 2, \ldots, n$, from (4.8), we have, for distinct numbers $\rho_1, \rho_2, \ldots, \rho_n$,

$$\mathbf{A}_\delta^\star(\rho_i)\frac{\mu\beta}{\beta - \rho_i}\tilde{\mathbf{m}}_1(\beta) = \mathbf{A}_\delta^\star(\rho_i)\tilde{\zeta}_1(\rho_i)\lambda\mathbf{e}_1.$$

Hence

$$\left[\mathbf{A}_\delta^\star(\rho_1)\frac{\mu\beta}{\beta - \rho_1} - \mathbf{A}_\delta^\star(\rho_2)\frac{\mu\beta}{\beta - \rho_2}\right]\tilde{\mathbf{m}}_1(\beta) = [\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1(\rho_1) - \mathbf{A}_\delta^\star(\rho_2)\tilde{\zeta}_1(\rho_2)]\lambda\mathbf{e}_1.$$

Namely

$$\left[\mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_1} - \mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_2} + \mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{\beta-\rho_2} - \mathbf{A}_\delta^\star(\rho_2)\tfrac{1}{\beta-\rho_2}\right]\mu\beta\tilde{\mathbf{m}}_1(\beta)$$
$$= [\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1(\rho_1) - \mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1(\rho_2) + \mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1(\rho_2) - \mathbf{A}_\delta^\star(\rho_2)\tilde{\zeta}_1(\rho_2)]\lambda\mathbf{e}_1.$$

Using the divided difference, we derive

$$\left[\mathbf{A}_\delta^\star(\rho_1)\tfrac{1}{(\beta-\rho_1)(\beta-\rho_2)} + \mathbf{A}_\delta^\star[\rho_1, \rho_2]\tfrac{1}{\beta-\rho_2}\right]\mu\beta\tilde{\mathbf{m}}_1(\beta)$$
$$= \{\mathbf{A}_\delta^\star(\rho_1)\tilde{\zeta}_1[\rho_1, \rho_2] + \mathbf{A}_\delta^\star[\rho_1, \rho_2]\tilde{\zeta}_1(\rho_2)\}\lambda\mathbf{e}_1.$$

We finally have by recursively deriving

(4.15)

$$\left(\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\frac{1}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\right)\mu\beta\tilde{\mathbf{m}}_1(\beta)=\lambda\left(\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n]\right)\mathbf{e}_1,$$

which leads to (4.13).

Similarly, we can obtain (4.14) from (4.9). $\qquad\square$

Applying the divided difference repeatedly to the numerators of (4.8) and (4.9), respectively, we obtain the following theorem.

4.4. Theorem. *The Laplace transforms of the expected discounted penalty function are given by*

$$
\begin{aligned}
\tilde{\mathbf{m}}_1(s) \;=\; & \frac{\prod\limits_{i=1}^{n}(s-\rho_i)}{det[\mathbf{A}_\delta(s)]}\left\{\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_n,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]\right.\\
& +\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\frac{1}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)\\
& \left.-\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1)\right\},
\end{aligned}
$$
(4.16)

and

$$
\begin{aligned}
\tilde{\mathbf{m}}_2(s) \;=\; & \frac{\prod\limits_{i=1}^{n}(s-\rho_i)}{det[\mathbf{A}_\delta(s)]}\left\{\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_n,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta)-\lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2\right]\right.\\
& +\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\frac{1}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_2(\beta)\right)\\
& \left.-\sum_{i=1}^{n}\mathbf{A}_{\delta}^{\star}[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2)\right\}.
\end{aligned}
$$
(4.17)

Proof. By the fact that $s=\rho_1$ is a root of the numerator in (4.8), we have

$$
\begin{aligned}
& \mathbf{A}_{\delta}^{\star}(s)\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]\\
=\; & \mathbf{A}_{\delta}^{\star}(s)\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]-\mathbf{A}_{\delta}^{\star}(\rho_1)\left[\tfrac{\mu\beta}{\beta-\rho_1}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(\rho_1)\mathbf{e}_1\right]\\
=\; & (s-\rho_1)\left\{\begin{array}{c}\mathbf{A}_{\delta}^{\star}[\rho_1,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]+\\ \mathbf{A}_{\delta}^{\star}(\rho_1)\frac{1}{(\beta-\rho_1)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)-\mathbf{A}_{\delta}^{\star}(\rho_1)\tilde{\zeta}_1[\rho_1,s](\lambda\mathbf{e}_1)\end{array}\right\}.
\end{aligned}
$$
(4.18)

Since $s=\rho_2$ is also a root of numerator in (4.8), it shows that $s=\rho_2$ is a zero of the expression within the brace in (4.18), namely

(4.19)

$$
\begin{aligned}
& \left(\mathbf{A}_{\delta}^{\star}[\rho_1,s]+\mathbf{A}_{\delta}^{\star}(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\left(\mathbf{A}_{\delta}^{\star}[\rho_1,s]\tilde{\zeta}_1(s)+\mathbf{A}_{\delta}^{\star}(\rho_1)\tilde{\zeta}_1[\rho_1,s]\right)\lambda\mathbf{e}_1\\
=\; & \left(\mathbf{A}_{\delta}^{\star}[\rho_1,s]+\mathbf{A}_{\delta}^{\star}(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\left(\mathbf{A}_{\delta}^{\star}[\rho_1,s]\tilde{\zeta}_1(s)+\mathbf{A}_{\delta}^{\star}(\rho_1)\tilde{\zeta}_1[\rho_1,s]\right)\lambda\mathbf{e}_1\\
& -\left(\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_2]+\mathbf{A}_{\delta}^{\star}(\rho_1)\tfrac{1}{\beta-\rho_1}\right)\tfrac{\mu\beta}{\beta-\rho_2}\tilde{\mathbf{m}}_1(\beta)+\left(\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_2]\tilde{\zeta}_1(\rho_2)+\mathbf{A}_{\delta}^{\star}(\rho_1)\tilde{\zeta}_1[\rho_1,\rho_2]\right)\lambda\mathbf{e}_1\\
=\; & (s-\rho_2)\left\{\begin{array}{c}\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_2,s]\left[\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)-\lambda\tilde{\zeta}_1(s)\mathbf{e}_1\right]+\\ \sum\limits_{i=1}^{2}\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_i]\frac{1}{\prod\limits_{l=i}^{2}(\beta-\rho_l)}\left(\tfrac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta)\right)-\sum\limits_{i=1}^{2}\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_i]\tilde{\zeta}_1[\rho_i,\rho_2,s](\lambda\mathbf{e}_1)\end{array}\right\},
\end{aligned}
$$

where we denote $\mathbf{A}_{\delta}^{\star}[\rho_1,\rho_i]=\mathbf{A}_{\delta}^{\star}(\rho_1)$, when $i=1$.

Substituting (4.19) into (4.18), recursively from the fact $s=\rho_3,\ldots,\rho_n$ are roots of the numerator in (4.8), (4.16) is derived.

By similar arguments, we obtain (4.17) from (4.9). $\qquad\square$

4.2. Closed forms for rational family claim-size distribution. Now, we restrict the further analysis to the case of the claim amount distributions $F(x)$ and $G(x)$ both with rational Laplace transforms, viz,

$$\tilde{f}(s) = \frac{f_{r_1-1}(s)}{f_{r_1}(s)}, \quad \tilde{g}(s) = \frac{g_{r_2-1}(s)}{g_{r_2}(s)}, \quad r_1, r_2 \in \mathbb{N}^+,$$

where $f_{r_1-1}(s)$, $g_{r_2-1}(s)$ are polynomials of degree $r_1 - 1$ and $r_2 - 1$ or less, respectively, while $f_{r_1}(s)$ and $g_{r_2}(s)$ are polynomials of degree r_1 and r_2 with only negative roots, and satisfy $f_{r_1-1}(0) = f_{r_1}(0)$, $g_{r_2-1}(0) = g_{r_2}(0)$. Without loss of generality, we assume that $f_{r_1}(s)$ and $g_{r_2}(s)$ have leading coefficient 1. This wide class of distributions includes the phase-type distributions, and in particular, it includes the Erlang, Coxian and exponential distribution and all the mixtures of them.

In what follows, let $h(s) = (s - \beta)^n [f_{r_1}(s)]^n g_{r_2}(s)$. Multiplying both numerator and denominator of (4.16) by $h(s)$, we get

(4.20)
$$\begin{aligned}
\tilde{\mathbf{m}}_1(s) &= \frac{\prod_{i=1}^{n}(s-\rho_i)}{h(s)det[\mathbf{A}_\delta(s)]} \left\{ \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]h(s) \left[\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1 \right] \right. \\
&\quad + h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod_{l=i}^{n}(\beta-\rho_l)} \left(\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) \right) \\
&\quad \left. - h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]\tilde{\zeta}_1[\rho_i, \cdots, \rho_n, s](\lambda\mathbf{e}_1) \right\}.
\end{aligned}$$

It is obvious that the factor $h(s)det[\mathbf{A}_\delta(s)]$ of the denominator is a polynomial of degree $n(r_1+1)+r_2$ with leading coefficient $\prod_{i=1}^{n}(\lambda_i^*+\delta)$. Therefore, the equation $h(s)det[\mathbf{A}_\delta(s)] = 0$ has $n(r_1+1)+r_2$ roots on the complex plane. We can factorize $h(s)det[\mathbf{A}_\delta(s)]$ as follows

(4.21) $$h(s)det[\mathbf{A}_\delta(s)] = \prod_{i=1}^{n}(\lambda_i^* + \delta) \prod_{j=1}^{n}(s - \rho_j) \prod_{j=1}^{nr_1+r_2}(s + R_j),$$

where R_j for each j has positive real part and we assume that all of them are distinct from each other.

Substituting (4.21) into (4.20) then canceling the same factor $\prod_{j=1}^{n}(s - \rho_{lj})$, we derive from (4.20) that

(4.22)
$$\begin{aligned}
\tilde{\mathbf{m}}_1(s) &= \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta) \prod_{j=1}^{nr_1+r_2}(s+R_j)} \left\{ \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]h(s) \left[\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) - \lambda\tilde{\zeta}_1(s)\mathbf{e}_1 \right] \right. \\
&\quad + h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i] \frac{1}{\prod_{l=i}^{n}(\beta-\rho_l)} \left(\frac{\mu\beta}{\beta-s}\tilde{\mathbf{m}}_1(\beta) \right) \\
&\quad \left. - h(s) \sum_{i=1}^{n} \mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]\tilde{\zeta}_1[\rho_i, \cdots, \rho_n, s](\lambda\mathbf{e}_1) \right\}.
\end{aligned}$$

It is easy to find that the elements in matrix $h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]$ are polynomials of degree less than $nr_1 + r_2$, of course, the elements in matrix $h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]\frac{1}{\beta-s}$ are polynomials of degree less than $nr_1 + r_2 - 1$, and each $\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_i]$ for $i = 1, 2, \cdots, n$ is constant. Therefore, we have the following partial fractions:

$$\frac{h(s)\mathbf{A}_\delta^\star[\rho_1, \cdots, \rho_n, s]}{\prod_{j=1}^{nr_1+r_2}(s + R_j)} = \sum_{j=1}^{nr_1+r_2} \frac{\mathbf{Q}_j}{s + R_j},$$

$$\frac{h(s)\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,s]\frac{1}{\beta-s}}{\prod\limits_{j=1}^{nr_1+r_2}(s+R_j)} = \sum_{j=1}^{nr_1+r_2}\frac{\mathbf{D}_j}{s+R_j},$$

and

$$\frac{h(s)\frac{1}{\beta-s}}{\prod\limits_{j=1}^{nr_1+r_2}(s+R_j)} = \sum_{j=1}^{nr_1+r_2}\frac{\varsigma_j}{s+R_j}, \quad \frac{h(s)}{\prod\limits_{j=1}^{nr_1+r_2}(s+R_j)} = 1 + \sum_{j=1}^{nr_1+r_2}\frac{\tau_j}{s+R_j},$$

where $\mathbf{Q}_j$, $\mathbf{D}_j$, τ_j and ς_j are given respectively by

$$(4.23) \quad \mathbf{Q}_j = \frac{h(-R_j)\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,-R_j]}{\prod\limits_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)},$$

$$(4.24) \quad \mathbf{D}_j = \frac{h(-R_j)\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_n,-R_j]\frac{1}{\beta+R_j}}{\prod\limits_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)},$$

and

$$(4.25) \quad \varsigma_j = \frac{h(-R_j)\frac{1}{\beta+R_j}}{\prod\limits_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)}, \quad \tau_j = \frac{h(-R_j)}{\prod\limits_{i=1,i\neq j}^{nr_1+r_2}(R_i-R_j)}.$$

In view of the above partial fractions, (4.22) can be rewritten as

$$(4.26) \quad
\begin{aligned}
\tilde{\mathbf{m}}_1(s) =\ & \frac{1}{\prod\limits_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{j=1}^{nr_1+r_2}\frac{1}{s+R_j}\Big\{\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_1(\beta) - \mathbf{Q}_j\lambda\tilde{\zeta}_1(s)\mathbf{e}_1 \\
& + \varsigma_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_1(\beta) \\
& -\tau_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1)\Big\} \\
& -\frac{1}{\prod\limits_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_1[\rho_i,\cdots,\rho_n,s](\lambda\mathbf{e}_1).
\end{aligned}$$

By the same arguments, we have

$$(4.27) \quad
\begin{aligned}
\tilde{\mathbf{m}}_2(s) =\ & \frac{1}{\prod\limits_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{j=1}^{nr_1+r_2}\frac{1}{s+R_j}\Big\{\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_2(\beta) - \mathbf{Q}_j\lambda_n\tilde{\zeta}_2(s)\mathbf{e}_2 \\
& + \varsigma_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod\limits_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_2(\beta) \\
& -\tau_j\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2)\Big\} \\
& -\frac{1}{\prod\limits_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\tilde{\zeta}_2[\rho_i,\cdots,\rho_n,s](\lambda_n\mathbf{e}_2).
\end{aligned}$$

From [4], we have the Laplace inverse of $\tilde{\zeta}[\rho_1,\rho_2,\cdots,\rho_n,s]$ as follows

$$(4.28) \quad \mathcal{L}^{-1}\left(\tilde{\zeta}[\rho_1,\rho_2,\cdots,\rho_n,s]\right) = (-1)^n\left(\prod_{i=1}^{n}T_{\rho_i}\right)\zeta(x).$$

Thus, by inverting (4.26) and (4.27) results in the following theorem

4.5. Theorem. *If the claim-size distributions $F(x)$ and $G(x)$ both belong to the rational family, the expected discounted penalty function are given by*

(4.29)
$$
\begin{aligned}
\mathbf{m}_1(u) = {} & \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{j=1}^{nr_1+r_2} \Big\{ e^{-R_j u}\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_1(\beta) - \mathbf{Q}_j\lambda[e^{-R_j u}\circledast\zeta_1(u)]\mathbf{e}_1 \\
& + \varsigma_j e^{-R_j u} \sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_1(\beta) \\
& + \tau_j e^{-R_j u}\circledast\left(\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i](-1)^{n-i}\left(\prod_{l=i}^{n}T_{\rho_l}\right)\zeta_1(u)\right)(\lambda\mathbf{e}_1)\Big\} \\
& + \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i](-1)^{n-i}\left(\prod_{l=i}^{n}T_{\rho_l}\right)\zeta_1(u)(\lambda\mathbf{e}_1), u\geq 0,
\end{aligned}
$$

and

(4.30)
$$
\begin{aligned}
\mathbf{m}_2(u) = {} & \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)} \sum_{j=1}^{nr_1+r_2} \Big\{ e^{-R_j u}\mathbf{D}_j\mu\beta\tilde{\mathbf{m}}_2(\beta) - \mathbf{Q}_j\lambda_n[e^{-R_j u}\circledast\zeta_2(u)]\mathbf{e}_2 \\
& + \varsigma_j e^{-R_j u} \sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i]\frac{\mu\beta}{\prod_{l=i}^{n}(\beta-\rho_l)}\tilde{\mathbf{m}}_2(\beta) \\
& + \tau_j e^{-R_j u}\circledast\left(\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i](-1)^{n-i}\left(\prod_{l=i}^{n}T_{\rho_l}\right)\zeta_2(u)\right)(\lambda_n\mathbf{e}_2)\Big\} \\
& + \frac{1}{\prod_{i=1}^{n}(\lambda_i^*+\delta)}\sum_{i=1}^{n}\mathbf{A}_\delta^\star[\rho_1,\cdots,\rho_i](-1)^{n-i}\left(\prod_{l=i}^{n}T_{\rho_l}\right)\zeta_2(u)(\lambda_n\mathbf{e}_2), u\geq 0,
\end{aligned}
$$

where $\circledast$ represents the convolution operator. $\mathbf{Q}_j$, $\mathbf{D}_j$, τ_j and ς_j are given respectively by (4.23)-(4.25).

5. Numerical illustrations

In this section, we present a numerical example to illustrate an application of the main results in this paper. We suppose that the claim amounts from class 1 and class 2 have density functions, respectively,

$$
f(x) = \mu_1 e^{-\mu_1 x}, \quad \mu_1 > 0, x > 0, \quad g(y) = \mu_2 e^{-\mu_2 y}, \quad \mu_2 > 0, y > 0.
$$

Hence, LTs $\tilde{f}(s) = \frac{\mu_1}{s+\mu_1}$, $\tilde{g}(s) = \frac{\mu_2}{s+\mu_2}$. The inter-claim times from class 1 occur following a Poisson process with parameter λ, and inter-claim times from class 2 occur following a generalized Erlang(2) distribution with parameters λ_1, λ_2. In addition, the number of insurer's premium income $M(t)$ follows a Poisson process with parameter $\mu > 0$ and the premium sizes are exponentially distributed with parameter $\beta > 0$.

In order to obtain the probability of ultimate ruin, we assume $\delta = 0$ and $w_1(x_1, x_2) = w_2(x_1, x_2) = 1$. Thus

$$
\mathbf{A}_0(s) = \begin{pmatrix} \frac{\mu\beta}{\beta-s} - \lambda_1^* + \lambda\tilde{f}(s) & \lambda_1 \\ \lambda_2\tilde{g}(s) & \frac{\mu\beta}{\beta-s} - \lambda_2^* + \lambda\tilde{f}(s) \end{pmatrix}.
$$

Now, $m_{kj}(u), k = 1, 2, j = 1, 2, \ldots, n$ simplify to the probability of ultimate ruin $\psi_{kj}(u), k = 1, 2, j = 1, 2, \ldots, n$. Eventually, we are only interested in $\psi_k(u) = \psi_{k1}(u), k = 1, 2$.

For illustration purpose, we set $\mu_1 = 1, \mu_2 = 2, \lambda = 2, \lambda_1 = 1, \lambda_2 = 3, \mu = 3, \beta = 1$. It is easy to check that the positive security loading conditions are satisfied. Under this hypothesis, the solutions of $h(s)det[\mathbf{A}_0(s)] = 0$ are $-R_1 = -1.9087, -R_2 = -0.7394, -R_3 = -0.1222, \rho_1 = 0, \rho_2 = 0.6037$. From Theorem 4.2, we have $\tilde{\mathbf{m}}_1(\beta) =$

$\mathbf{\tilde{m}}_1(1) = \begin{pmatrix} 0.6906 \\ 0.5948 \end{pmatrix}$ and $\mathbf{\tilde{m}}_2(\beta) = \mathbf{\tilde{m}}_2(1) = \begin{pmatrix} 0.0911 \\ 0.2267 \end{pmatrix}$. Substituting $\mathbf{\tilde{m}}_1(1)$, $\mathbf{\tilde{m}}_2(1)$ into (4.29) and (4.30), respectively, we obtain the probability of ruin due to a claim from class k,

$$(5.1) \quad \psi_1(u) = -0.0214e^{-1.9087u} - 0.2504e^{-0.7394u} + 0.8202e^{-0.1222u} + 0.3333e^{-u}, u \geq 0,$$

$$(5.2) \quad \psi_2(u) = 0.0040e^{-1.9087u} - 0.0303e^{-0.7394u} + 0.0958e^{-0.1222u}, u \geq 0.$$

Thus, in view of $\psi(u) = \psi_1(u) + \psi_2(u)$, we can obtain the probability of ruin $\psi(u)$. Figure 1 shows the probabilities of ruin $\psi_1(u)$, $\psi_2(u)$ and $\psi(u)$ for different values of $u \in [0, 10]$.
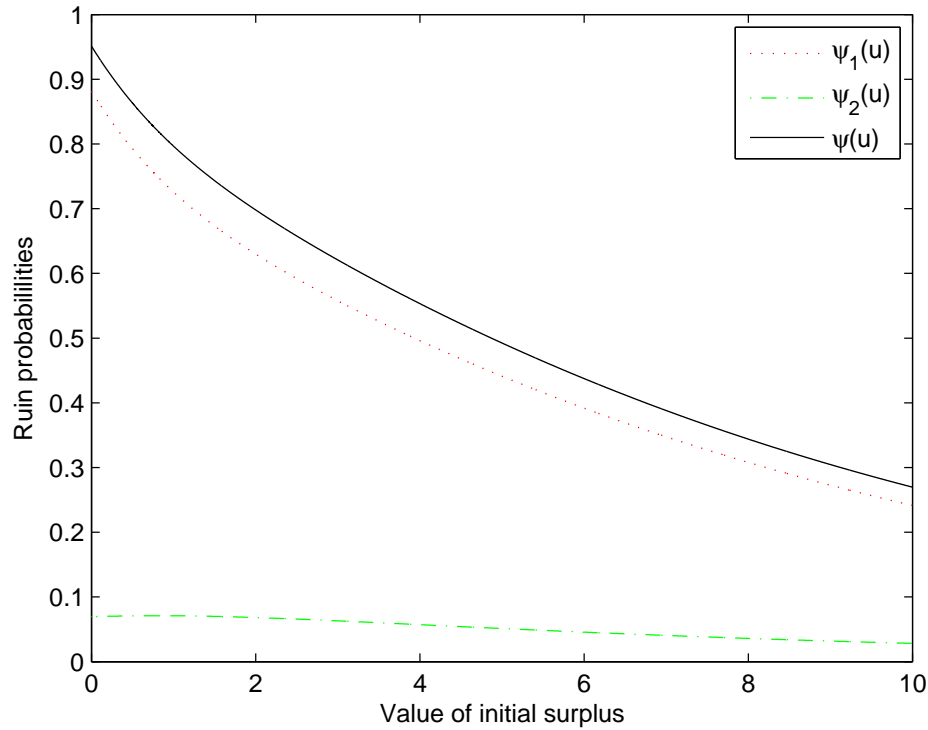

Figure 1. Ruin probabilities for different values of $u \in [0, 10]$.

6. Concluding remarks

In present paper, we investigate the expected discounted penalty functions in a risk model involving two independent classes of risks with stochastic income, in which the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. Namely, we extend the model in [13] by assuming that the premium income arrival process is a Poisson process. The integro-differential equations for the expected discounted penalty functions are established. By aid of Dickson-Hipp operator and divided difference, the Laplace transforms for the expected discounted penalty functions

are obtained, and explicit expressions are derived when the claim amount distributions belong to the rational family.

The results in our paper can be extended. For example, the premium income arrival process may be a renewal process, the model can also be perturbed by diffusion. We remark that it is very challenging to obtain closed form solutions for the expected discounted penalty functions if we move away from the exponential assumption for the premium sizes. Of course, we can find the solutions numerically for some complicated premium size distributions.

Acknowledgements

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