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A Related Fixed Point Theorem for F-Contractions on Two Metric Spaces

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Abstract

Recently, Wardowski in [Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012] introduced the concept of F-contraction on complete metric space which is a proper generalization of Banach contraction principle. In the present paper, we proved a related fixed point theorem with F-contraction mappings on two complete metric spaces.

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1. Introduction and preliminaries

The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory and its significance lies in its vast applicability in a number of branches of mathematics. There are a lot of generalization of Banach contraction mapping principle in the literature. One of a different way of this generalization is to consider two metric space. In 1981, Fisher defined related fixed points of mappings on two metric spaces and obtained some related fixed point theorems. Let (X, d) and (Y, ρ) be two metric space, $T : X \to Y$ and $S : Y \to X$ be two mappings. If there exist $x \in X$ and $y \in Y$ such that $Tx = y$ and $Sy = x$, then the pair of (T, S) is said to be has related fixed points. Thereafter many authors obtained some related fixed point theorems (see [1, 3–5, 10]).

In 1994, Namdeo et al. [9] proved the following:

Theorem 1.1. *Let (X, d) and (Y, ρ) be two complete metric spaces, $T : X \to Y$ and $S : Y \to X$ mappings satisfying the following equations:*

$$\begin{aligned} d(Sy, STx) &\leq c\phi(x, y) \\ \rho(Tx, TSy) &\leq c\psi(x, y) \end{aligned}$$

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for all $x \in X$ and $y \in Y$ for which

$$g(x,y) \neq 0 \neq h(x,y)$$

where $0 \leq c < 1$

$$\phi(x,y) = \frac{f(x,y)}{g(x,y)}, \ \psi(x,y) = \frac{f(x,y)}{h(x,y)} \tag{1.1}$$

and

$$\begin{aligned} f(x,y) &= \max\{d(x,Sy)\rho(y,Tx), d(x,STx)\rho(y,TSy), d(Sy,STx)\rho(y,Tx)\} \\ g(x,y) &= \max\{d(x,STx), \rho(y,TSy), d(x,Sy)\} \\ h(x,y) &= \max\{d(x,STx), \rho(y,TSy), \rho(y,Tx)\}. \end{aligned}$$

Then, ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further, $Tz = w$ and $Sw = z$.

In this paper, by taking into account the recent proof technique, which is first used by Wardowski [16], we will present a related fixed point result for two single valued mappings on two complete metric spaces. For the sake of completeness, we consider the following notion due to [16].

Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following:

(F1) F is stricly increasing, that is for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of the functions belonging to $\mathcal{F}$ are $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \alpha + \ln\alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$.

Definition 1.2 ([16]). Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then, we say that T is an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x,y \in X \ [d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))]. \tag{1.2}$$

If we take $F(\alpha) = \ln\alpha$ in Definition 1.2, the inequality (1.2) turns to

$$d(Tx,Ty) \leq e^{-\tau} d(x,y), \text{ for all } x,y \in X, Tx \neq Ty. \tag{1.3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx,Ty) \leq e^{-\tau}d(x,y)$ also holds. Thus T is a Banach contraction with contractive constant $L = e^{-\tau}$. Therefore, every Banach contraction is also F-contraction, but the converse may not be true as shown in the Example 2.5 of [16]. If we choose some different functions from $\mathcal{F}$ in (1.2), we can obtain some new as well as existing contractive conditions. In addition, Wardowski showed that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx,Ty) < d(x,y), \text{ for all } x,y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous map. We can find some important properties about F-contractions in [2, 6–8, 11–15, 17]. In the light of these informations, we can see that the following theorem is a proper generalization of Banach Contraction Principle.

Theorem 1.3 ([16]). *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then, T has a unique fixed point.*

2. Main result

In this section, we present a new kind of related fixed point theorems using the concept of F-contraction.

Theorem 2.1. *Let (X,d) and (Y,ρ) be two complete metric spaces, $T : X \to Y$ and $S : Y \to X$ be two mappings. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(Sy,STx) > 0 \Rightarrow \tau + F(d(Sy,STx)) \leq F(\phi(x,y)) \quad (2.1)$$

$$\rho(Tx,TSy) > 0 \Rightarrow \tau + F(\rho(Tx,TSy)) \leq F(\psi(x,y)) \quad (2.2)$$

hold for all $x \in X$ and $y \in Y$ for which

$$g(x,y) \neq 0 \neq h(x,y),$$

where ϕ and ψ are as in Theorem 1.1. Then, ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further, $Tz = w$ and $Sw = z$.

Proof. Let $x \in X$ be an arbitrary point. Define sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ by

$$(ST)^n x = x_n, \quad T(ST)^{n-1}x = y_n$$

and define $\alpha_n = d(x_n,x_{n+1})$ and $\beta_n = \rho(y_n,y_{n+1})$, $n = 1,2,3,...$

If there exist $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ or $y_{n_0+1} = y_{n_0}$ then the proof is finished. Indeed, if $x_{n_0+1} = x_{n_0}$, then $(ST)^{n_0+1}x = (ST)^{n_0}x$ and so $(ST)(ST)^{n_0}x = (ST)^{n_0}x$. Therefore, $(ST)^{n_0}x := z$ is a fixed point of ST. Also, if $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0+1} = Tx_{n_0}$ and so $T(ST)^{n_0+1}x = T(ST)^{n_0}x$ or equivalently we have

$$TST(ST)^{n_0}x = T(ST)^{n_0}x.$$

Therefore, $T(ST)^{n_0}x := w$ is a fixed point TS. In this case we have $Tz = w$ and $Sw = z$. Similar result can be obtained when $y_{n_0+1} = y_{n_0}$ for some n_0.

Now suppose that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for every $n \in \mathbb{N}$. Applying inequality (2.1) we get

$$d(x_n,x_{n+1}) = d(Sy_n,STx_n) > 0$$

so we can write

$$F(d(Sy_n,STx_n)) \leq F(\phi(x_n,y_n)) - \tau$$

from which it follows that

$$F(\alpha_n) \leq F(\beta_n) - \tau. \quad (2.3)$$

Applying inequality (2.2) we get

$$\rho(y_n,y_{n+1}) = \rho(Tx_{n-1},TSy_n) > 0$$

so we can write

$$F(\rho(Tx_{n-1},TSy_n)) \leq F(\psi(x_{n-1},y_n)) - \tau$$

from which it follows that

$$F(\beta_n) \leq F(\alpha_{n-1}) - \tau. \quad (2.4)$$

From (2.3) and (2.4) we get

$$\begin{aligned} F(\alpha_n) &\leq F(\beta_n) - \tau \\ &\leq F(\alpha_{n-1}) - 2\tau \\ &\leq \\ &\quad \vdots \\ &\leq F(\alpha_0) - 2n\tau \\ &\leq F(\beta_0) - (2n+1)\tau \end{aligned} \quad (2.5)$$

for all $n \in \mathbb{N}$. From (2.5) we obtain $\lim_{n\to\infty} F(\alpha_n) = -\infty$ and with (F2) we get

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.6}$$

Similarly, we get $\lim_{n\to\infty} F(\beta_n) = -\infty$ from (2.4) and with (F2) we find

$$\lim_{n\to\infty} \beta_n = 0. \tag{2.7}$$

From (F3) there exist $k \in (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0 \text{ and } \lim_{n\to\infty} \beta_n^k F(\beta_n) = 0. \tag{2.8}$$

By (2.5) the following holds for all $n \in \mathbb{N}$

$$\begin{aligned} \alpha_n^k F(\alpha_n) &\leq \alpha_n^k[F(\alpha_{n-1}) - 2\tau] \\ &\leq \\ &\quad \vdots \\ &\leq \alpha_n^k[F(\alpha_0) - 2n\tau] \end{aligned}$$

and so

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -2\alpha_n^k n\tau \leq 0. \tag{2.9}$$

Letting $n \to \infty$ in (2.9), using (2.6) and (2.8) we obtain

$$\lim_{n\to\infty} \alpha_n^k n = 0 \tag{2.10}$$

Similarly by (2.5) we get

$$\beta_n^k F(\beta_n) - \beta_n^k F(\beta) \leq -\beta_n^k (2n+1)\tau \leq 0. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), using (2.7) and (2.8) we obtain

$$\lim_{n\to\infty} (2n+1)\beta_n^k = \lim_{n\to\infty} n\beta_n^k = 0. \tag{2.12}$$

Now let us observe that from (2.10) there exist $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$ and from (2.12) there exist $n_2 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$, then we have for all $n > n_0$

$$\alpha_n^k \leq \frac{1}{n} \text{ and } \beta_n^k \leq \frac{1}{n}. \tag{2.13}$$

In order to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences consider $m, n \in \mathbb{N}$ such that $m > n > n_0$. From (2.13) and triangular inequality, we write

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &< \sum_{i=n}^{\infty} \alpha_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \end{aligned}$$

and

$$\begin{aligned} \rho(y_n, y_m) &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \dots + \rho(y_{m-1}, y_m) \\ &< \sum_{i=n}^{\infty} \beta_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

From the convergence of the serie $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ we receive that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits $z \in X$ and $w \in Y$ respectively.

Now, suppose $z \neq STz$ and $w \neq TSw$. The following two cases arise:

Case 1. Let $z = Sw$ and $w = Tz$. Then, $w = TSw$ and $z = STz$, which is a contradiction.

Case 2. Let $z \neq Sw$ or $w \neq Tz$. If $z \neq Sw$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $d(Sw, x_{n(k)}) > 0$ for all $k \in \mathbb{N}$. Therefore, applying inequality (2.1) we have

$$F(d(Sw, STx_{n(k)})) \leq F(\phi(x_{n(k)}, w)) - \tau \tag{2.14}$$

where $\phi(x_{n(k)}, w) = \dfrac{f(x_{n(k)}, w)}{g(x_{n(k)}, w)}$. Since

$$\lim_{n\to\infty} g(x_{n(k)}, w) = \lim_{n\to\infty} \max\{d(x_{n(k)}, x_{n(k)+1}), \rho(w, TSw), d(x_{n(k)}, Sw)\} > 0$$

and

$$\begin{aligned}
\lim_{n\to\infty} f(x_{n(k)}, w) &= \lim_{n\to\infty} \max\left\{\begin{array}{l} d(x_{n(k)}, Sw)\rho(w, Tx_{n(k)}), \\ d(x_{n(k)}, STx_{n(k)})\rho(w, TSw), \\ d(Sw, STx_{n(k)})\rho(w, Tx_{n(k)}) \end{array}\right\} \\
&= \lim_{n\to\infty} \max\left\{\begin{array}{l} d(x_{n(k)}, Sw)\rho(w, y_{n(k)+1}), \\ d(x_{n(k)}, x_{n(k)+1})\rho(w, TSw), \\ d(Sw, x_{n(k)+1})\rho(w, y_{n(k)}) \end{array}\right\} \\
&= 0
\end{aligned}$$

we get $\lim_{n\to\infty} \phi(x_{n(k)}, w) = 0$. Therefore, from (2.14) and (F2) we have

$$\lim_{n\to\infty} d(Sw, STx_{n(k)}) = 0$$

and so $Sw = z$, which is a contradiction. If $w \neq Tz$, then similar contradiction can be occur.

Therefore, either $z = STz$ or $w = TSw$. If $z = STz$, then z is a fixed point of ST and Tz is a fixed point of TS. Similarly, if $w = TSw$, then w is a fixed point of TS and Sw is a fixed point of ST.

To prove uniqueness, suppose that z and z' are two fixed points of ST. Then, since

$$\phi(z', Tz) = \frac{f(z', Tz)}{g(z', Tz)} = \rho(Tz, Tz')$$

and

$$\psi(z', Tz) = \frac{f(z', Tz)}{h(z', Tz)} = d(z, z'),$$

it follows from inequality (2.1) and (2.2) that

$$\begin{aligned}
F(d(STz, STz')) &\leq F(\phi(z', Tz)) - \tau \\
&= F(\rho(Tz, Tz')) - \tau \\
&\leq F(\psi(z, Tz')) - 2\tau \\
&= F(d(z, z')) - 2\tau,
\end{aligned}$$

which is a contradiction. Therefore, ST (similarly TS) has a unique fixed point in X. □

We can obtain the following corollaries.

Corollary 2.2. *Theorem 1.1 is immediate from Theorem 2.1.*

Proof. The proof is clear, by taking $F(\alpha) = \ln \alpha$ in Theorem 2.1. □

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(Ty, T^2x) > 0 \Rightarrow \tau + F(d(Ty, T^2x)) \leq F(\phi(x, y))$$

holds for all $x, y \in X$ *for which* $\max\{d(x,T^2x), d(y,T^2y), d(x,Ty)\} > 0$, *where*

$$\phi(x,y) = \frac{\max\{d(x,Ty)d(y,Tx), d(x,T^2x)d(y,T^2y), d(Ty,T^2x)d(y,Tx)\}}{\max\{d(x,T^2x), d(y,T^2y), d(x,Ty)\}}.$$

Then, T *has a unique fixed point.*

Proof. Take $X = Y$, $d = \rho$ and $T = S$ in Theorem 2.1. □

Corollary 2.4. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\begin{aligned} d(y,Tx) &> 0 \Rightarrow \tau + F(d(y,Tx)) \leq F(\phi(x,y)) \\ d(Tx,Ty) &> 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(\psi(x,y)) \end{aligned}$$

hold for all $x, y \in X$ *for which*

$$g(x,y) \neq 0 \neq h(x,y),$$

where

$$\phi(x,y) = \frac{f(x,y)}{g(x,y)}, \ \psi(x,y) = \frac{f(x,y)}{h(x,y)}$$

and

$$\begin{aligned} f(x,y) &= \max\{d(x,y)d(y,Tx), d(x,Tx)d(y,Ty), d^2(y,Tx)\} \\ g(x,y) &= \max\{d(x,Tx), d(y,Ty), d(x,y)\} \\ h(x,y) &= \max\{d(x,Tx), d(y,Ty), d(y,Tx)\}. \end{aligned}$$

Then, T *has a unique fixed point.*

Proof. Take $X = Y$, $d = \rho$ and $S = I$ (the identity mapping) in Theorem 2.1. □

Example 2.5. Let $X = Q$ and $Y = I$, where Q is the set of rational numbers and I is the set of irrational numbers. Consider the discrete metric d on X, and a metric defined by

$$\rho(x,y) = \begin{cases} 0 & , \quad x = y \\ 1 + |x-y| & , \quad x \neq y \end{cases}$$

on Y. Then, it is clear that (X,d) and (Y,ρ) are complete metric spaces. Define two mappings $T : X \to Y$ by $Tx = \sqrt{2}$ and $S : Y \to X$ by $Sy = 0$. Then, for all $x \in X$ and $y \in Y$, we have

$$d(Sy, STx) = 0 = \rho(Tx, TSy).$$

This shows that the conditions (2.1) and (2.2) are satisfied for all $F \in \mathcal{F}$ and $\tau > 0$. Therefore, by Theorem 2.1, ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further, $Tz = w$ and $Sw = z$.

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