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The transmuted exponentiated Weibull geometric distribution: Theory and applications

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Abstract

A generalization of the exponentiated Weibull geometric model called the transmuted exponentiated Weibull geometric distribution is proposed and studied. It includes as special cases at least ten models. Some of its structural properties including order statistics, explicit expressions for the ordinary and incomplete moments and generating function are derived. The estimation of the model parameters is performed by the maximum likelihood method. The use of the new lifetime distribution is illustrated with an example. We hope that the proposed distribution will serve as a good alternative to other models available in the literature for modeling positive real data in several areas.

Keywords: Exponentiated Weibull geometric distribution, Goodness–of–fit statistic, Maximum likelihood estimation, Moment, Order statistic, Survival function, Transmutation map.

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1. Introduction

Several compounding distributions have been proposed in the literature to model lifetime data. Adamidis and Loukas [2] pioneered the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus [16] defined the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Adamidis *et al.* [1] proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its structural properties along with its reliability features. The hazard rate function (hrf) of the EEG distribution can be monotone decreasing, increasing or constant. Lai *et al.* [17] introduced a modified Weibull distribution capable of modeling a bathtub-shaped hazard rate function (hrf). Mahmoudi and Shiran [19] proposed an exponentiated Weibull-geometric (EWG) distribution by compounding the EW and geometric distributions more flexible than the EW distribution and studied some of its properties. Wang and Elbatal [35] discussed a modified Weibull geometric distribution having monotonically increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped hazard rate functions. Finally, Saboor *et al.* [31] introduced a transmuted exponential Weibull distribution which have a bathtub-shaped and upside-down bathtub-shaped hazard rate functions.

The modeling of lifetime data by compounding a life model and a discrete distribution has been used to construct new lifetime models in the last few years. For some references, see Silva et al. [27]. *In practice, the exponential and Weibull are the most used baseline models. Suppose that a company has N systems functioning independently and producing a certain product at a given time, where N is a random variable, which is often determined by economy, customers demand, etc. The reason for considering N as a random variable comes from a practical viewpoint in which failure (of a device for example) often occurs due to the present of an unknown number of initial defects in the system. In this paper, we focus on the case in which N is a geometric random variable with probability mass function (pmf) $P(N=n)=(1-p)\,p^{n-1}$, for $0<p<1$ and $n=1,2,\cdots$. We can also consider that N follows other discrete distributions, such as the binomial, Poisson, etc, whereas they require to be truncated at zero since $N\geq 1$. Another reason by taking N to be a geometric random variable is that the "optimum" number can be interpreted as the "number to event", matching up with the definition of a geometric random variable as suggested by Nadarajah et al.* [22]. *Other motivations can also be found in Nadarajah et al.* [22]. Readers are referred to [34].

Suppose that $\{Z_i\}_{i=1}^{N}$ are independent and identically distributed (iid) random variables having the EW(α,β,θ) distribution with cumulative distribution function (cdf) given by

$$F(x;\alpha,\beta,\theta)=(1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}})^{\theta},\ \mathrm{x}>0,$$

and N a discrete random variable having a geometric distribution defined before. Let $Z_{(n)}=\max\{Z_i\}_{i=1}^{N}$. The cdf and probability density function (pdf) of $Z_{(n)}$ are given by

$$G(x;\alpha,\beta,\theta,p)=\frac{(1-p)\,(1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}})^{\theta}}{1-p(1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}})^{\theta}} \tag{1.1}$$

and

$$g(x;\alpha,\beta,\theta,p)=(1-p)\theta\beta\alpha^{\beta}x^{\beta-1}\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}(1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}})^{\theta-1}\,[1-\mathrm{p}(1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}})^{\theta}]^{-2}, \tag{1.2}$$

respectively, where α, β, $\theta>0$ and $p\in[0,1)$. The lifetime model defined by (1.1) and (1.2) is called the *exponentiated Weibull geometric* (EWG) distribution [19]. Hereafter, let Y be a random variable having the density (1.2) and write $Y \backsim$EWG(α,β,θ,p).

In this paper, we define and study a new lifetime model called the *transmuted exponentiated Weibull-geometric* ("TEWG" for short) distribution. The main feature of this model is that a transmuted parameter is inserted in (1.2) to give greater flexibility in the form of the generated distribution. Using the quadratic rank transmutation map studied by [32], we construct the five-parameter TEWG model. We give a comprehensive description of some mathematical properties of the new distribution with the hope that it will attract wider applications in reliability, engineering and other areas of research. The concept of transmuted generator is explained below.

A Quadratic Rank Transmutation Map (QRTM) is defined by $G_{R12}(u) = u + \lambda u\,(1-u), |\lambda| \leq 1$, from which the cdf's satisfy $F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2$. By differentiating $F_2(x)$, we have

$$f_2(x) = f_1(x)\ \left[(1+\lambda) - 2\lambda F_1(x)\right], \tag{1.3}$$

where $f_1(x)$ and $f_2(x)$ are the pdf's corresponding to the cdf's $F_1(x)$ and $F_2(x)$, respectively. For $\lambda = 0$, we have $f_2(x) = f_1(x)$.

1.1. Lemma. *The function* $f_2(x)$ *given by* (1.3) *is a well-defined density function.*

Proof. Rewriting $f_2(x)$ as $f_2(x) = f_1(x)[1 - \lambda\{2F_1(x) - 1\}]$, we note that $f_2(x)$ is nonnegative. We prove that the integration over its support is equal to one. Considering that the support of $f_1(x)$ is $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} f_2(x)dx = (1+\lambda)\int_{-\infty}^{\infty} f_1(x)\mathrm{d}x - 2\lambda\int_{-\infty}^{\infty} f_1(x)F_1(x)\mathrm{d}x = 1.$$

Similarly, for other cases, where the support of $f_1(x)$ is a part of the real line, the previous lemma holds. Hence, $f_2(x)$ is a well-defined pdf. We call $f_2(x)$ the transmuted pdf of a random variable with baseline density $f_1(x)$. This proves the current result.

Many authors constructed generalizations of some well-known distributions by using the *transmuted construction*. Aryal and Tsokos [4, 3] defined the transmuted generalized extreme value and transmuted Weibull distributions. Aryal [5] proposed and studied various structural properties of the transmuted log-logistic distribution, Shuaib and King [28] introduced the transmuted modified Weibull distribution, which extends the transmuted Weibull distribution by [3], and studied some of its mathematical properties and maximum likelihood estimation of the unknown parameters. Elbatal and Aryal [11] discussed the transmuted additive Weibull distribution. Elbatal [12, 13] presented the transmuted generalized inverted exponential and transmuted modified inverse Weibull distributions. Further, Merovci and Elbatal [20] proposed the transmuted Lindley-geometric distribution, Merovci *et al.* [20] defined the transmuted generalized inverse Weibull distribution and Elbatal *et al.* [10] studied the transmuted exponentiated Fréchet distribution.

The rest of the paper is organized as follows. In Section 2, we provide the pdf, cdf and survival function (sf) of the new distribution. Some special cases are given in Section 3. The density of the order statistics is given in Section 4. A mixture representation for the new pdf is derived in Section 5, where some of its structural properties can be easily obtained. Section 7 is related to the maximum likelihood estimates (MLEs) and the asymptotic confidence intervals for the unknown parameters. Finally, in Section 8, we present a real data analysis to illustrate the flexibility of the new lifetime model. Some conclusions are given in Section 9.

2. The TEWG Distribution

Let $\phi = (\alpha, \beta, \theta, p, \lambda)^T$. By inserting (1.1) and (1.2) in equation (1.3), the cdf and pdf of the TEWG distribution are given by

$$F_{TEWG}(x;\phi) = \frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}} \times \left\{1+\lambda-\lambda\left[\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}}\right]\right\} \tag{2.1}$$

and

$$f_{TEWG}(x;\phi) = \theta\beta\alpha^{\beta}(1-p)x^{\beta-1}\mathrm{e}^{-(\alpha x)^{\beta}} \times \left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta-1}\left[1-p\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}\right]^{-2} \times \left\{(1+\lambda)-2\lambda\left[\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{\theta}}\right]\right\}, \tag{2.2}$$

respectively, where $p \in [0,1), \alpha,\ \beta,\ \theta > 0$ and $|\lambda| \leq 1$. If X is a random variable with pdf (2.2), we use the notation $X \backsim$TEWG(ϕ).

We emphasize that the new model (2.2) is obtained by using the transmuted construction applied to a compounding life distribution from the exponentiated Weibull and geometric distributions.

The sf of X is given by $S_{TEWG}(x;\phi) = 1 - F_{TEWG}(x;\phi)$, whereas its hazard rate function (hrf) becomes $h_{TEWG}(x;\phi) = f_{TEWG}(x;\phi)/S_{TEWG}(x;\phi)$, which is an important quantity to characterize life phenomenon. The reversed hazard rate function (rhrf) of X is given by $\tau_{TEWG}(x;\phi) = f_{TEWG}(x;\phi)/F_{TEWG}(x;\phi)$.

2.1. Shapes of density and hazard function. The TEWG density (2.2) allows for greater flexibility of the tails. This function can exhibit different behavior depending on the parameter values as shown in Figures 1, 2 and 3. They display plots of the pdf of X for selected parameter values. Figure 1(a,b) and Figure 2(d) reveal that the mode of the pdf increases as λ, α and θ increases, respectively. Figure 2(c) and 3(e) indicate that the parameters β and p behave somewhat as scale parameters. Figure 3(f) and 4(g) display the increasing and bathtub-shaped of the hrf's, respectively.

3. Special Models

The TEWG distribution is a very flexible model that provides different distributions when its parameters are changed. It contains the following ten special models:

- For $\lambda = 0$, then (2.2) reduces to the EWG distribution pioneered by [19].
- The case $\theta = 1$ refers to the transmuted Weibull-geometric distribution.
- For $\lambda = 0$ and $\theta = 1$, we have the Weibull-geometric distribution given by [6].
- The transmuted generalized exponential geometric distribution arises as a special case of the TEWG distribution by taking $\theta = \beta = 1$.
- The case $\beta = 1$ refers to the transmuted exponentiated exponential geometric distribution.
- Setting $\lambda = 0$ and $\beta = 1$, we have the exponentiated exponential geometric distribution given by [18].

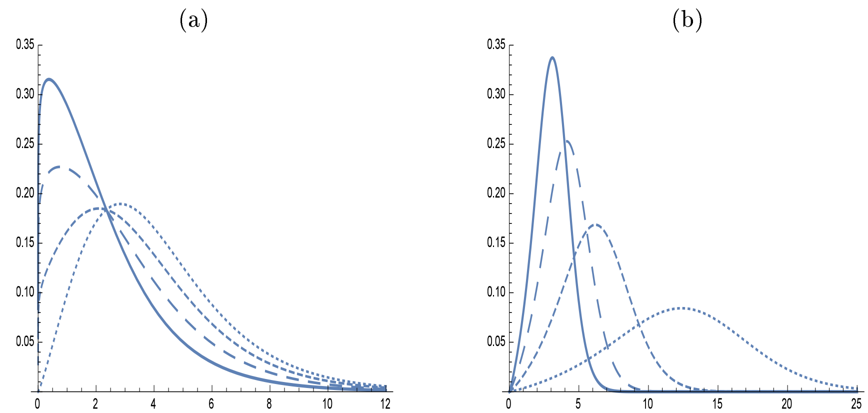

Figure 1. The TEWG density function: (a) $\alpha = 0.5$, $\theta = 1$, $\beta = 1.1$, $p = 0.5$ and $\lambda = -1$ (dotted line), $\lambda = -0.5$ (small dashed line), $\lambda = 0$ (long dashed line), $\lambda = 0.5$ (thick line). (b) $\lambda = -0.5$, $\theta = 2$, $\beta = 1.1$, $p = 0.5$ and $\alpha = 0.1$ (dotted line), $\alpha = 0.2$ (small dashed line), $\alpha = 0.3$ (long dashed line), $\alpha = 0.4$ (thick line).

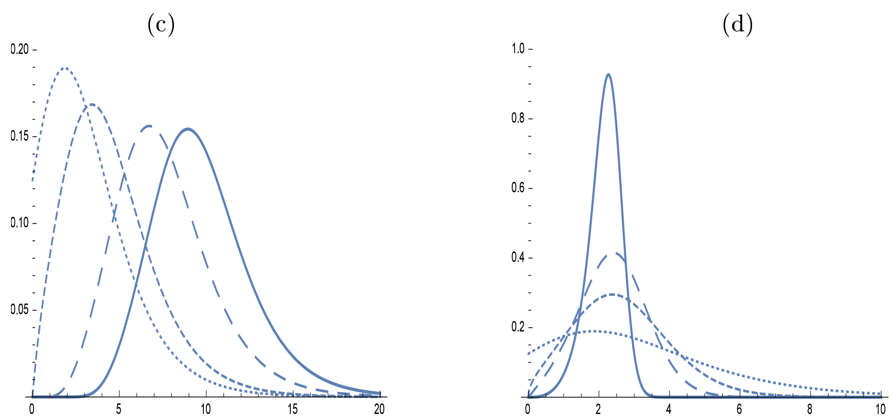

Figure 2. The TEWG density function: (c) $\alpha = 0.5$, $\theta = 1$, $p = 0.5$, $\lambda = -0.5$ and $\beta = 1$ (dotted line), $\beta = 2$ (small dashed line), $\beta = 10$ (long dashed line), $\beta = 30$ (thick line). (d) $\lambda = -0.5$, $\alpha = 0.5$, $\beta = 1$, $p = 0.5$ and $\theta = 1$ (dotted line), $\theta = 1.5$ (small dashed line), $\theta = 2$ (long dashed line), $\theta = 4$ (thick line).

- For $\theta = \beta = 1$, it follows the transmuted exponential geometric distribution.
- For $\lambda = 0$ and $\theta = \beta = 1$, we obtain the exponential geometric distribution given by [2].
- For $\beta = 2$, we have the transmuted generalized Rayleigh geometric distribution.
- The case $\beta = 2$ and $\theta = 1$ refers to the transmuted Rayleigh geometric distribution.

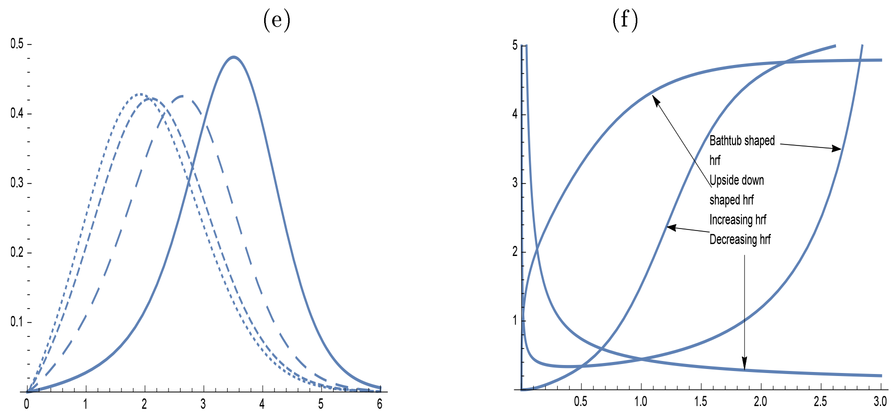

Figure 3. The TEWG density function: (e) $\alpha = 0.5, \theta = 2, \lambda = -0.5, \beta = 1.1$ and $p = 0$ (dotted line), $p = 0.3$ (small dashed line), $p = 0.6$ (long dashed line), $p = 0.9$ (thick line). The TEWG hazard rate function: (f) Increasing $(\alpha = 2.45, \beta = 1.2, \theta = 2.9, p = 0.9, \lambda = 0.15)$, decreasing $(\alpha = 0.5, \beta = 0.4, \theta = 0.1, p = 0.2, \lambda = 0.1)$, bathtub $(\alpha = 0.3, \beta = 3.3, \theta = 0.1, p = 0.8, \lambda = 1.2)$ and upside-down bathtub $(\alpha = 2.4, \beta = 1, \theta = 1.3, p = 0.01, \lambda = 0.5)$.

4. Order statistics

In this section, we derive closed-form expressions for the pdf of the rth order statistic of X. Let $X_1, \ldots, X_n$ be a simple random sample from the TEWG distribution with pdf and cdf given by (2.1) and (2.2), respectively. Let $X_{(1)} \le X_{(2)} \le, \ldots, \le X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{i:n}$, say $f_{i:n}(x;\phi)$, is given by

$$f_{i:n}(x,\phi) = \frac{1}{B(i, n-i+1)}\, F(x;\phi)^{i-1}\left[1 - F(x;\phi)\right]^{n-i} f(x;\phi), \tag{4.1}$$

where $F(x;\phi)$ and $f(x;\phi)$ are the cdf and pdf of X given by (2.1) and (2.2), respectively, and $\mathrm{B}(\cdot,\cdot)$ is the beta function. We have

$$\begin{aligned} f_{i:n}(x;\phi) = {} & \frac{\theta\beta\alpha^{\beta}(1-p)}{B(i,n-i+1)}\, x^{\beta-1}\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}\mathrm{h}^{\theta-1} \\ & \times \left[1 - ph^{\theta}\right]^{-2}\left\{(1+\lambda) - 2\lambda\left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\} \\ & \times \left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\left\{1+\lambda-\lambda\left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\}\right]^{i-1} \\ & \times \left[1 - \frac{(1-p)h^{\theta}}{1-ph^{\theta}}\left\{1+\lambda-\lambda\left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\}\right]^{n-i}, \end{aligned}$$

where $h = 1 - \mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}$.

5. Mixture Representation

Based on equation (1.3), we can write

$$f(x) = \sum_{k=0}^{\infty} \left[w_{1k}\, h_{k+1}(x) + w_{2k}\, h_{k+2}(x) \right], \tag{5.1}$$

where $w_{1k} = (1+\lambda)(1-p)p^k$ and $w_{2k} = -(k+1)\lambda(1-p)^2 p^k$. Equation (5.1) reveals that the density function of X is a mixture of EW densities.

5.1. Moments. Using the mixture representation, we obtain

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} \left[w_{1k}\, E(Y_{k+1}^r) + w_{2k}\, E(Y_{k+2}^r) \right]. \tag{5.2}$$

We now provide two explicit expressions for $E(Y_{k+1}^r)$. First, Choudhury [7] derived the closed-form expression

$$E(Y_{k+1}^r) = \frac{(k+1)\theta}{\alpha^r}\, \Gamma\left(\frac{r}{\beta}+1\right) \left[1 + \sum_{i=1}^{\infty} \frac{(-1)^i\, a_i((k+1)\theta)}{(i+1)^{r/\beta+1}} \right],$$

where $a_i = a_i(\gamma) = (-1)^i\,(\gamma-1)\cdots(\gamma-i)/i!$ for $i = 1, 2, \ldots$. The infinite series on the right hand side converges for all $\theta > 0$.

Second, Nadarajah and Gubta [24] derived an infinite series representation applicable for any $r > -\beta$ real or integer given by

$$E(Y_k^r) = \frac{(k+1)\theta}{\alpha^r}\, \Gamma\left(\frac{r}{\beta}+1\right) \sum_{i=0}^{\infty} \frac{(1-(k+1)\theta)_i}{i!\,(i+1)^{(r+\beta)/\beta}}.$$

Inserting the last two expressions in (5.2) gives $E(X^r)$

5.2. Incomplete moments. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well.

For lifetime models, it is of interest to known the rth lower and upper incomplete moments of X defined by $m_r(x) = \int_0^x x^r\, f(x)dx$ and $v_r(x) = \int_x^{\infty} x^r f(x)dx$, respectively, for any real $r > 0$. Clearly, these rth incomplete moments are related by $v_r(x) = \mu_r' - m_r(x)$.

Based on equation (5.1), we have

$$m_r(x) = \sum_{k=0}^{\infty} \left[w_{1k}\, m_r^{(k+1)}(x) + w_{2k}\, m_r^{(k+2)}(x) \right], \tag{5.3}$$

where $m_r^{(k+1)}(x) = \int_x^{\infty} x^r\, h_{k+1}(x)dx$ is the rth lower incomplete moment of Y_{k+1}.

Following a result of [23], we obtain

$$\begin{aligned} m_r^{(k+1)}(x) &= (k+1)\,\theta\,\alpha^{-r\,\beta} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)^{(r+1)\beta}} \binom{(k+1)\theta - 1}{j} \\ &\quad \times \gamma\left(\frac{r}{\beta}+1; (j+1)(\alpha x)^{\beta}\right), \end{aligned} \tag{5.4}$$

where $\gamma(s;t) = \int_0^t x^{s-1}\, \mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the lower incomplete gamma function. Equation (5.4) gives $m_r(x)$ as a linear combination of incomplete gamma functions evaluated at different points.

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $B(\pi)=m_1(q)/(\pi\,\mu_1')$ and $L(\pi)=m_1(q)/\mu_1'$, respectively, where $m_1(q)$ can be determined from (5.3) with $r=1$ and $q=Q(\pi)$ is calculated by inverting numerically (2.1).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1=\int_0^\infty|x-\mu_1'|f(x)dx$ and $\delta_2(x)=\int_0^\infty|x-M|f(x)dx$, respectively, where $\mu_1'=E(X)$ is the mean and $M=Q(0.5)$ is the median. These measures can be determined using the relationships $\delta_1=2\mu_1'F(\mu_1';\phi))-2m_1(\mu_1')$ and $\delta_2=\mu_1'-2m_1(M)$, where $m_1(\mu_1')$ comes from (5.3) with $r=1$.

5.3. Generating function. Let $M_{k+1}(t)$ be the moment generating function (mgf) of Y_{k+1}. We obtain the mgf of X, say $M(t)$, from equation (5.1) as

$$M(t)=\sum_{k=0}^{\infty}\left[w_{1k}\,M_{k+1}(t)+w_{2k}\,M_{k+2}(t)\right].$$

We provide an explicit expression for $M_{k+1}(t)$ when $\beta>1$, which requires the complex parameter Wright generalized hypergeometric function with p numerator and q denominator parameters (Kilbas *et al.*, 2006, Equation (1.9)) defined by

$$(5.5)\qquad {}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{array};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_jn\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_jn\right)}\frac{z^n}{n!}$$

for $z\in\mathbb{C}$, where $\alpha_j,\ \beta_k\in\mathbb{C},\ A_j,\ B_k\neq0,\ j=\overline{1,p},\ k=\overline{1,q}$ and the series converges for $1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0$.

The mgf of Y_{k+1} (when $\beta>1$) is given by

$$(5.6)\qquad M_{k+1}(t)=(k+1)\theta\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\binom{(k+1)\theta-1}{j}\,{}_1\Psi_0\left[\begin{array}{c}\left(1,\beta^{-1}\right)\\-\end{array};\alpha\,t\,(j+1)^{-1/\beta}\right].$$

Generalized hypergeometric functions are included as in-built functions in most analytical softwares, so the special function in (5.5) and hence (5.6) can be evaluated by the softwares Maple, Matlab and Mathematica using known procedures.

6. Residual life and reversed failure rate functions

Given that a component survives up to time $t\geq0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X-t|X>t$. In reliability, it is well-known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely [15]. Therefore, we obtain the rth order moment of the residual life using the general formula

$$\mu_r(t)=\frac{1}{\overline{F}(t)}\int_t^\infty(x-t)^r f(x;\varphi)\,dx,\ r\geq1.$$

Applying the binomial expansion of $(x-t)^r$ and substituting $f(x;\varphi)$ given by (2.2) into the above formula and using the generalized binomial power series gives

$$\begin{aligned}\mu_r(t) &= \frac{\theta\beta\alpha^{\beta}(1-p)}{\overline{F}(t)} \sum_{m=0}^{r}\sum_{k,j=0}^{\infty}(-1)^{m+k}\binom{r}{m}(j+1)p^j\Bigg\{(1+\lambda)\tbinom{(j+1)\theta-1}{k}\,t^m \\ &\quad - \lambda(1-p)(j+2)\binom{(j+2)\theta-1}{k}\Bigg\}\int_t^{\infty} x^{r+\beta-m-1}\,\mathrm{e}^{-(\mathrm{k}+1)(\alpha\mathrm{x})^{\beta}}\mathrm{dx} \\ &= \frac{\theta(1-p)}{\overline{F}(t)} \sum_{m=0}^{r}\sum_{k,j=0}^{\infty}(-1)^{m+k}\binom{r}{m}(j+1)\,p^j\,t^m\Bigg\{(1+\lambda)\binom{(j+1)\theta-1}{k} \\ &\quad - \lambda(1-p)(j+2)\binom{(j+2)\theta-1}{k}\Bigg\}\left[\frac{\Gamma(\frac{r-m}{\beta}+1;(k+1)(\alpha t)^{\beta})}{\alpha^{r-m}(k+1)^{\frac{r-m}{\beta}+1}}\right],\end{aligned}\tag{6.1}$$

where $\Gamma(s;t)=\int_t^{\infty}x^{s-1}\,\mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the upper incomplete gamma function.

Another important characteristic of the TEWG model is the mean residual life (MRL) function obtained by setting $r=1$ in equation (6.1). The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate (FR) function. Lifetimes can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). The MRL function that first decreases (increases) and then increases (decreases) is usually called bathtub (upside-down bathtub) shaped, BMRL (UMRL). Ghitany [14], Mi [21], Park [30] and Tang *et al.* [33], among others, studied the relationship between the behaviors of the MLR and FR functions of a distribution.

7. Estimation and Inference

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the new distribution from complete samples only. Let $x_1,\ldots,x_n$ be a random sample of size n from the TEWG$(x;\phi)$ model, where $\phi=(\alpha,\beta,\theta,p,\lambda)^T$. The log likelihood function for the vector of parameters ϕ can be expressed as

$$\begin{aligned}\ell(\phi) &= n\log\theta + n\log\beta + n\beta\log\alpha + n\log(1-p) + (\beta-1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\alpha x_i)^{\beta} \\ &\quad - 2\sum_{i=1}^{n}\log\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right] + (\theta-1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right) \\ &\quad + \sum_{i=1}^{n}\log\left\{(1+\lambda) - 2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\right)\right\}.\end{aligned}$$

The corresponding score function is given by

$$U_n(\varphi) = \left(\frac{\partial\ell(\phi)}{\partial\alpha},\frac{\partial\ell(\phi)}{\partial\beta},\frac{\partial\ell(\phi)}{\partial\theta},\frac{\partial\ell(\phi)}{\partial p},\frac{\partial\ell(\phi)}{\partial\lambda}\right)^T.\tag{7.1}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained from (7.1), namely:

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\alpha} &= \tfrac{n\beta}{\alpha}-\beta\alpha^{\beta-1}\sum_{i=1}^{n}(x_i)^{\beta}+(\theta-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\beta\alpha^{\beta-1}(\mathrm{x_i})^{\beta}}{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}\\
&\quad+2\theta p\sum_{i=1}^{n}\frac{e^{-(\alpha x_{(i)})^{\beta}}\beta\alpha^{\beta-1}(x_i)^{\beta}\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]}\\
&\quad-2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\quad\times\left[\frac{(1-p)\theta\beta\alpha^{\beta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\mathrm{x_i})^{\beta}\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\beta} &= \tfrac{n}{\beta}+n\log\alpha+\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i)\\
&\quad+2p\theta\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x_i})^{\beta}\log(\alpha\mathrm{x_i})}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]}\\
&\quad+(\theta-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x_i})^{\beta}\log(\alpha\mathrm{x_i})}{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}\\
&\quad-2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\quad\times\left[\frac{\theta(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x_i})^{\beta}\log(\alpha\mathrm{x_i})}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\theta} &= \tfrac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)+2p\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\\
&\quad+\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\quad\times\left[\frac{(1-p)\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\frac{\partial \ell(\phi)}{\partial p} = \frac{-n}{1-p} + 2\sum_{i=1}^{n} \frac{\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{\left[1-p\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right]}$$

$$-2\lambda \sum_{i=1}^{n} \frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}}$$

$$\times \left[\frac{\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\left(1-\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right)}{\left[1-p\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right]^{2}}\right] = 0,$$

and

$$\frac{\partial \ell(\phi)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}} = 0.$$

The above equations cannot be solved analytically but statistical software can be used to solve them numerically, for example, through the R-language or any iterative methods such as the NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing) and L-BFGS-B (Limited-Memory Quasi-Newton code for Bound-Constrained Optimization).

The modified Anderson-Darling (A^*) and the modified Cramér-von Mises (W^*) statistics are widely used to determine how closely a specific cdf $F(\cdot)$ fits the empirical distribution for a given data set. The statistics A^* and W^* are given by

$$A^* = \left(\frac{2.25}{n^2} + \frac{0.75}{n} + 1\right)\left[-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(z_i\left(1-z_{n-i+1}\right)\right)\right],$$

and

$$W^* = \left(\frac{0.5}{n} + 1\right)\left[\sum_{i=1}^{n}\left(z_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}\right],$$

respectively, where $z_i = F(y_{(i)})$, and the $y_{(i)}$'s are the ordered observations.

The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of them were tabulated by [25].

8. Application to the carbon fibres

We provide an application to a real data set to prove the flexibility of the TEWG distribution. We fit the gamma exponentiated exponential (GEE) [29], exponentiated Weibull–geometric (EWG) [19], extended Weibull (ExtW) [26], Kumaraswamy modified Weibull (KwMW) [9] and TEWG distributions to a real data on "carbon fibres" [25]. The parameters of the following distributions are estimated by maximizing the log-likelihood using the *NMaximize* procedure in the symbolic computational package *Mathematica*. The density functions (for $x > 0$) associated to these models are given by:

- The GEE density function,

$$f(x) = \frac{\lambda\,\alpha^{\delta}\,\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}\left(-\log\left(1-\mathrm{e}^{-\lambda x}\right)\right)^{\delta-1}}{\Gamma(\delta)}, \quad \lambda, \alpha, \delta, x > 0.$$

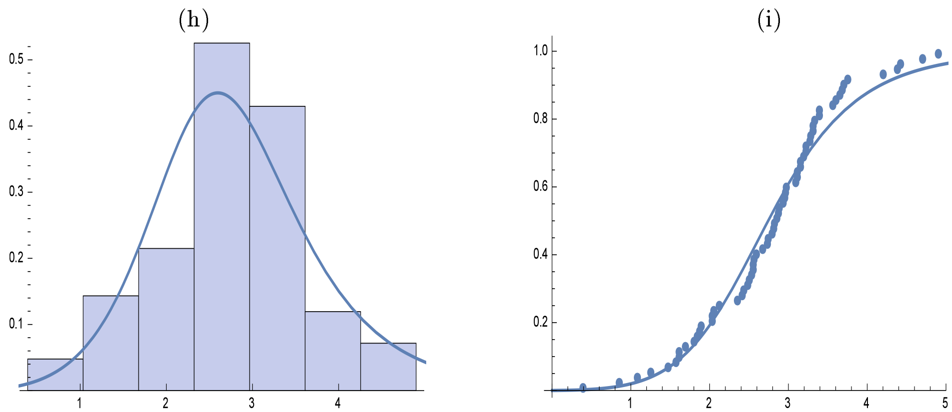

Figure 4. (a) The estimated TEWG density superimposed on the histogram for the carbon fibres. (b) The empirical cdf and the estimated TEWG cdf.

Table 1. MLEs of the parameters (standard errors in parentheses) for the carbon fibres

Distributions	Estimates				
GEE $(\lambda, \alpha, \delta)$	0.26555	10.0365	7.23658		
	(0.21621)	(2.59504)	(7.05288)		
EWG $(\alpha, \theta, \beta, p)$	520.24	0.35943	177.132	0.999778	
	(332.051)	(0.02509)	(207.54)	(0.00262)	
ExtW (a, b, c)	16.1979	0.001	8.05671		
	(25.7118)	(0.938764)	(1.65309)		
KwMW$(\alpha, \gamma, \lambda, a, b)$	0.14981	1.7994	0.49987	0.64975	0.171114
	(0.326517)	(2.40813)	(0.616749)	(1.13328)	(0.529126)
TEWG $(\alpha, \theta, \beta, p, \lambda)$	59.2556	0.455874	1.42577	0.999917	-0.447535
	(27.5648)	(0.03366)	(1.60102)	(0.00937)	(0.49717)

- The ExtW density function,

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\,\mathrm{e}^{-c/x - a x^{b} \mathrm{e}^{-c/x}}, \quad a > 0\,,\ b > 0\,,\ c \geq 0\,,\ x > 0.$$

- The KwMW density function,

$$f(x) = a\,b\,\alpha\,x^{\gamma-1}(\gamma + \lambda\,x)\,\exp\left(\lambda\,x - \alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\,\left[1 - \exp\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\right]^{a-1}$$
$$\times\,\left\{1 - \left[1 - \exp\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\right]^{a}\right\}^{b-1},$$

where $a > 0\,, b > 0,\ \alpha > 0\,, \gamma > 0\,, \lambda \geq 0.$

The estimated pdf and cdf of the TEWG distribution fitted to the uncensored breaking stress of carbon fibres (in Gba) reported by [8] are displayed in Figure 4. The estimates of the parameters and their standard errors (SEs) are listed in Table 1. The values of the statistics A^* and W^*, Akaike Information Criterion (AIC), Bayesian Information Criterioon (BIC), Hannan–Quinn Information Criterion (HQIC) and Consistent Akaike

Table 2. Goodness-of-fit statistics for the carbon fibres

Distributions	A^*	W^*	AIC	BIC	HQIC	CAIC
GEE (λ, α, δ)	1.43415	0.266823	189.787	196.356	192.383	190.175
EWG (α, β, θ, p)	0.789187	0.121661	118.164	127.922	122.625	119.82
ExtW (a , b , c)	2.26745	0.416152	207.471	214.04	210.067	207.858
KwMW($\alpha\,\gamma$, λ , a , b)	1.28891	0.212227	180.676	191.624	185.002	181.676
TEWG (α, β, θ, p, λ)	0.77199	0.12016	117.586	128.534	121.912	118.586

Information Criterion (CAIC) are also given in Table 2. We note that the TEWG model provides the best fit among these models.

To compare the TEWG model with its EWG sub-model, the likelihood-ratio (LR) statistic is given by $w = 4.54198$ with p-value 0.033. The value if the LR statistic suggests that the TEWG model performs significantly better than its sub-model EWG.

9. Conclusions

We propose a new compounding lifetime model named the transmuted exponentiated Weibull geometric distribution, and study some of its general structural properties. The proposed model includes at least ten special lifetime models. A very useful mixture representation for its density function is derived. We provide explicit expressions for the moments and incomplete moments, generating and quantile functions, mean deviations and order statistics. These expressions are manageable using analytic and numerical computer resources, which may turn into adequate tools comprising the arsenal of applied statisticians. The model parameters are estimated by maximum likelihood. We prove that the proposed model can be superior to some models generated from other know families in terms of model fitting by means of an application to a real data set.

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