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Ulam-Hyers-Stability for nonlinear fractional neutral differential equations

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Abstract

We discuss Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability for a class of nonlinear fractional functional differential equations with delay involving Caputo fractional derivative by using Picard operator. An example is also given to show the applicability of our results.

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1. Introduction

Fractional differential equations is the area of concentration of recent research and there has been significant progress in this area. However, the concept of fractional derivative is as old as differential equations. L' Hospital in 1695 wrote a letter to Leibniz related to his generalization of differentiation and raised the question about fractional derivative. Nowadays the fractional order differential equations has proved to be the most valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find many applications in electromagnetic, control, electrochemistry etc. (see[5–8]). For more details on this area, one can see the monograph of Kilbas et al. [14], I. Podulbny [23], Miller and Ross [17], Li et al. [15, 16], Rehman et al. [25] , Chen et al. [1–4], Saeed [27] and the references therein.

Over last three decades, the stability theory for functional equations developed and it got popularity so quickly. It started in 1940, when the stability of functional equations were originally raised by Ulam at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping"? (for more details see [29]). The first answer to the question of Ulam [9] was given by Hyers in 1941 in the case of Banach spaces. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [24] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables.

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Subsequently, a large number of mathematicians took these two types of stabilities, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability to carry on their researches and the study of this area has grown to be one of the central and most essential subjects in the mathematical analysis area. For more details on the recent advances on the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of differential equations, one can see the monographs [10, 11] and the research papers [12, 13, 18–21, 26, 28, 30, 32–35]. However, to the best of our knowledge, most of the authors discuss the stability for implicit functions but in our paper we take neutral functions and as far as we know Ulam's type stability results for a class of nonlinear neutral functional differential equations involving Caputo fractional derivatives have not been investigated yet. Wherefore, motivated by the above articles, we have discussed Ulam-Hyers stability for initial value problems of fractional differential equations with delay

$$\begin{cases} {}^cD_0^\gamma x(t) = f(t, x_t, {}^cD_0^\delta x_t), \ t \in I, \\ x(t) = \psi(t), \ t \in [-\tau, 0] \\ x(0) = x_0, \ x'(0) = x_1. \end{cases}$$

where ${}^cD_0^\gamma$ and ${}^cD_0^\delta$ are Caputo derivatives with $I = [0,1]$, $1 < \gamma < 2$, $0 < \delta < 1$, and x_0, x_1 are real constants, $f : [0,1] \times C_\tau \times C_\tau \longrightarrow \mathbb{R}$ and $\psi : [-\tau, 0] \longrightarrow \mathbb{R}$ are continuous, we denote C_τ the Banach space of all continuous functions $\phi : [-\tau, 1] \to \mathbb{R}$, endowed with the maximum norm $||\phi|| = \max\{|\phi(s)|; -\tau \leq s \leq 1\}$. If $x : [-\tau, 1] \to \mathbb{R}$, then for any $t \in I$, and $x_t \in C_\tau$ we denote x_t by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-\tau, 0]$, $\tau > 0$.

The paper is arranged as follows: In Section 2 we review some basic definitions and lemmas used throughout this paper. In the third section we establish Ulam-Hyers stability, Ulam-Hyers-Rassias stability, Generalized Ulam-Hyers-Rassias stability for the above initial value problem and in the last section an example is given to show the applicability of our results.

2. Preliminaries

This part includes some basic definitions and results used throughout this paper.

Definition 2.1. [14] The Gamma function is defined as,

$$\Gamma(\gamma) = \int_0^\infty e^{-t} t^{\gamma - 1} dt, \quad \gamma > 0.$$

One of the basic property of Gamma function is that it satisfies the following functional equation: $\Gamma(\gamma + 1) = \gamma \Gamma(\gamma)$.

Definition 2.2. [14] The fractional integral for a function f with lower limit t_0 and order γ can be defined as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad \gamma > 0, \ t > t_0.$$

where Γ is the Gamma function, and right hand side is point-wise defined on $\mathbb{R}^+$.

Definition 2.3. [14] The left Caputo fractional derivative of order γ is given by

$${}^cD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds.$$

where $n = [\gamma] + 1$ ($[\gamma]$ stands for the bracket function of γ). Here we define one of the important property of Caputo derivative as the composition of the fractional integration operator I_t^γ with the fractional differentiation operator ${}^cD_t^\gamma$. Let $\gamma > 0$, $n = [\gamma] + 1$ and let $f_{n-\gamma}(t) = {}^c D_t^{n-\gamma} f(t)$ then if $f \in C^n[a, b]$ then

$$I_t^\gamma {}^cD_t^\gamma f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

Lemma 2.4. (Gronwall lemma)[22] *Let $\mu, \nu \in C([0,1], \mathbb{R}^+)$. Suppose that μ is increasing. If $x \in C([0,1], \mathbb{R}^+)$ is a solution to the inequality*

$$x(t) \leq \mu(t) + \int_0^t \nu(s)x(s)ds, \quad t \in [0,1],$$

then

$$x(t) \leq \mu(t) \exp\left(\int_0^t \nu(s)ds\right), \quad t \in [0,1].$$

Definition 2.5. [22] Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $u^* \in X$ such that

(i) $F_A = \{u^*\}$ where $F_A = \{\mu \in X : A(\mu) = \mu\}$ is the fixed point set of A.
(ii) The sequence $(A^n(\mu_0))_{n \in N}$ converges to u^* for all $\mu_0 \in X$.

Definition 2.6. [22] Let $(X, d, \leq)$ be an ordered metric space. An operator $A : X \to X$ is an increasing Picard operator $F_A = \mu^*$, then for $\mu \in X$, $\mu \leq A(\mu) \Rightarrow \mu \leq \mu^*$ while $\mu \geq A(\mu) \Rightarrow \mu \geq \mu^*$.

3. Stability

In this section, we will discuss Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability for a class of fractional neutral differential equations. Let ϵ be a positive real number, $T : X \to X$ is a continuous operator and $f : [0,1] \times C_\tau \times C_\tau \to \mathbb{R}$ is a continuous function, we consider the following differential equation

$$\begin{cases} {}^cD_0^\gamma x(t) = f(t, x_t, {}^cD_0^\delta x_t), \ t \in [0,1], \\ x(t) = \psi(t), \ t \in [-\tau, 0], \\ x(0) = x_0, \ x'(0) = x_1. \end{cases} \tag{3.1}$$

For equation (3.1), for some $\epsilon > 0$, $\phi \in C([-\tau, 1], \mathbb{R}^+)$, we focus on the following inequalities:

$$|{}^cD_0^\gamma y(t) - f(t, y_t, {}^cD_0^\delta y_t)| \leq \epsilon, \ t \in [0,1]. \tag{3.2}$$

$$|{}^cD_0^\gamma y(t) - f(t, y_t, {}^cD_0^\delta y_t)| \leq \phi(t), \ t \in [0,1]. \tag{3.3}$$

$$|{}^cD_0^\gamma y(t) - f(t, y_t, {}^cD_0^\delta y_t)| \leq \epsilon\phi(t), \ t \in [0,1]. \tag{3.4}$$

Definition 3.1. [31] Equation(3.1) is Ulam-Hyers stable if there exists a positive real number c_1 such that for each positive ϵ and for every solution $y \in C^1([-\tau, 1], \mathbb{R})$ of (3.2) there exists a solution $x \in C^1([-\tau, 1], \mathbb{R})$ of (3.1) with $|y(t) - x(t)| \leq c_1\epsilon, \ t \in [-\tau, 1]$.

Definition 3.2. [31] Equation(3.1) is Generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists $c_{1\phi} > 0$ such that for each solution $y \in C^1([-\tau, 1], \mathbb{R})$ to (3.3) there exists a solution $x \in C^1([-\tau, 1], \mathbb{R})$ to (3.1) with $|y(t) - x(t)| \leq c_{1\phi}\phi(t)$, $t \in [-\tau, 1]$.

Definition 3.3. [31] Equation(3.1) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists $c_{1\phi} > 0$ such that for each solution $y \in C^1([-\tau, 1], \mathbb{R})$ to (3.4) there exists a solution $x \in C^1([-\tau, 1], \mathbb{R})$ to (3.1) with $|y(t) - x(t)| \leq c_{1\phi}\epsilon\phi(t)$, $t \in [-\tau, 1]$.

Remark 3.4. A solution of differential equation is stable (asymptotically stable) if it attracts all other solutions with sufficiently close initial values.

On the other hand, in Hyers-Ulam stability, we compare solution of given differential equation with the solution of differential inequality. We say solution of differential equation is stable if it stays close to solution of differential inequality.

Hyers-Ulam stability may not imply the asymptotic stability.

Remark 3.5. [31] A function $y \in C^1([0,1],\mathbb{R})$ is a solution of the inequality (3.2) if and only if there exist $h \in C^1([0,1],\mathbb{R})$ such that

(i) $|h(t)| \le \epsilon, \quad t \in [0,1]$,
(ii) ${}^cD_0^\gamma y(t) = f(t, y_t, {}^cD_0^\delta y_t) + h(t), \quad t \in [0,1]$.

Also a function $y \in C^1([0,1],\mathbb{R})$ is a solution of the inequality (3.3) if and only if there exist $\tilde{h} \in C^1([0,1],\mathbb{R})$ such that

(i) $|\tilde{h}(t)| \le \phi(t), \quad t \in [0,1]$,
(ii) ${}^cD_0^\gamma y(t) = f(t, y_t, {}^cD_0^\delta y_t) + \tilde{h}(t), \quad t \in [0,1]$.

Similarly for (3.4) there exist a function $g \in C^1([0,1],\mathbb{R})$ such that

(i) $|g(t)| \le \epsilon\phi(t), \quad t \in [0,1]$,
(ii) ${}^cD_0^\gamma y(t) = f(t, y_t, {}^cD_0^\delta y_t) + g(t), \quad t \in [0,1]$.

Remark 3.6. Let $1 < \gamma < 2$ and $0 < \delta < 1$ if $y \in C^1([0,1],\mathbb{R})$ is a solution of inequality (3.2) then y is a solution of the following inequality

$$\left| y(t) - y(0) - y'(0)t - \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^cD_0^\delta y_s)ds \right| \le \frac{t^\gamma \epsilon}{\Gamma(\gamma+1)}, \quad t \in [0,1].$$

From Remark(3.5) we have

$${}^cD_0^\gamma y(t) = f(t, y_t, {}^cD_0^\delta y_t) + h(t).$$

Then

$$\begin{aligned} y(t) - y(0) - y'(0)t &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^cD_0^\delta y_s)ds \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} h(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned} \left| y(t) - y(0) - y'(0)t - \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^cD_0^\delta y_s)ds \right| &\le \frac{1}{\Gamma(\gamma)}\frac{t^\gamma \epsilon}{\gamma} \\ &\le \frac{t^\gamma \epsilon}{\Gamma(\gamma+1)}. \end{aligned}$$

If $y \in C^1([0,1],\mathbb{R})$ is a solution of inequality (3.4) then y is a solution of the following inequality

$$\begin{aligned} \Big| y(t) - y(0) - y'(0)t &- \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^cD_0^\delta y_s)ds \Big| \\ &\le \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\phi(s)ds, \quad t \in [0,1]. \end{aligned}$$

And for inequality (3.4)

$$\begin{aligned} \Big| y(t) - y(0) - y'(0)t &- \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^cD_0^\delta y_s)ds \Big| \\ &\le \frac{\epsilon}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\phi(s)ds, \quad t \in [0,1]. \end{aligned}$$

In the following theorems we will prove the Ulam-Hyers stability, Generalized Ulam-Hyers-Rassias stability and Ulam-Hyers-Rassias stability for equation(3.1) on the interval $I = [0,1]$.

Theorem 3.7. *Suppose that*

(a) $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $|{}^cD_0^\delta x(t)| \le \frac{1}{\Gamma(2-\delta)}|x(t)|$;

(b) there exists $Q > 0$ such that for every $t \in [0,1]$, $\mu_i, \nu_i \in \mathbb{R}$, $i = 1, 2$

$$|f(t,\mu_1,\mu_2) - f(t,\nu_1,\nu_2)| \leq Q \sum_{i=1}^{2} |\mu_i - \nu_i|,$$

and $\frac{Q}{\Gamma(\gamma)}[\frac{1}{\gamma} + \frac{1}{\Gamma(2-\delta)}] = k < 1$*. Then*

(i) Problem (3.1) has a unique solution in $C^1([-\tau,1],\mathbb{R}) \cap C^1([0,1],\mathbb{R})$.
(ii) Equation (3.1) is Ulam-Hyers stable.

Proof. (i) Under the condition (a), (3.1) is equivalent to the integral equation

$$x(t) = \begin{cases} x_0 + x_1 t + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x_s, {}^c D_0^\delta x_s) ds, & t \in [0,1] \\ \psi(t), & t \in [-\tau, 0]. \end{cases}$$

Let $X = \{x \in C[-\tau,1]; {}^c D_0^\delta x \in C^1[-\tau,1]\}$ with $||x|| = \max_{t\in I} |x(t)| + \max_{t \in I} |{}^c D_0^\delta x(t)|$, here $C[-\tau,1]$ and $C^1[-\tau,1]$ are denoted as continuous and continuously differentiable sets and $T : X \to X$ be given by

$$Tx(t) = \begin{cases} x_0 + x_1 t + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x_s, {}^c D_0^\delta x_s) ds, & t \in [0,1] \\ \psi(t), & t \in [-\tau, 0]. \end{cases}$$

Here we will show that T is a contraction on X.
$|Tx(t) - Ty(t)| = 0, \quad x, y \in C([-\tau,1],\mathbb{R}), \quad t \in [-\tau, 0]$. And for $t \in [0,1]$, by using

$$\begin{aligned} \max_{0\leq s\leq t} |x_s - y_s| &= \max_{0\leq s\leq t} |x(s+\theta) - y(s+\theta)| \\ &= \max_{\theta\leq s+\theta\leq t+\theta} |x(s+\theta) - y(s+\theta)| \\ &\leq \max_{-\tau\leq \bar{s}\leq t} |x(\bar{s}) - y(\bar{s})|, \text{where } s+\theta = \bar{s}, \text{ and } -\tau \leq \theta < 0 \\ &\leq \max_{-\tau\leq \bar{s}\leq 1} |x(\bar{s}) - y(\bar{s})| \\ &= ||x - y||. \end{aligned}$$

Thus

$$\begin{aligned} &|Tx(t) - Ty(t)| \\ &\leq \frac{1}{\Gamma(\gamma)} \left| \int_0^t (t-s)^{\gamma-1} f(s, x_s, {}^c D_0^\delta x_s) ds - \int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^c D_0^\delta y_s) ds \right| \\ &\leq \frac{Q}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Big(|x_s - y_s| + |{}^c D_0^\delta x_s - {}^c D_0^\delta y_s| \Big) ds \\ &\leq \frac{Q}{\Gamma(\gamma)} \Big(\max_{0\leq s\leq t} |x_s - y_s| + \max_{0\leq s\leq t} |{}^c D_0^\delta x_s - {}^c D_0^\delta y_s| \Big) \int_0^t (t-s)^{\gamma-1} ds \\ &\leq \frac{Q}{\Gamma(\gamma+1)} \Big(\max_{-\tau\leq \bar{s}\leq 1} |x(\bar{s}) - y(\bar{s})| + {}^c D_0^\delta \max_{-\tau\leq \bar{s}\leq 1} |x(\bar{s}) - y(\bar{s})| \Big) \\ &\leq \frac{Q}{\Gamma(\gamma+1)} \Big(||x - y|| + {}^c D_0^\delta ||x - y|| \Big) \\ &\leq \frac{Q}{\Gamma(\gamma+1)} ||x - y||. \end{aligned}$$

Also

$$\begin{aligned}
&|D^{\delta}Tx(t)-D^{\delta}Ty(t)|\\
&\leq \frac{1}{\Gamma(1-\delta)}\Big|\int_0^t (t-s)^{-\delta}\Big(\frac{\gamma-1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-2}f(s,x_s,{}^cD_0^{\delta}x_s)ds\\
&-\frac{\gamma-1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-2}f(s,y_s,{}^cD_0^{\delta}y_s)ds\Big)ds\Big|\\
&\leq \frac{Q}{\Gamma(1-\delta)\Gamma(\gamma)}\int_0^t (t-s)^{-\delta}ds||x-y||\\
&\leq \frac{Q}{\Gamma(2-\delta)\Gamma(\gamma)}||x-y||.
\end{aligned}$$

So,

$$\begin{aligned}
||Tx-Ty|| &\leq \frac{Q}{\Gamma(\gamma+1)}||x-y||+\frac{Q}{\Gamma(2-\delta)\Gamma(\gamma)}||x-y||\\
&\leq \frac{Q}{\Gamma(\gamma)}[\frac{1}{\gamma}+\frac{1}{\Gamma(2-\delta)}]||x-y||.
\end{aligned}$$

Therefore $||Tx(t)-Ty(t)|| \leq k||x-y||$. Hence by Banach contraction principle T is a contraction.

(ii) Let $y \in C^1([0,1],\mathbb{R})$ be the solution of (3.2), let us denote by $x \in C^1([0,1],\mathbb{R})$ the unique solution of equation(3.1) i.e

$$\begin{cases}
{}^cD_0^{\gamma}x(t)=f(t,x_t,{}^cD_0^{\delta}x_t),\ t\in[0,1],\\
x(t)=y(t),\ t\in[-\tau,0],\\
x(0)=x_0,\ x'(0)=x_1.
\end{cases}$$

then we have

$$\begin{aligned}
x(t) &= x(0)+x'(0)t+\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s,x_s,{}^cD_0^{\delta}x_s)ds\\
&= y(0)+y'(0)t+\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s,x_s,{}^cD_0^{\delta}x_s)ds.
\end{aligned}$$

We can see that $|y(t)-x(t)|=0$ for $t\in[-\tau,0]$. For $t\in[0,1]$ we have

$$\begin{aligned}
&|y(t)-x(t)| \\
\leq\ & \left|y(t)-y(0)-y'(0)t-\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x_s,{}^c D_0^\delta x_s)ds\right| \\
\leq\ & \left|y(t)-y(0)-y'(0)t-\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,y_s,{}^c D_0^\delta y_s)ds\right. \\
&+\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,y_s,{}^c D_0^\delta y_s)ds \\
&\left.-\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x_s,{}^c D_0^\delta x_s)ds\right| \\
\leq\ & \frac{t^\gamma \epsilon}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|f(s,y_s,{}^c D_0^\delta y_s) \\
&\qquad -f(s,x_s,{}^c D_0^\delta x_s)|ds.
\end{aligned}$$

Using (b)

$$\begin{aligned}
\leq\ & \frac{t^\gamma \epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Big(|y_s-x_s|+|{}^c D_0^\delta y_s-{}^c D_0^\delta x_s|\Big)ds \\
\leq\ & \frac{t^\gamma \epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\Big(\int_0^t (t-s)^{\gamma-1}|y_s-x_s|ds \\
+\ & \int_0^t (t-s)^{\gamma-1}|{}^c D_0^\delta y_s-{}^c D_0^\delta x_s|ds\Big).
\end{aligned} \tag{3.5}$$

According to the last inequality for $\mu\in C([-\tau,1],\mathbb{R}^+)$. We consider the operator $A: C([-\tau,1],\mathbb{R}^+)\to C([-\tau,1],\mathbb{R}^+)$ defined by

$$A\mu(t)=\begin{cases} 0, & t\in[-\tau,0], \\ \frac{t^\gamma\epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mu_s ds & \\ +\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}{}^c D_0^\delta \mu_s ds, \quad t\in[0,1]. & \end{cases}$$

For proving A is a Picard operator, we prove that A is a contraction.

$$\begin{aligned}
&|A\mu(t)-A\nu(t)| \\
\leq\ & \frac{Q}{\Gamma(\gamma)}\Big(\int_0^t (t-s)^{\gamma-1}|\mu_s-\nu_s|ds+\int_0^t (t-s)^{\gamma-1}|{}^c D_0^\delta \mu_s-{}^c D_0^\delta \nu_s|ds\Big) \\
\leq\ & \frac{Qt^\gamma}{\gamma\Gamma(\gamma)}\Big(\max_{0\leq s\leq t}|\mu_s-\nu_s|+\max_{0\leq s\leq t}|{}^c D_0^\delta \mu_s-{}^c D_0^\delta \nu_s|\Big) \\
\leq\ & \frac{Q}{\Gamma(\gamma+1)}||\mu-\nu||.
\end{aligned}$$

And

$$\begin{aligned}
&|D^{\delta}A\mu(t)-D^{\delta}A\nu(t)|\\
&\leq \frac{1}{\Gamma(1-\delta)}\Big[\int_0^t (t-s)^{-\delta}\Big(\frac{Q(\gamma-1)}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-2}|\mu_s-\nu_s|ds\Big)ds\\
&+\int_0^t (t-s)^{-\delta}\Big(\frac{Q(\gamma-1)}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-2}|^cD_0^{\delta}\mu_s -^c D_0^{\delta}\nu_s|ds\Big)ds\Big]\\
&\leq \frac{Q(\gamma-1)}{\Gamma(1-\delta)(\gamma-1)\Gamma(\gamma)}||\mu-\nu||\int_0^t (t-s)^{-\delta}ds\\
&\leq \frac{Q}{\Gamma(2-\delta)\Gamma(\gamma)}||\mu-\nu||.
\end{aligned}$$

So

$$\begin{aligned}
||A(\mu)-A(\nu)|| &\leq \Big[\frac{Q}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(2-\delta)\Gamma(\gamma)}\Big]||\mu-\nu||\\
&\leq \frac{Q}{\Gamma(\gamma)}[\frac{1}{\gamma}+\frac{1}{\Gamma(2-\delta)}]||\mu-\nu||.
\end{aligned}$$

for all $\mu,\nu \in C([-\tau,1],\mathbb{R}^+)$. Therefore $||A(\mu)-A(\nu)|| \leq k||\mu-\nu||$ for all $\mu,\nu \in C([-\tau,1],\mathbb{R}^+)$. Hence A is a contraction with respect to the norm on X. By applying the Banach contraction principle, we can say that A is a Picard operator and $F_A=\{\mu^*\}$, then

$$u^*(t) \leq \frac{t^{\gamma}\epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u_s^*ds+\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1c}D_0^{\delta}u_s^*ds.$$

The solution $u^*(t)$ is increasing and ${}^cD_0^{\delta}u^*>0$, also

$$\begin{aligned}
u^*(t) &\leq \frac{t^{\gamma}\epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u^*(s)ds\\
&\quad+\frac{Q}{\Gamma(\gamma)\Gamma(2-\delta)}\int_0^t (t-s)^{\gamma-1}u^*(s)ds\\
u^*(t) &\leq \frac{t^{\gamma}\epsilon}{\Gamma(\gamma+1)}+\frac{Q}{\Gamma(\gamma)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big)\int_0^t (t-s)^{\gamma-1}u^*(s)ds.
\end{aligned}$$

therefore by using Gronwall lemma, we can say that

$$\begin{aligned}
u^*(t) &\leq \frac{t^{\gamma}\epsilon}{\Gamma(\gamma+1)}\exp\frac{Q}{\Gamma(\gamma)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big)\int_0^t (t-s)^{\gamma-1}ds, \quad t\in[0,1]\\
&\leq \frac{\epsilon}{\Gamma(\gamma+1)}\exp\frac{Q}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big)\\
u^*(t) &\leq c_1\epsilon, \quad \text{where } c_1=\frac{1}{\Gamma(\gamma+1)}\exp\frac{Q}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big).
\end{aligned}$$

So particularly, from (3.5) if $u=|y-x|$, then $u(t)\leq Au(t)$ and by applying the abstract Gronwall lemma we get $u(t)\leq u^*(t)$. Thus it follows

$$|y(t)-x(t)|\leq c_1\epsilon, \quad t\in[-\tau,1].$$

Hence equation (3.1) is Ulam-Hyers stable. □

Theorem 3.8. *If*

(a) $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $\tilde{h} \in C^1([0,1], \mathbb{R})$. $|\tilde{h}(t)| \leq \phi(t)$, $\tilde{h} > 0$;

(b) there exists $Q_f \in L^1([0,1], \mathbb{R}^+)$ *such that for all* $t \in [0,1]$, $\mu_i, \nu_i \in \mathbb{R}$, $i = 1, 2$

$$|f(t, \mu_1, \mu_2) - f(t, \nu_1, \nu_2)| \leq Q_f(t) \sum_{i=1}^{2} |\mu_i - \nu_i|.$$

(c) the function $\phi \in C([0,1], \mathbb{R}^+)$ *is an increasing function and there exists* $\lambda_\phi > 0$ *such that*

$$\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \phi(s) ds \leq \lambda_\phi \phi(t); \text{ for all } t \in [0,1].$$

Then equation (3.1) has a unique solution in $C^1([-\tau, 1], \mathbb{R}) \cap C^1([0,1], \mathbb{R})$ *which is generalized Ulam-Hyers-Rassias stable with respect to* ϕ.

Proof. The proof follows the same steps as in Theorem (3.7). Let $y \in C^1([-\tau, 1], \mathbb{R}) \cap C^1([0,1], \mathbb{R})$ be a solution to (3.3) then by previous theorem $x \in C^1([-\tau, 1), \mathbb{R}) \cap C^1([0,1], \mathbb{R})$ is a unique solution to the Cauchy problem

$$\begin{cases} {}^cD_0^\gamma x(t) = f(t, x_t, {}^c D_0^\delta x_t), t \in [0,1], \\ x(t) = y(t), t \in [-\tau, 0], \\ x(0) = x_0, \ x'(0) = x_1. \end{cases}$$

So

$$x(t) = \begin{cases} x(0) + x'(0)t + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x_s, {}^c D_0^\delta x_s) ds, & t \in [0,1] \\ y(t). & t \in [-\tau, 0]. \end{cases}$$

Remarks (3.5) and (3.6) imply

$$\begin{aligned} \left| y(t) - y(0) - y'(0)t \right. &- \left. \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^c D_0^\delta y_s) ds \right| \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \phi(s) ds \\ &\leq \lambda_\phi \phi(t), \quad t \in [0,1]. \end{aligned}$$

From Theorem (3.7) we can see that $|y(t) - x(t)| = 0$, for $t \in [-\tau, 0]$. For $t \in [0,1]$ we have

$$\begin{aligned} & |y(t) - x(t)| \\ \leq & \left| y(t) - y(0) - y'(0)t - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^c D_0^\delta y_s) ds \right. \\ + & \left. \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, y_s, {}^c D_0^\delta y_s) ds - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x_s, {}^c D_0^\delta x_s) ds \right| \\ \leq & \lambda_\phi \phi(t) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} |f(s, y_s, {}^c D_0^\delta y_s) - f(s, x_s, {}^c D_0^\delta x_s)| ds \\ \leq & \lambda_\phi \phi(t) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} Q_f(s) |y_s - x_s| ds \\ + & \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} Q_f(s) |{}^c D_0^\delta y_s - {}^c D_0^\delta x_s| ds. \end{aligned}$$

From the proof of Theorem (3.7) it follows that

$$\begin{aligned}|y(t)-x(t)| &\leq \lambda_\phi \phi(t) \exp \frac{Q_f(s)}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big)(t)^\gamma, \quad t\in[0,1]\\ &\leq c_{1\phi}\phi(t), \quad \text{where } c_{1\phi}=\lambda_\phi \exp\frac{Q_f(s)}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big).\end{aligned}$$

Hence equation (3.1) is generalized Ulam-Hyers-Rassias stable. □

Theorem 3.9. *Suppose that*

(a) $f\in C(I\times\mathbb{R}^2,\mathbb{R})$, $g\in C^1([0,1],\mathbb{R})$, $|g(t)|\leq \epsilon\phi(t)$, $g>0$;

(b) *there exists* $l_f>0$ *such that for all* $t\in[0,1]$, $\mu_i,\nu_i\in\mathbb{R}$, $i=1,2$

$$|f(t,\mu_1,\mu_2)-f(t,\nu_1,\nu_2)|\leq l_f\sum_{i=1}^{2}|\mu_i-\nu_i|$$

with $\frac{l_f}{\Gamma(\gamma)}[\frac{1}{\gamma}+\frac{1}{\Gamma(2-\delta)}]<1$.

(c) *the function* $\phi\in C([0,1],\mathbb{R}^+)$ *is an increasing function there exists* $\lambda_\phi>0$ *such that*

$$\frac{\epsilon}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\phi(s)ds\leq \lambda_\phi\phi(t);\ \ for\ all\ t\in[0,1].$$

Then equation (3.1) has a unique solution in $C^1([-\tau,1],\mathbb{R})\cap C^1([0,1],\mathbb{R})$ *and is Ulam-Hyers-Rassias stable with respect to* ϕ.

Proof. Following the same steps as in Theorems (3.7) and (3.8), we can find both the results, i.e here we will get

$$\begin{aligned}&|y(t)-x(t)|\\ \leq\ & \epsilon\lambda_\phi\phi(t)+\frac{l_f}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|y_s-x_s|ds\\ &+\frac{l_f}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|{}^cD_0^\delta y_s-{}^c D_0^\delta x_s|ds\\ \leq\ & \epsilon\lambda_\phi\phi(t)\exp\frac{l_f}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big)(t)^\gamma, \quad t\in[0,1]\\ \leq\ & c_{1\phi}\epsilon\phi(t), \quad \text{where } \ c_{1\phi}=\lambda_\phi\exp\frac{l_f}{\Gamma(\gamma+1)}\Big(1+\frac{1}{\Gamma(2-\delta)}\Big).\end{aligned}$$

So equation (3.1) is Ulam-Hyers-Rassias stable. □

4. Examples

In this section, we present an example to explain the applicability of main results.

Example 4.1. Consider the initial value problem

$$\begin{cases} D_0^{\frac{3}{2}}x(t)=Ax(t)+Bx(t-0.1)+CD_0^{\frac{1}{2}}x(t-0.1), t\in[0,1]\\ x(t)=0.2, t\in[-0.1,0],\\ x(0)=x'(0)=0.\end{cases} \tag{4.1}$$

and the inequalities

$$|D_0^{\frac{3}{2}}y(t)-Ay(t)-By(t-0.1)-CD_0^{\frac{1}{2}}y(t-0.1)| \ \leq\ \epsilon_1, \quad t\in[0,1], \tag{4.2}$$

$$|D_0^{\frac{3}{2}}y(t)-Ay(t)-By(t-0.1)-CD_0^{\frac{1}{2}}y(t-0.1)| \ \leq\ \phi(t), \ t\in[0,1], \tag{4.3}$$

$$|D_0^{\frac{3}{2}}y(t) - Ay(t) - By(t-0.1) - CD_0^{\frac{1}{2}}y(t-0.1)| \leq \epsilon_1\phi(t), \ t \in [0,1], \quad (4.4)$$

where $A = \frac{1}{9}, \ B = \frac{1}{9}, \ C = \frac{1}{9}$. For proving that equation (4.1) is Ulam-Hyers stable, we take the conditions as in Theorem (3.7) i.e a function $y \in C^1([0,1],\mathbb{R})$ is a solution of the inequality (3.6) if and only if there exists $h \in C^1([0,1],\mathbb{R})$ such that

$$\begin{cases} |h(t)| \leq \epsilon_1, \quad t \in [0,1], \\ D_0^{\frac{3}{2}}y(t) = Ay(t) + By(t-0.1) + CD_0^{\frac{1}{2}}y(t-0.1) + h(t), \quad t \in [0,1]. \end{cases} \quad (4.5)$$

Here $\gamma = \frac{3}{2}$, $\delta = \frac{1}{2}$, and $Q = \frac{1}{3}$ also $\frac{Q}{\Gamma(\gamma)}[\frac{1}{\gamma} + \frac{1}{\Gamma(2-\delta)}] \approx 0.6752 < 1$. Furthermore all the assumptions of Theorem (3.7) are satisfied, thus problem (4.1) has a unique solution and is Ulam-Hyers stable with

$$|y(t) - x(t)| \leq c_1\epsilon, \quad t \in [-0.1, 1].$$

where $c_1 \approx 1.2823 > 0$.

Remark 4.2. If we replace equation(4.5) by the inequality

$$\begin{aligned} |\tilde{h}(t)| &\leq \phi(t), \quad t \in [0,1], \\ D_0^{\frac{3}{2}}y(t) &= Ay(t) + By(t-0.1) + CD_0^{\frac{1}{2}}y(t-0.1) + \tilde{h}(t), \quad t \in [0,1]. \end{aligned}$$

By repeating the same process as in above example one can easily verify the main results of Theorem (3.8). Similarly replace ϵ by $\epsilon\phi(t)$ and $h(t)$ by $g(t)$ we can get the results for Theorem (3.9).

5. Conclusions

We present some new results about stability of a class of fractional neutral differential equations with Caputo fractional derivative by using Picard operator. We discuss the Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability, which maybe provide a new way for the researchers to discuss such interesting problems in the mathematical analysis area.

The current concepts have significant applications since it means that if we are studying Hyers-Ulam-Rassias stable (or Hyers-Ulam stable) system then one does not have to reach the exact solution. We just need to get a function which satisfies a suitable approximation inequality. In other words, Hyers-Ulam-Rassias stability (or Hyers-Ulam stability) guarantees that there exists a close exact solution. This is altogether useful in many applications where finding the exact solution is quite difficult such as optimization, numerical analysis, biology and economics. It also helps, if the stochastic effects are small, to use deterministic model to approximate a stochastic one.

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