A multi-item EPQ model with imperfect production process for time varying demand with shortages

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Abstract

In this paper, economic production quantity(EPQ)models for break-able or deteriorating items are developed with time dependent linear variable demands. Here rate of production and holding cost are time dependent and unit production cost is a function of both production reliability indicator and production rate. Set-up cost is also partially production rate dependent. Here two models are developed in optimal control framework considering the effect of time value of money and inflation. Shortages are allowed for both the models. The problems are solved using Eulers-Lagrangian function based on variational calculus and applying generalized reduced gradient method using LINGO 13.0 software to determine the optimal reliability indicator (r) and then corresponding production rates and total profits. Numerical experiments are performed for both the models to illustrate the models both numerically and graphically.

Keywords: Optimal control, shortages, time dependent demand.

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1. Introduction

In real life, we are accustomed with two categories of items mainly-damageable and non-damageable items. Again damageable items can be divided into two sub-categories namely-breakable item and deteriorating items. Deteriorating items deteriorate with time like seasonal fruits, different vegetable items etc. Since the items are deteriorated with time ,as a result the holding cost of the items is increased. For example, the fruits like grapes are available in the market from march to July in every year. Therefore the business time of that type of fruit is finite. Naturally, demand of the grapes increases with time and it exist in the market for a short period of time i.e. the business time horizon is finite. Also, fruits like mango, apple,vegetable like ladies finger, cabbage, beet and carrot are available in the market for a finite period and their demands increase with time. Some research works already have been investigated so far by several researchers on EOQ and EPQ/EMQ models with time dependent demand (cf. Dave and Patel[15], Dutta and Pal[14],Cheng[9],Lee and Hsu[25],Sana[47],Sarkar et al.[48], Maihami and Kamalabadi[32], Guchhait et al.[20]).

On the other hand, items made of glass, clay, ceramic, etc. belong to breakable category. Mainly fashionable/decorating items are made of glass, ceramic, etc., and demand of these types of items exists over finite time only. As sale of these fashionable products increases with the exhibition of stock, manufacturers of these items face a conflicting situation in their business. To stimulate the demand, they are tempted to go for huge number of production to have a large display and in this process, invites more damage to his units, as breakability increases with the increase of piled stock and the duration of stress due to the stock. So, breakability depend on huge stock and duration of accumulated stress due to stock. In the literature, there are only very few articles with this type of items(cf. Maiti and Maiti[[35],[36],[37]], Mandal[[38],[39]],Lee[[26]]). Still there is a scope to develop/modify some inventory models in this area considering time dependent breakability specially in different environments.

In real life, basically in metropolitan cities, holding cost increases with time due to non availability of space, bank interest etc. Also set-up is cost partially dependent on production rate. The researchers gave the less attention for research in this area. A notable remarks have been highlighted in inventory control problems with variable holding cost (cf. Alfares[1], Urban[56]) and Set-up cost (cf. Matsuyama[31], Darwish[13]). As per our knowledge, no one has formulated an inventory model for breakable/demegable items with the assumption of variable set-up cost or time dependent holding cost.

In recent times, the economy of developing countries like India, Bangladesh, Nepal, Bhutan, Pakistan etc. changes rigorously due to high inflation. The effect of inflation and time value of money are also well established in inventory problems. Initially, Buzacott[4] used the inflation subject to different types of pricing policies. Then consequently in the subsequent years, Mishra[33] , Padmanavan and Vrat[42] , Hariga and Ben-daya[21] , Bierman and Thomas[6] , Chen[8] , Moon and Lee[34] , Dey et al.[16] , Shah [50] etc. have worked in this area. Liao et al.[28] investigated the model of Aggarwal and Jaggi[2] with the assumption of inflation and stock dependent demand rate. Chen and Kang[7] presented integrated models with permissible delay in payments and variant pricing energy.

In most of the previous production inventory models, the researchers considered that all the produced items are of perfect quality. But, in real life, due to complex design of mechinaries items, it is not possible to produce all the items of perfect quality and is directly affected by the reliability of the production process.Recently some research works have been done in an imperfect production process like as Bazan[5] , Paul[43], Dey[17], Sarkar[[51],[52]], Mohammadia[40], Haidar[22] etc. In the literature there are

few research publications in the two ware-house inventory model with defective items like as Rad[45], Pal[44] etc . In the literature there are some notable works in the area of rework of the imperfect product such as Cardenas-Barron [[10][11][12]], Taleizadeh [[54],[55]], Sarkar[53], Wee[58] etc. In imperfect production-inventory models, reliability of the production process is considered in different ways. Firstly a fraction r of produced units are considered as good product and remaining (1-r) defective units. Some authors consider r as crisp (cf. Cheng[9], Maiti and Maiti[29]) and others consider r as uncertain (cf. Yoo et al.[57],Liao and Sheu[28]) and they tried to determine optimal r so as to optimize cost or profit.In reality if r is maximum, the manufactures are highly satisfied. Considering this fact some research works have been done in this area (cf. Sana[47] and Sarkar et al.[48] ,Guchhait[20]) . In this research work, we consider this approach.

Nearly all inventory models are formulated with constant holding cost (cf. Sana[47], Sarkar et al.[48], Maiti,[30]). In reality, due to rental charges, inflation, preservation cost, bank interest, etc., holding cost increases with time. Thus some factors contributing to the holding cost change with time (cf. Giri et al.[18]) . Also set-up cost depends on production rate as high production rate require sophisticated modern mechanism. In this paper set-up and holding costs are considered as functions of production rate and time respectively.

Variational principle is a straightforward process for the analysis of optimal control problems. Few researchers have formulated the production- inventory models as optimal control problems and solved using this method (cf. Sana[47],Sarkar et al.[48] and Guchhait[20]). But all the researchers formulated their models with single item.To the best of our knowledge, none has considered the multi-item with shortages via variational principle. The present problem has been solved under the assumption of multi-item with shortages using variational principle.

Most of the EPQ models, unit production cost is taken as a constant. But in reality production cost varies with production rate, raw material cost, labour charge, wear and tear cost and reliability of the production process (cf. Khouja[24]). In this study, unit production cost is dependent on production rate, reliability indicator, raw material, labour charge and wear-and-tear costs.

Thus, major contributions of the present investigation is as follows:

• A notable remark has been put in the area of production-inventory research where the models are developed with the assumption of infinite time horizon (cf. Porteous[41], Cheng[9],Maiti and Maiti[29], Yoo et al. [57]etc). According to their assumptions, demand of an item remains unchanged for interminably. But, in real life, Gurnani[19] pointed out that rapid development of technology leads to the change in product specification with latest feature which in turn, motivates the customers to go for buy new products. For this reason, many researchers have investigated and analysised the inventory models with finite time horizon (cf. Khanra and Choudhuri[23], Maiti[30] etc). But in the existing literature of inventory model with demagable/deteriorating items,they overlooked this phenomenon (cf. Maiti and Maiti[29], Guchhait et al.[20]). For this reason, here a finite time horizon multi-item production manufacturing model of a damageable items with shortages has been formulated and solved.

• In this paper, due to this reasons mentioned above holding and set-up costs are considered as functions of time and production rate for both the item respectively.

• In imperfect production inventory control problem,reliability factors play an important role in manufacturing process. But in the competitive market, due to existence in the market, managers of the production firms are highly satisfied if r i.e. reliability (also called process reliability)reaches its maximum levels and they can not allow the reliability to fall below a minimum level. Following, this approach, recently some works have been done by Sana[47],Sarker et al.[48], Sarkar[46] and Guchait et al.[20]. In the present

investigation, the authors have considered this approach for both the items.

• But in the common business practices that customers are allured with displayed stock and for that, demand is considered as stock-dependent. Some works have been done by Levin et al.[27], Baker and Urban[3], Alfares[1], Stavrulaki[49], etc. In recent market policy of the big departmental stores like Big Bazar, Metro Bazar, Bazar Kolkata, Wall Mart, TESCO, Carrefour etc.,where the items are displayed in huge stock and for the breakable items, huge stocks invites more breakability / damageability along with more sales. Hence, a balanced is to be maintained between increased breakability and sale for maximum profit. Till now, inventory practitioners have been paid a little attention in this area of inventory problem with damageable items (cf. Maiti and Maiti[29]). In this present investigation, optimum reliability indicator and the inventory level of breakable items made a balance between the process reliability and increased sale so as to maximize the profit.

• Due to simplicity and effectiveness of the variational principle as mentioned above , the present models are solved using Variational principle method by considering the augmented profit function.

• Thus, here an attempt has made to formulate and solve multi-item EPQ models incorporating all the features. As per the above arguments,in this present investigation, unit production cost taken depends on production rate, reliability indicator, raw material cost, etc.

•Till now, none has considered all the above features into account in a single model.

In this paper, a multi-item production-inventory model with imperfect production process is formulated for a breakable or deteriorating items over a finite time horizon. Here we formulate two models with shortages. First model is for two items with shortages and the second model is for single item with shortages. The unit production cost is a function of production rate, raw material cost, labour charge, wear and tear cost and product reliability indicator. The first model is formulated as optimal control problems for the maximization of total profits over the planning horizon with budget constraint and optimum profit with profits along with optimum reliability indicator(r) are obtained using Euler-Lagrange equation based on variational principle. The second model is of single item also solved under the same assumptions and technique. Both problems have been solved using a non-linear optimization technique -GRG (LINGO-13.0) and illustrated with some numerical data. Several particular cases are derived and the results are presented in both tabular and graphical forms. Finally, some sensitivity analyses can be made with respect to different parameters.

The rest of the research paper is structured as follows. Some notations and assumptions are given by section 2. Section 3 is followed by the mathematical development and description of the proposed model with shortages through optimal control framework. Here three lemmas are proposed and proved. Also, the mathematical development and description of the model with single item are proposed in section 4. Section 5 proposed the solution procedure. Section 6 represents the numerical data and results of different models and pictographic representation of the effect of different parameters. Discussion and managerial insights are discussed in section 7. After that a summarization of this study is included in section 8 by naming it as conclusion and future research work. At last, the list references that are used to make this study possible.

2. Notations and assumptions for the proposed model

2.1. Notations:

(i) $q_1(t)$ and $q_2(t)$ be the inventory at any time t of item-1 and -2 respectively.

(ii) $\dot{q_1}(t)$ and $\dot{q_2}(t)$ are the derivative of $q_1(t)$ and $q_2(t)$ with respect to time t respectively.

(iii) $B_1(q_1, t)$ and $B_2(q_2, t)$ be the breakability or damageability function of item-1 and item- 2 respectively.

(iv) $P_1(t)$ and $P_2(t)$ are the production rate of item-1 and item-2 respectively at any time t.

(v) r_1 and r_2 are the production reliability indicator for item-1 and item-2 respectively,
$0 \leq r_1, r_2 \leq 1$.

(vi) r_{1min}, r_{2min}. and r_{1max}, r_{2max} are the minimum and maximum value of r_1 and r_2 respectively,
$0 \leq r_{1min}, r_{2min} \leq 1, 0 \leq r_{1max}, r_{2max} \leq 1$.

(vii) λ_1 and λ_2 are the variation constant of tool or die costs for item-1 and-2 respectively, $\lambda_1 > 0$, $\lambda_2 > 0$.

(viii) $\chi(r_1)$ and $\chi(r_2)$ are the development cost of item-1 and-2 respectively.

(ix) C_{p_1} and C_{p_2} are the unit production cost of item-1 and item-2 respectively.

(x) C_{d_1} and C_{d_2} are the rework cost per defective item-1 and item-2 respectively.

(xi) $C_{h1}(t)$ and $C_{h2}(t)$ are the unit holding cost of item-1 and item-2 respectively.

(xii) C_3 and C_4 are the setup cost of item-1 and item-2 respectively.

(xiii) S_{p_1} and S_{p_2} is the unit selling price for the item-1 and item-2 respectively, $S_{p_1} > C_{p_1}$,$S_{p_2} > C_{p_2}$.

(xiv) S_{h_1} and S_{h_2} is the unit shortages cost for the item-1 and item-2 respectively.

2.2. Assumptions:

(i) The imperfect production-inventory system involves single and multi-item and which are to be sold .

(ii) The planning horizon for both the models are limited i.e. T is finite.

(iii) Here, it is assume that the inventory levels at $t = 0$ is $-S_1$ for item-1 and $-S_2$ for item-2 and both the inventory reaches to 0 at $t = T$.

(iv) In the show-rooms, the items made of China-clay, mud, glass,ceramic, etc., are kept in a heaped stocks. Due to this reason, the items at the bottom are under stress due to weight and for a long time, items are get damaged and break. Therefore, the breakability or damageability rate depends upon the stock of item and as well as how many times is under stress. Therefore the breakability rate of item-1 can be expressed as a function of stock levels and time and is of the form: $B_1(q_1, t) = b_{10}q_1 + b_{11}t$ for $q_1 > 0$ where b_{10} and b_{11} are the parameters can be chosen for best fit for the reliability function. Similarly, $B_2(q_2, t) = b_{20}q_2 + b_{21}t$ for $q_2 > 0$ where b_{20} and b_{21} are the parameters can be chosen for best fit for the reliability function.

(v) For the seasonal fruits like mango, apple etc., theirs demand is increases with time though their business period is limited and finite. Here demand rate is linear time-dependent for both the item.

(vi) Production rate for both items increases with time.

(vii) r_1 and r_2 indicates the defective rate of the production. Therefore, $r_1 P_1(t)$, $r_2 P_2(t)$ are the rate of producing defective item-1 and -2 respectively.

(viii) λ_1 and λ_2 are the variation constant of tool or die costs for item-1 and-2 respectively.

(ix) $\chi(r_1)$ and $\chi(r_2)$ depends upon the production reliability indicator, r_1 and r_2 respectively and are represented as $\chi(r_1) = N_1 + N_2 e^{C_A(r_{1max} - r_1)/(r_1 - r_{1min})}$ and $\chi(r_2) = N_3 + N_4$

$e^{C_A(r_{2max}-r_2)/(r_2-r_{2min})}$ where N_1 and N_3 are the fixed cost like labour, energy, etc., and is independent of r_1 and r_2. N_2 and N_4 are the cost of modern technology, resource and design complexity for production when $r_1 = r_{1max}$, $r_2 = r_{2max}$,. Also, C_A represents the difficulties in increasing reliability, which depends on the design complexity, technology and resource limitations, etc for both the items.

(x) Unit production cost,C_{p_1} and C_{p_2} are the function of production rate $P_1(t)$,$P_2(t)$ respectively and production reliability and can be expressed in the form
$C_{p_1}(r_1,t) = C_{r_1} + \frac{\chi(r_1)}{P_1(t)} + \lambda_1 P_1(t)$ for item-1 and $C_{p_2}(r_2,t) = C_{r_2} + \frac{\chi(r_2)}{P_2(t)} + \lambda_2 P_2(t)$ for item-2, where C_{r_1} and C_{r_2} are the fixed material cost for item-1 and 2 respectively. Second term is the development cost which is equally distributed over the production $P_1(t)$,$P_2(t)$ at any time t. Also, the third term $\lambda_1 P_1(t)$ and $\lambda_2 P_2(t)$ are the tool/ die cost which is proportional to the production rate respectively for both the item.

(xi) Now-a-days, due to inflation, bank interest, hiring charge, etc., holding cost increases with time. For this reason the holding cost changes with time and other factors remain constant. Hence the holding cost $C_{h1}(t)$ and $C_{h2}(t)$ can be expressed as $C_{h1}(t) = C_{10} + C_{11}t$ and $C_{h2}(t) = C_{20} + C_{21}t$ respectively for item-1 and item-2, where C_{10}, C_{11}, C_{20} and C_{21} are constants.

(xii) Set-up cost, C_3 and C_4, are normally constant with time for both the items. But, , if dynamic production rate is considered, some machineries, etc., are to be set-up and maintained in such a way that the production system can stand with the pressure of increasing demand. Thus, a part of C_3,C_4 are linearly proportional to production rate and hence C_3, C_4 are of the form: $C_3(P_1(t)) = C_{30} + C_{31}P_1(t)$ and $C_4(P_2(t)) = C_{40} + C_{41}P_2(t)$, where C_{30}, C_{31}, C_{40} andC_{41} are the constants.

(xiii) In the developing countries, inflation is predominant and interest rate depends on the inflation value. Thus $\mu = R - i$, where R and i are the interest and inflation per unit currency, respectively,$\mu > 0$.

(xiv) All inventory costs are positive.

3. Mathematical formulation of the proposed multi-item model:

3.1. Model-1: Model with stock and time dependent breakable items: In real life, a production company not only produce one item but produce different types of item i.e. multi-item. Due to continuous long operation of machinery units and over duty of the workers , the production firm produces good quality item as well as imperfect quality items.These defective or imperfect quality items are instantly reworked at a per unit cost to make the product as new as perfect one to maintain the brand image of the manufacturer. The production of the defective items increases with time and reliability parameter of the produced item. The parameters r_1 and r_2 are the reliability indicator of the item-1 and -2 respectively. The production system became more stable and reliable , if r_1 and r_2 decreases i.e. smaller value of r_1 and r_2 provides the better quality product and produced smaller imperfect quality unites.

The inventory levels decreases due to demand and breakability/deterioration. Thus, the rate of change of inventory level at any time t for the item -1 can be represented by the following differential equation:

$$\frac{dq_1(t)}{dt} = P_1 - D_1 - B_1(q_1,t)$$

(3.1) $\qquad$ i.e $P_1(t) = \dot{q}_1 + D_1 + B_1(q_1,t), \text{with } q_1(0) = -S_1 \text{ and } q_1(T) = 0,$

$\qquad$ where $\quad D_1 \equiv D_1(t)$

Thus, the rate of change of inventory level at any time t for the item -2 can be represented by the following differential equation:

$$\frac{dq_2(t)}{dt} = P_2 - D_2 - B_2(q_2, t)$$

(3.2) i.e. $P_2(t) = \dot{q}_2 + D_2 + B_2(q_2, t)$ with $q_2(0) = -S_2$ and $q_2(T) = 0,$

where $D_2 \equiv D_2(t)$

where D_1 and D_2 are the demand function of time t and is of the form $D_1(t) = a_1 + b_1 t$ and $D_2(t) = a_2 + b_2 t$ for item-1 & 2 respectively.

The end condition $q_1(0) = -S_1$, $q_2(0) = -S_2$ and $q_1(T) = 0$ and $q_2(T) = 0$ indicate that at time $t = 0$ the maximum shortages is $-S_1$ for item-1 and $-S_2$ for item-2 i.e. the inventory starts with shortages at time $t = 0$. As P_1 and D_1 are the function of time t and combined effect of theses two the shortages reaches to zero and the inventory build-up as $P_1(t) > D_1 + B_1(q_1, t)$ in the first part of the cycle. After some time, as demand is a function of time t, D_1 is more than the combined effect of $D_1 + B_1(q_1, t)$ i.e. the accumulated stock decreases as $P_1(t) < D_1 + B_1(q_1, t)$ and ultimately the stock reaches to zero. Similar process is also followed for the item-2.

Since the production firm manufacturers two different types of items, then a budget constraint is imposed for procurement of the raw materials cost. Here C_{r_1} and C_{r_2} are the fixed material cost for item-1 and -2 respectively and if M be the maximum available budget for both the items, then the budget constraint can be expressed as

(3.3) $$C_{r_1} q_1 + C_{r_2} q_2 \leq M$$

The corresponding profit function for both the items, incorporation the inflation and time value of money during the time duration $[0, T]$ is given by

$$Z_p = \int_0^T \left\{ e^{-\mu t} \left[S_{p_1} D_1 - C_{p_1}(r_1, t) P_1(t) - C_{d_1} r_1 P_1(t) - C_{h_1}(t) q_1 - \frac{C_3(P_1(t))}{T} - \right. \right.$$

$$\left. S_{h_1} S_1 \right] + e^{-\mu t} \left[S_{p_2} D_2 - C_{p_2}(r_2, t) P_2(t) - C_{d_2} r_2 P_2(t) - C_{h_2}(t) q_2 - \frac{C_4(P_2(t))}{T} - \right.$$

$$\left. \left. S_{h_2} S_2 \right] \right\} dt$$

$$= \int_0^T e^{-\mu t} \left[S_{p_1} D_1 + S_{p_2} D_2 - (C_{r_1} + C_{d_1} r_1)(\dot{q}_1 + D_1 + B_1) - (C_{r_2} + C_{d_2} r_2) \right.$$

$$(\dot{q}_2 + D_2 + B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q}_1 + D_1 + B_1)^2 - \lambda_2(\dot{q}_2 + D_2 + B_2)^2$$

$$-(C_{10} + C_{11} t) q_1 - (C_{20} + C_{21} t) q_2$$

$$\left. -\{C_{30} + C_{31}(\dot{q}_1 + D_1 + B_1])\}/T - \{C_{40} + C_{41}(\dot{q}_2 + D_2 + B_2)\}/T - S_{h_1} S_1 - S_{h_2} S_2 \right] dt$$

$$= \int_0^T f(q_1, q_2, \dot{q}_1, \dot{q}_2, t) dt$$

$$\text{where } f(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = e^{-\mu t} \left[S_{p_1} D_1 + S_{p_2} D_2 - (C_{r_1} + C_{d_1} r_1)(\dot{q}_1 + D_1 + B_1) \right.$$

$$-(C_{r_2} + C_{d_2} r_2)(\dot{q}_2 + D_2 + B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q}_1 + D_1 + B_1)^2 - \lambda_2(\dot{q}_2 + D_2 + B_2)^2$$

$$-(C_{10} + C_{11}t)q_1 \quad - \quad (C_{20} + C_{21}t)q_2 - \{C_{30} + C_{31}(\dot{q_1} + D_1 + B_1])\}/T - \{C_{40} + C_{41}$$

$$(3.4) \qquad\qquad (\dot{q_2} + D_2 + B_2)\}/T \quad - \quad S_{h_1}S_1 - S_{h_2}S_2 \bigg]$$

Now our problem is to find the path of $q_1(t)$, $q_2(t)$, $P_1(t)$ and $P_2(t)$ such that Z_p is maximum with respect to the budget constraint. Since the problem is involved with a constraint, then to find the optimal solution of the optimal control problem, we construct the augmented profit functional as

$$(3.5) \qquad\qquad Z_T = \int_0^T \bigg[f(q_1, q_2, \dot{q_1}, \dot{q_2}, t) + \lambda e^{-\mu t}(C_{r_1}q_1 + C_{r_2}q_2 - M) \bigg] dt$$

where,$F(q_1, q_2, \dot{q_1}, \dot{q_2}, t) = f(q_1, q_2, \dot{q_1}, \dot{q_2}, t) + \lambda e^{-\mu t}(C_{r_1}q_1 + C_{r_2}q_2 - M)$ and λ is the Lagrange multiplier having any real value.

3.1. Lemma. Z_T *has a maximum value for a path* $q_1 = q_1(t)$ *and* $q_2 = q_2(t)$*in the interval* $[0, T]$

Proof. Proof of the Lemma 3.1 . we consider a path(curve)$q_1 = q_1(t)$ and $q_2 = q_2(t)$ such that the functional Z_T is maximum in that path in the interval $[0, T]$ i.e. $t = 0$ and $t = T$. Let us consider a path q_0 which is given by the path $q = q_0$ for which Z_T has a maximum value.We consider a class of neighboring curves p_ρ which is given by $q_1 = q_{1\rho}(t) = q_0(t) + \rho_1\eta_1(t)$ and $q_2 = q_{2\rho}(t) = q_0(t) + \rho_2\eta_2(t)$, where ρ_1 and ρ_2 is a very small constant and $\eta_1(t)$ and $\eta_2(t)$ (> 0, for all values of t) is any two differential functions of t.Therefore, the value of Z_T for the path p_ρ is given by the relation $Z_T(\rho) = \int_0^T F_{\rho_1\rho_2}$ dt, where $F_{\rho_1\rho_2} = F(q_0(t) + \rho_1\eta_1(t), \dot{q_0}(t) + \rho_1\dot{\eta_1}(t), q_0(t) + \rho_2\eta_2(t), \dot{q_0}(t) + \rho_2\dot{\eta_2}(t), t)$
For maximum value of Z_T, we must have $\frac{\partial}{\partial\rho_1}(Z_T(\rho_1, \rho_2)) \mid_{\rho_1=0} = 0$

and $\frac{\partial}{\partial\rho_2}(Z_T(\rho_1, \rho_2)) \mid_{\rho_2=0} = 0$ and $\left[\frac{\partial^2 Z_T}{\partial\rho_1^2} \frac{\partial^2 Z_T}{\partial\rho_2^2} - \frac{\partial^2 Z_T}{\partial\rho_1\partial\rho_2} \right] > 0$ and $\frac{\partial^2}{\partial\rho_1^2}(Z_T(\rho_1, \rho_2)) < 0$
Now,

$$
\begin{aligned}
\frac{\partial}{\partial\rho_1}(Z_T(\rho_1, \rho_2)) &= \int_0^T \left\{ \eta_1(t)\frac{\partial F_\rho}{\partial q_1} + \dot{\eta_1}(t)\frac{\partial F_\rho}{\partial \dot{q_1}} \right\} dt \\
&= \int_0^T \left[\eta_1(t)\frac{\partial F_\rho}{\partial q_1} \right] dt + \left[\eta_1(t)\frac{\partial F_\rho}{\partial \dot{q_1}} \right]_0^T - \int_0^T \eta_1(t)\frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_1}} \right) dt \\
&= \int_0^T \eta_1(t)\left\{ \frac{\partial F_\rho}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_1}} \right) \right\} dt
\end{aligned}
$$

As $q_1(t)$ is fixed at the end points $t = 0$ and $t = T$, so, $\eta_1(0) = \eta_1(T) = 0$. Therefore, $\frac{d}{d\rho_1}(Z_T(\rho_1, \rho_2)) \mid_{\rho_1=0} = 0$ gives

$$(3.6) \qquad\qquad \frac{\partial F_\rho}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_1}} \right) = 0$$

Similarly,

$$
\begin{aligned}
\frac{\partial}{\partial\rho_2}(Z_T(\rho_1, \rho_2)) &= \int_0^T \left\{ \eta_2(t)\frac{\partial F_\rho}{\partial q_2} + \dot{\eta_2}(t)\frac{\partial F_\rho}{\partial \dot{q_2}} \right\} dt \\
&= \int_0^T \left[\eta_2(t)\frac{\partial F_\rho}{\partial q_2} \right] dt + \left[\eta_2(t)\frac{\partial F_\rho}{\partial \dot{q_2}} \right]_0^T - \int_0^T \eta_2(t)\frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_2}} \right) dt \\
(3.7) \qquad &= \int_0^T \eta_2(t)\left\{ \frac{\partial F_\rho}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_2}} \right) \right\} dt
\end{aligned}
$$

As $q_2(t)$ is fixed at the end points $t = 0$ and $t = T$, so, $\eta_2(0) = \eta_2(T) = 0$. Therefore, $\frac{\partial}{\partial \rho_2}(Z_T(\rho_1, \rho_2))\mid_{\rho_2=0}= 0$ gives

$$(3.8) \qquad \frac{\partial F_\rho}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial F_\rho}{\partial \dot{q_2}}\right) = 0$$

Equations (3.6) and (3.8) are the necessary conditions for extreme value of P_T.

Again, to find the maximum value of Z_T we must have, $\left[\frac{\partial^2 Z_T}{\partial \rho_1^2} \frac{\partial^2 Z_T}{\partial \rho_2^2} - \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] > 0$ and $\frac{\partial^2 Z_T}{\partial \rho_1^2} < 0$

Now,

$$\begin{aligned}
\frac{\partial^2 Z_T}{\partial \rho_1^2} &= \int_0^T \left\{\eta_1^2 \frac{\partial^2 Z_p}{\partial q_1^2} + 2\eta_1 \dot{\eta_1} \frac{\partial^2 Z_p}{\partial q_1 \partial \dot{q_1}} + \dot{\eta_1}^2 \frac{\partial^2 Z_p}{\partial \dot{q_1}^2}\right) dt \\
&= -2\lambda_1 e^{-\mu t}\left\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta_1} b_{10} + \dot{\eta_1}^2\right\} < 0 \text{ as } 2\lambda_1 e^{-\mu t} > 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 Z_T}{\partial \rho_2^2} &= \int_0^T \left\{\eta_2^2 \frac{\partial^2 Z_p}{\partial q_2^2} + 2\eta_2 \dot{\eta_2} \frac{\partial^2 Z_p}{\partial q_2 \partial \dot{q_2}} + \dot{\eta_2}^2 \frac{\partial^2 Z_p}{\partial \dot{q_2}^2}\right) dt \\
&= -2\lambda_2 e^{-\mu t}\left\{\eta_2^2 b_{20}^2 + 2\eta_2 \dot{\eta_2} b_{20} + \dot{\eta_2}^2\right\} < 0 \text{ as } 2\lambda_2 e^{-\mu t} > 0
\end{aligned}$$

$$\text{Finally,} \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2} = 0$$

Therefore,

$$\left[\frac{\partial^2 Z_T}{\partial \rho_1^2} \frac{\partial^2 Z_T}{\partial \rho_2^2} - \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] = 2\lambda_1 e^{-\mu t}\left\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta_1} b_{10} + \right.$$

$$\left. \dot{\eta_1}^2\right\} 2\lambda_2 e^{-\mu t}\left\{\eta_2^2 b_{20}^2 + 2\eta_2 \dot{\eta_2} b_{20} + \dot{\eta_2}^2\right\} > 0$$

$$\text{and} \frac{\partial^2 Z_T}{\partial \rho_1^2} = -\lambda_1 e^{-\mu t}\left\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta_1} b_{10} + \dot{\eta_1}^2\right\} < 0 \text{ as } \lambda_1 e^{-\mu t} > 0$$

Hence the sufficient condition, $\left[\frac{\partial^2 Z_T}{\partial \rho_1^2} \frac{\partial^2 Z_T}{\partial \rho_2^2} - \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] > 0$ and $\frac{\partial^2 Z_T}{\partial \rho_1^2} < 0$ shows that Z_T has a maximum in $[0, T]$. $\qquad \square$

3.2. Lemma. $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} = 0$ *must have at least one solution in* $[r_{1min}, r_{1max}]$, *if* $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} < 0$, *provided* $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} \to \propto$ *at* $r_1 = r_{1min}$ *for all* r_2, *otherwise* $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} = 0$ *may have or may not have a solution in* $[r_{1min}, r_{1max}]$. *The solution gives a maximum value of* Z_T, *if* $\frac{\partial^2 Z_T}{\partial r_1^2} < 0$ *and* $\frac{\partial^2 Z_T}{\partial r_1^2} \frac{\partial^2 Z_T}{\partial r_2^2} - \left(\frac{\partial^2 Z_T}{\partial r_1 \partial r_2}\right)^2 > 0$ *in the rectangle* $[r_{1min}, r_{1max} : r_{2min}, r_{2max}]$

Proof. Proof of the Lemma 3.2 .For maximization of the associate profit for both the items, $Z_T(r_1, r_2)$, differentiating $Z_T(r_1, r_2)$ with respect to r_1, we have

$$\frac{\partial Z_T}{\partial r_1} = N_2 e^{C_A(r_{1max} - r_1)/(r_1 - r_{1min})} C_A \frac{r_{1min} - r_{1max}}{(r_1 - r_{1min})^2} \frac{e^{-\mu T} - 1}{\mu}$$

As $r_1 \to r_{1min}$, then $\dfrac{\partial Z_T}{\partial r_1} \to \propto$

Again,

$$\frac{\partial^2 Z_T}{\partial r_1^2} = \frac{N_2 C_A e^{-\mu T - 1}}{\mu} \Big[e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})} \frac{r_{1min} - r_{1max}}{(r_1 - r_{1min})^4} + e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}$$
$$\frac{r_{1min} - r_{1max}}{(r_1 - r_{1min})^3} \Big]$$

As $r_1 \to r_{1min}$ then $\dfrac{\partial Z_T}{\partial r_1} \to \propto$, therefore $\dfrac{\partial Z_T}{\partial r_1}$ has at least one solution if $\dfrac{\partial Z_T}{\partial r_1} \to \propto$ holds; otherwise $\dfrac{\partial Z_T(r_1, r_2)}{\partial r_1} = 0$ may have or may not have a solution in $[r_{1min}, r_{1max}]$. If $\dfrac{\partial Z_T}{\partial r_1} \mid_{r_1 = r_1^*} = 0$ for $r_1^* \in [r_{1min}, r_{1max}]$ and $\dfrac{\partial^2 Z_T}{\partial r_1^2} < 0$ and $\dfrac{\partial^2 Z_T}{\partial r_1^2} \dfrac{\partial^2 Z_T}{\partial r_2^2} - (\dfrac{\partial^2 Z_T}{\partial r_1 \partial r_2})^2 > 0$, then $P_T(r_1^*)$ is maximum.

Similarly, **Lemma 3.3** can be written as, $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3. Lemma. $\dfrac{\partial Z_T(r_1, r_2)}{\partial r_2} = 0$ *must have at least one solution in* $[r_{2min}, r_{2max}]$, *if* $\dfrac{\partial Z_T(r_1, r_2)}{\partial r_2} < 0$, *provided* $\dfrac{\partial Z_T(r_1, r_2)}{\partial r_2} \to \propto$ *at* $r_2 = r_{2min}$ *for all* r_1 *otherwise* $\dfrac{\partial Z_T(r_1, r_2)}{\partial r_2} = 0$ *may have or may not have a solution in* $[r_{2min}, r_{2max}]$. *The solution gives a maximum value of* Z_T, *if* $\dfrac{\partial^2 Z_T}{\partial r_2^2} < 0$ *and* $\dfrac{\partial^2 Z_T}{\partial r_1^2} \dfrac{\partial^2 Z_T}{\partial r_2^2} - (\dfrac{\partial^2 Z_T}{\partial r_1 \partial r_2})^2 > 0$ *in the rectangle* $[r_{1min}, r_{1max} : r_{2min}, r_{2max}]$

Proof. Proof of the Lemma 3.3. we can proof the Lemma 3.3 following the same of Lemma 3.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, for find the optimal path, we have from the Euler-Lagranges equation for the maximum value of $F(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$ is

$$(3.9) \qquad\qquad\qquad\qquad \frac{\partial F}{\partial q_1} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}_1}) = 0$$

$$(3.10) \qquad\qquad\qquad\qquad \frac{\partial F}{\partial q_2} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}_2}) = 0$$

Firstly, we consider the first Euler-Lagrangian equation (3.9) and the boundary condition (3.1), we have

$$(3.11) \qquad\qquad\qquad\qquad \ddot{q}_1 - \mu \dot{q}_1 - (b_{10} + \mu)b_{10}q_1 = H_1(t)$$

where

$$H_1(t) = (\mu + b_{10})D_1 - b_1 - b_{11}(b_{10}t - 1 + \mu t)$$
$$+ \frac{(C_{r_1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10}) + (C_{10} + C_{11})t}{2\lambda_1} - \frac{\lambda}{2\lambda_1}C_{r_1}$$
$$= K_1 + K_2 t + K_1'$$
$$\text{where } K_1 = a_1(\mu + b_{10}) - b_1 - b_{11} + \frac{(C_{r_1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10}) + C_{10}}{2\lambda_1}$$

$$(3.12) \qquad K_2 = \Big[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1} \Big], K_1' = -\frac{\lambda}{2\lambda_1}C_{r_1}$$

The complementary function of the Eq. (3.11) is $C_1 e^{(b_{10}+\mu)t}+C_2 e^{-b_{10}t}$, where C_1 and C_2 are arbitrary constants and the particular integral is given by the $\frac{1}{D^2-\mu D-(\mu+b_{10})b_{10}}\{K_1+K_2 t+K_1'\}$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(3.11) can be represented as

$$(3.13) \quad q_1(t) \;=\; C_1 e^{(b_{10}+\mu)t}+C_2 e^{-b_{10}t}-\frac{1}{K_3^2}\big[K_1 K_3+K_2\big(K_3 t-\mu\big)\big]-\frac{K_1'}{K_3}$$

$$(3.14) \qquad\qquad P_1(t) \;=\; K_4 e^{(b_{10}+\mu)t}+K_5 t+K_6+K_7,$$

where, $K_4 \;=\; C_1(2b_{10}+\mu)$, $K_3 = b_{10}(b_{10}+\mu)$, $K_5 = (b_1+b_{11}-\dfrac{b_{10}K_2}{K_3})$, $K_6 = -\dfrac{K_1'}{K_3}b_{10}$

$$K_7 \;=\; \frac{1}{K_3^2}\left[a_1 K_3^2-K_2 K_3-b_{10}(K_1 K_3-K_2\mu),\right]$$

$$C_2 = \frac{1}{[e^{(b_{10}+\mu)T}-e^{-b_{10}T}]}\Big(\frac{1}{K_3^2}[(K_1 K_3-K_2\mu)e^{(b_{10}+\mu)T}-(K_1 K_3+K_2(K_3 T-\mu))]$$

$$-\frac{K_1'}{K_3}(e^{(b_{10}+\mu)T}-1)-S_1 e^{(b_{10}+\mu)T}\Big)$$

$$C_1 = -S_1-C_2+\frac{1}{K_3^2}(K_1 K_3-K_2\mu)+\frac{K_1'}{K_3}$$

Substituting the value of $q_1(t)$ and $P_1(t)$ in the expression of (3.4), the corresponding profit function for the item-1 can be expressed as

$$Z_{p_1} = \int_0^T e^{-\mu t}\big[S_{p_1}D_1-(C_{r_1}+C_{d_1}r_1)(\dot{q}_1+D_1+B_1)-\chi(r_1)-\lambda_1(\dot{q}_1+D_1+B_1)^2$$

$$-(C_{10}+C_{11}t)q_1-\{C_{30}+C_{31}(\dot{q}_1+D_1+B_1)\}/T-S_{h_1}S_1\big]dt$$

$$= \; S_{p_1}\Big[-a_1\frac{(e^{-\mu T}-1)}{\mu}-b_1\frac{(e^{-\mu T}-1+\mu T e^{-\mu T})}{\mu^2}\Big]-(C_{r_1}+r_1 C_{d_1})\Big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}}$$

$$+ \; K_5\Big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\Big)+\frac{K_6}{\mu}(1-e^{-\mu T})+\frac{K_7}{\mu}(1-e^{-\mu T})\Big]+\big[N_1+N_2$$

$$e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\big]\Big(\frac{e^{-\mu T}-1}{\mu}\Big)+\lambda_1\Big[(K_4 e^{(b_{10}+\mu)T}+K_5 T+K_6+K_7)^2\frac{e^{-\mu T}}{\mu}$$

$$- \; \frac{(K_4+K_6+K_7)^2}{\mu}+\frac{2}{\mu^2}\Big(\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4 K_5}{b_{10}}(e^{b_{10}T}-1)+$$

$$\frac{K_4 K_5(b_{10}+\mu)}{b_{10}}\Big\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\Big\}+\frac{(K_7+K_6)K_4(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1)$$

$$- \; K_5^2\Big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\Big)+\frac{(K_6+K_7)K_5}{\mu}(1-e^{-\mu T})\Big)\Big]-C_{10}\Big[\frac{C_1}{b_{10}}(e^{b_{10}T}-1)$$

$$- \; \frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1'}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1 K_3+K_2(K_3 T-\mu))\frac{e^{-\mu T}}{\mu}$$

$$+ \; \frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{K_3^2\mu}(K_1 K_3-K_2\mu)\Big]-C_{11}\Big[C_1\Big(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\Big)+$$

$$C_2\left(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)} - \frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2} + \frac{1}{(b_{10}+\mu)^2}\right) - \frac{K_1}{K_3}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right)$$

$$- \frac{K_2\mu}{K_3^2}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right) + \frac{K_2}{K_3}\left(\frac{-T^2e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3}e^{-\mu T} + \frac{2}{\mu^3}\right)$$

$$- \frac{K_1^{'}}{K_3}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right)\Big] - \frac{1}{T}\Big[\frac{C_{30}}{\mu}(1 - e^{-\mu T}) + C_{31}\left(\frac{K_4}{b_{10}}(e^{b_{10}T} - 1)\right)$$

$$(3.15) + \quad K_5\left(-\frac{Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right) + \frac{(K_6 + K_7)}{\mu}(1 - e^{-\mu T}))\Big] - S_{h_1}S_1\left(\frac{1 - e^{-\mu T}}{\mu}\right)$$

From the second Euler-Lagranges Equation and using the boundary condition, we have

$$(3.16) \quad \ddot{q}_2 - \mu\dot{q}_2 - (b_{20} + \mu)b_{20}q_2 = H_2(t)$$

where

$$\begin{aligned}
H_2(t) &= (\mu + b_{20})D_2 - b_2 - b_{21}(b_{20}t - 1 + \mu t) \\
&\quad + \frac{(C_{r_2} + r_2C_{d_2} + C_{41}/T)(\mu + b_{20}) + (C_{20} + C_{21})t}{2\lambda_2} - \frac{\lambda}{2\lambda_2}C_{r_2} \\
&= K_{11} + K_{22}t + K_{11}^{'} \\
\text{where, } K_{11} &= a_2(\mu + b_{20}) - b_2 - b_{21} + \frac{(C_{r_2} + r_2C_{d_2} + C_{41}/T)(\mu + b_{20}) + C_{20}}{2\lambda_2} \\
K_{22} &= \left[b_2(\mu + b_{20}) + b_{21}(b_{20} + \mu) + \frac{C_{21}}{2\lambda_2}, \quad K_{11}^{'} = -\frac{\lambda}{2\lambda_2}C_{r_2}
\end{aligned}$$

and $K_{33} = b_{20}(b_{20} + \mu)$

The complementary function of the Eq. (3.16) is $C_3e^{(b_{20}+\mu)t} + C_4e^{-b_{20}t}$, where C_3 and C_4 are arbitrary constants and the particular integral is given by the $\frac{1}{D^2 - \mu D - (\mu + b_{20})b_{20}}\left\{K_{11} + K_{22}t + K_{11}^{'}\right\}$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(16) can be represented as

$$(3.17) \quad q_2(t) = C_3e^{(b_{20}+\mu)t} + C_4e^{-b_{20}t} - \frac{1}{K_{33}^2}\left[K_{11}K_{33} + K_{22}\left(K_{33}t - \mu\right)\right]$$

$$-\frac{K_{11}^{'}}{K_{33}}$$

$$(3.18) \text{ and } P_2(t) = K_8e^{(b_{20}+\mu)t} + K_9t + K_{10} + K_{12},$$

$$\text{where, } K_8 = C_3(2b_{20} + \mu), K_9 = (b_2 + b_{21} - b_{20}\frac{K_{22}}{K_{33}}), K_{10} = -\frac{K_{11}^{'}b_{20}}{K_{33}},$$

$$K_{12} = \frac{1}{K_{33}^2}[a_2K_{33}^2 - K_{22}K_{33} - b_{20}(K_{11}K_{33} - K_{22}\mu)]$$

$$C_4 = \frac{1}{[e^{(b_{20}+\mu)T} - e^{-b_{20}T}]}\left(\frac{1}{K_{33}^2}[(K_{11}K_{33} - K_{22}\mu)e^{(b_{20}+\mu)T} - (K_{11}K_{33} + K_{22}(K_{33}T - \mu))]\right.$$

$$\left. -\frac{K_{11}^{'}}{K_{33}}(e^{(b_{20}+\mu)T} - 1) - S_2e^{(b_{20}+\mu)T}\right)$$

$$C_3 = -S_2 - C_4 + \frac{1}{K_{33}^2}(K_{11}K_{33} - K_{22}\mu) + \frac{K_{11}^{'}}{K_{33}}$$

Substituting the value of $q_2(t)$ and $P_2(t)$ in the expression of (3.4), the corresponding profit function for the item-2 can be expressed as

$$Z_{p_2} = \int_0^T e^{-\mu t} \big[S_{p_2} D_2 - (C_{r_2} + C_{d_2} r_2)(\dot{q}_2 + D_2 + B_2) - \chi(r_2) - \lambda_2(\dot{q}_2 + D_2 + B_2)^2$$

$$- (C_{20} + C_{21} t)q_2 - \{C_{40} + C_{41}(\dot{q}_2 + D_2 + B_2]\})/T - S_{h_2} S_2 \big] dt$$

$$= S_{p_2} \Big[- a_2 \frac{(e^{-\mu T} - 1)}{\mu} - b_2 \frac{(e^{-\mu T} - 1 + \mu T e^{-\mu T})}{\mu^2} \Big] - (C_{r_2} + r_2 C_{d_2}) \Big[\frac{K_8(e^{b_{20}T} - 1)}{b_{20}}$$

$$+ K_9 \Big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) + \frac{K_{10}}{\mu}(1 - e^{-\mu T}) + \frac{K_{12}}{\mu}(1 - e^{-\mu T}) \Big] + \Big[N_3$$

$$+ N_4 e^{C_A(r_{2max} - r_2)/(r_2 - r_{2min})} \Big] \frac{e^{-\mu T} - 1}{\mu} + \lambda_1 \Big[(K_8 e^{(b_{20} + \mu)T} + K_{10}T + K_{10} + K_{12})^2$$

$$\frac{e^{-\mu T}}{\mu} - \frac{(K_8 + K_{10} + K_{12})^2}{\mu} + \frac{2}{\mu^2} \Big(\frac{K_8^2(b_{20} + \mu)}{(2b_{20} + \mu)}(e^{(2b_{20} + \mu)T} - 1) + \frac{K_8 K_9}{b_{20}}(e^{b_{20}T}$$

$$- 1) + \frac{K_8 K_9(b_{20} + \mu)}{b_{20}} \Big\{ \frac{Te^{b_{20}T}}{b_{20}} - \frac{e^{b_{20}T}}{b_{20}^2} + \frac{1}{b_{20}^2} \Big\} + \frac{(K_{12} + K_{10})K_8(b_{20} + \mu)}{b_{20}}(e^{b_{20}T} -$$

$$1) - K_9^2 \Big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) + \frac{(K_{10} + K_{12})K_9}{\mu}(1 - e^{-\mu T})) \Big] - C_{20} \Big[\frac{C_3}{b_{20}}(e^{b_{20}T}$$

$$- 1) - \frac{C_4}{(b_{20} + \mu)}(e^{-(b_{20} + \mu)T} - 1) + \frac{K'_{11}}{K_{33}\mu}(e^{-\mu T} - 1) + \frac{1}{K_{33}^2}(K_{11}K_{33} + K_{22}(K_{33}T$$

$$-\mu)) \frac{e^{-\mu T}}{\mu} + \frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T} - 1) - \frac{1}{K_{33}^2\mu}(K_{11}K_{33} - K_{22}\mu) \Big] - C_{21} \Big[C_3 \Big(\frac{Te^{b_{20}T}}{b_{20}} -$$

$$\frac{e^{b_{20}T}}{b_{20}^2} + \frac{1}{b_{20}^2} \Big) + C_4 \Big(\frac{-Te^{-(b_{20} + \mu)T}}{(b_{20} + \mu)} - \frac{e^{-(b_{20} + \mu)T}}{(b_{20} + \mu)^2} + \frac{1}{(b_{20} + \mu)^2} \Big) - \frac{K_{11}}{K_{33}} \Big(\frac{-Te^{-\mu T}}{\mu}$$

$$- \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) - \frac{K_{22}\mu}{K_{33}^2} \Big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) + \frac{K_{22}}{K_{33}} \Big(\frac{-T^2 e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3}$$

$$e^{-\mu T} + \frac{2}{\mu^3} \Big) - \frac{K'_{11}}{K_{33}} \Big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) \Big] - \frac{1}{T} \Big[\frac{C_{40}}{\mu}(1 - e^{-\mu T}) + C_{41} \Big(\frac{K_8}{b_{20}}(e^{b_{20}T} -$$

$$(3.19) \quad 1) + K_9 \Big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2} \Big) + \frac{(K_{10} + K_{12})}{\mu}(1 - e^{-\mu T}) \Big) \Big] - S_{h_2} S_2 \Big(\frac{1 - e^{-\mu T}}{\mu} \Big)$$

Therefore total profit for item-1 and -2 can be expressed as $Z_p = Z_{p_1} + Z_{p_2}$, where Z_{p_1} and Z_{p_2} are given by (3.15)&(3.19) respectively, Therefore,

$$Z_p = \int_0^T e^{-\mu t} \Big[S_{p_1} D_1 + S_{p_2} D_2 - (C_{r_1} + C_{d_1} r_1)(\dot{q}_1 + D_1 + B_1) - (C_{r_2} + C_{d_2} r_2)(\dot{q}_2 + D_2$$

$$+B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q}_1 + D_1 + B_1)^2 - \lambda_2(\dot{q}_2 + D_2 + B_2)^2 - (C_{10} + C_{11}t)q_1 - (C_{20}$$

$$+C_{21}t)q_2 - \{C_{30} + C_{31}(q_1 + D_1 + B_1)\}/T - \{C_{40} + C_{41}(\dot{q}_2 + D_2 + B_2)\}/T - S_{h_1} S_1$$

$$-S_{h_2} S_2 + \lambda(C_{r_1} q_1 + C_{r_2} q_2 - M) \Big] dt$$

$$
= \ S_{p_1}\Big[-a_1\frac{(e^{-\mu T}-1)}{\mu} - b_1\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\Big] - (C_{r_1}+r_1 C_{d_1})\Big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}}
$$

$$
+ \ K_5\Big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\Big)+\frac{K_6}{\mu}(1-e^{-\mu T})+\frac{K_7}{\mu}(1-e^{-\mu T})\Big]+\Big[N_1+N_2
$$

$$
e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\Big](\frac{e^{-\mu T}-1}{\mu})+\lambda_1\Big[(K_4 e^{(b_{10}+\mu)T}+K_5 T+K_6+K_7)^2\frac{e^{-\mu T}}{\mu}
$$

$$
- \ \frac{(K_4+K_6+K_7)^2}{\mu}+\frac{2}{\mu^2}\Big(\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4 K_5}{b_{10}}(e^{b_{10}T}-1)
$$

$$
+ \ \frac{K_4 K_5(b_{10}+\mu)}{b_{10}}\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\}+\frac{(K_7+K_6)K_4(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1)
$$

$$
- \ K_5^2(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_6+K_7)K_5}{\mu}(1-e^{-\mu T}))\Big]-C_{10}\Big[\frac{C_1}{b_{10}}(e^{b_{10}T}-1)-
$$

$$
\frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1^{'}}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1 K_3+K_2(K_3 T-\mu))\frac{e^{-\mu T}}{\mu}
$$

$$
+ \ \frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{K_3^2\mu}(K_1 K_3-K_2\mu)\Big]-C_{11}\Big[C_1(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2})
$$

$$
+ \ C_2(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)}-\frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2}+\frac{1}{(b_{10}+\mu)^2})-\frac{K_1}{K_3}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})
$$

$$
- \ \frac{K_2\mu}{K_3^2}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{K_2}{K_3}(\frac{-T^2 e^{-\mu T}}{\mu}-\frac{2Te^{-\mu T}}{\mu^2}-\frac{2}{\mu^3}e^{-\mu T}+\frac{2}{\mu^3})-
$$

$$
\frac{K_1^{'}}{K_3}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})\Big]-\frac{1}{T}\Big[\frac{C_{30}}{\mu}(1-e^{-\mu T})+C_{31}(\frac{K_4}{b_{10}}(e^{b_{10}T}-1)+
$$

$$
K_5(-\frac{Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_6+K_7)}{\mu}(1-e^{-\mu T}))\Big]-S_{h_1}S_1(\frac{1-e^{-\mu T}}{\mu})
$$

$$
+ \ S_{p_2}\Big[-a_2\frac{(e^{-\mu T}-1)}{\mu}-b_2\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\Big]-(C_{r_2}+r_2 C_{d_2})\Big[\frac{K_8(e^{b_{20}T}-1)}{b_{20}}
$$

$$
+ \ K_9\Big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\Big)+\frac{K_{10}}{\mu}(1-e^{-\mu T})+\frac{K_{12}}{\mu}(1-e^{-\mu T})\Big]+\Big[N_3+N_4
$$

$$
e^{C_A(r_{2max}-r_2)/(r_2-r_{2min})}\Big]\frac{e^{-\mu T}-1}{\mu}+\lambda_2\Big[(K_8 e^{(b_{20}+\mu)T}+K_{10}T+K_{10}+K_{12})^2
$$

$$
\frac{e^{-\mu T}}{\mu}-\frac{(K_8+K_{10}+K_{12})^2}{\mu}+\frac{2}{\mu^2}\Big(\frac{K_8^2(b_{20}+\mu)}{(2b_{20}+\mu)}(e^{(2b_{20}+\mu)T}-1)+\frac{K_8 K_9}{b_{20}}
$$

$$
(e^{b_{20}T}-1)+\frac{K_8 K_9(b_{20}+\mu)}{b_{20}}\{\frac{Te^{b_{20}T}}{b_{20}}-\frac{e^{b_{20}T}}{b_{20}^2}+\frac{1}{b_{20}^2}\}+\frac{(K_{12}+K_{10})K_8(b_{20}+\mu)}{b_{20}}-
$$

$$
(e^{b_{20}T}-1)K_9^2(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_{10}+K_{12})K_9}{\mu}(1-e^{-\mu T}))\Big]-C_{20}\Big[\frac{C_3}{b_{20}}
$$

$$
(e^{b_{20}T}-1)-\frac{C_4}{(b_{20}+\mu)}(e^{-(b_{20}+\mu)T}-1)+\frac{K_{11}^{'}}{K_{33}\mu}(e^{-\mu T}-1)+\frac{1}{K_{33}^2}(K_{11}K_{33}+K_{22}
$$

$$
(K_{33}T-\mu))\frac{e^{-\mu T}}{\mu}+\frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T}-1)-\frac{1}{K_{33}^2\mu}(K_{11}K_{33}-K_{22}\mu)\Big]-C_{21}\Big[C_3(\frac{Te^{b_{20}T}}{b_{20}}
$$

$$
- \ \frac{e^{b_{20}T}}{b_{20}^2}+\frac{1}{b_{20}^2})
$$

$$+ \quad C_4\left(\frac{-Te^{-(b_{20}+\mu)T}}{(b_{20}+\mu)} - \frac{e^{-(b_{20}+\mu)T}}{(b_{20}+\mu)^2} + \frac{1}{(b_{20}+\mu)^2}\right) - \frac{K_{11}}{K_{33}}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right)$$

$$- \quad \frac{K_{22}\mu}{K_{33}^2}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right) + \frac{K_{22}}{K_{33}}\left(\frac{-T^2e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3}e^{-\mu T} + \frac{2}{\mu^3}\right) -$$

$$\frac{K_{11}'}{K_{33}}\left(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right)] - \frac{1}{T}[\frac{C_{40}}{\mu}(1 - e^{-\mu T}) + C_{41}(\frac{K_8}{b_{20}}(e^{b_{20}T} - 1)$$

$$+ \quad K_9\left(-\frac{Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\right) + \frac{(K_{10} + K_{12})}{\mu}(1 - e^{-\mu T}))] - S_{h_2}S_2\frac{1 - e^{-\mu T}}{\mu} + \lambda$$

$$\left[C_{r_1}\left\{\frac{C_1}{b_{10}}(e^{b_{10}T} - 1) - \frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T} - 1) + \frac{1}{K_3^2\mu}(K_1K_3 + K_2(K_3T - \mu))\right.\right.$$

$$\left.e^{-\mu T} + \frac{K_2}{K_3\mu^2}(e^{-\mu T} - 1) - \frac{1}{K_3^2\mu}(K_1K_3 - K_2\mu) + \frac{K_1'}{K_3\mu}(e^{-\mu T} - 1)\right\} + C_{r_2}\left\{\frac{C_3}{b_{20}}\right.$$

$$(e^{b_{20}T} - 1) - \frac{C_4}{(b_{20}+\mu)}(e^{-(b_{20}+\mu)T} - 1) + \frac{1}{K_{33}^2\mu}(K_{11}K_{33} + K_{22}(K_{33}T - \mu))e^{-\mu T}$$

$$+ \quad \frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T} - 1) - \frac{1}{K_{33}^2\mu}(K_{11}K_{33} - K_{22}\mu) + \frac{K_{11}'}{K_{33}\mu}(e^{-\mu T} - 1)\right\}$$

$$(3.20) \qquad\qquad\qquad\qquad + \quad \frac{M}{\mu}(e^{-\mu T} - 1)\Bigg]$$

3.2. Model-1a: Model with two stock-dependent breakable items.

In the above Model-1, if we take the the parametric values of breakability/deterioration which are directly related to the time equal to zero i.e. $b_{11} = 0$ and $b_{21} = 0$, then we get another Model-1a. Therefore, the Model-1 reduces to a production-inventory model for deteriorating items with stock dependent breakability/deterioration. So, the total profit can be obtain by optimizing the Eq. (3.20) with $b_{11} = 0$ and $b_{21} = 0$

3.3. Model-1b: Model with two items without breakability.

In the above Model-1, if we take the parametric value of deterioration which is directly related to stock and time is equal to zero i.e. $b_{10} = 0$, $b_{11} = 0, b_{20} = 0$, $b_{21} = 0$, then we get a another Model-1b. Therefore, the Model-1 reduces to a production-inventory model with out deteriorating item. As $b_{10}, b_{11}, b_{20}, b_{21}$ appears in the denominator of the expression of (3.20) So, the total profit can not obtain by optimizing the Eq. (3.20) by directly putting with $b_{10} = 0, b_{11} = 0$, $b_{20} = 0$, $b_{21} = 0$. Thus, for the total profit of Model-1b can be obtain by omitting the breakability term from the expression of 1 and 2 and processing the same way as before in Model-1.

3.4. Model-1c: Model with two breakable items with constant demand.

In the above Model-1, if we take the the parametric value of demand which is directly related to the time is equal to zero i.e. $b_1 = 0, b_2 = 0$, then we get a another Model-1c. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant demand . So, the total profit can be obtain by optimizing the Eq. (3.20) with $b_1 = 0$ and $b_2 = 0$.

3.5. Model-1d: Model two breakable items with constant holding cost. In the above Model-1, if we take the the parametric value of holding cost which is directly related to the time is equal to zero i.e. $C_{11} = 0, C_{21} = 0$, then we get a another Model-1d. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant holding cost. So, the total profit can be obtain by optimizing the Eq. (3.20) with $C_{11} = 0$ and $C_{21} = 0$.

3.6. Model-1e: Model with two breakable items with constant set-up cost. In the above Model-1, if we take the the parametric value of setup cost which is directly related to the production rate is equal to zero i.e. $C_{31} = 0, C_{41} = 0$, then we get a another Model-1e. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant set up cost. So, the total profit can be obtain by optimizing the Eq. (3.20) with $C_{31} = 0$ and $C_{41} = 0$.

4. Mathematical formulation of the proposed model with single item:

4.1. Model-2: Model with single item. In real life, the manager of a production firm always wants to produce more quantity through a long-run process by imposing over-time to its labour as well as machinery items. As a result, there may aries different types of difficulties in the production process which results the production of perfect quality item as well as defective item. These defective items are reworked instantly at a per unit cost to make the product as new as perfect one to maintain the brand image of the manufacturer. The production of the defective items increases with time and the reliability parameter of the produced item. The parameter r_1 is the reliability indicator of the item-1. The production system became more stable and reliable , if r_1 decreases i.e. smaller value of r_1 provides the better quality product and produced smaller imperfect quality unites.

The inventory levels decreases due to demand and deterioration. Thus, the change of inventory level at any time t can be represented by the following differential equation:

$$\frac{dq_1(t)}{dt} = P_1 - D_1 - B_1(q_1, t)$$
$$\text{i.e. } P_1(t) = \dot{q_1} + D_1 + B_1(q_1, t)$$

(4.1) $\quad\quad\quad$ with $q_1(0) = -S_1$ and $q_1(T) = 0$, $where\ D_1 \equiv D_1(t)$

where D_1 is the demand function of time t and is of the form $D_1(t) = a_1 + b_1 t$.

The end condition $q_1(0) = -S_1$ and $q_1(T) = 0$ indicate that at time $t = 0$ the maximum shortages is $-S_1$ i.e. the inventory starts with shortages at time $t = 0$. As P_1 and D_1 are the function of time t and combined effect of theses two the shortages reaches to zero and the inventory build-up as $P_1(t) > D_1 + B_1(q_1, t)$ in the first part of the cycle. After some time, as demand is a function of time t, D_1 is more than the combined effect of $D_1 + B_1(q_1, t)$ i.e. the accumulated stock decreases as $P_1(t) < D_1 + B_1(q_1, t)$ and ultimately the stock reaches to zero.

The corresponding profit function, incorporation the inflation and time value of money during the time duration $[0, T]$ is given by

$$Z_p = \int_0^T e^{-\mu t} \big[S_{p_1} D_1 - C_{p_1}(r_1, t) P_1(t) - C_{d_1} r_1 P_1(t) - C_{h_1}(t) q_1 - C_3(P_1(t))/T - S_{h_1} S_1 \big] dt$$

$$= \int_0^T e^{-\mu t} \big[S_{p_1} D_1 - (C_{r_1} + C_{d_1} r_1)(\dot{q_1} + D_1 + B_1) - \chi(r_1) - \lambda_1(\dot{q_1} + D_1 + B_1)^2 - (C_{10}$$

(4.2) $\quad\quad + C_{11} t) q_1 - \{ C_{30} + C_{31}(\dot{q_1} + D_1 + B_1) \}/T - S_{h_1} S_1 \big] dt$

$$= \int_0^T F(q_1, \dot{q}_1, t)dt$$

where $F(q_1, \dot{q}_1, t) = e^{-\mu t}\big[S_{p1}D_1 - (C_{r1} + C_{d1}r_1)(\dot{q}_1 + D_1 + B_1) - \chi(r_1) - \lambda_1$
$(\dot{q} + D_1 + B_1)^2 - (C_{10} + C_{11}t)q - \{C_{30} + C_{31}(\dot{q}_1 + D_1 + B_1)\}/T - S_{h_1}S_1\big]$

Now our problem is to find the path of $q_1(t)$ and $P_1(t)$ such that $F(q_1, \dot{q}_1, t)$ is to be maximized. Now, for find the optimal path, we have from the Euler-Lagranges equation for the maximum value of $F(q_1, \dot{q}_1, t)$ is

(4.3)
$$\frac{\partial F}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_1}\right) = 0$$

using (4.2), we have,

(4.4)
$$\ddot{q}_1 - \mu\dot{q}_1 - (b_{10} + \mu)b_{10}q_1 = H_1(t)$$

where

$$H_1(t) = a_1(\mu + b_{10}) - b_1 - b_{11} + \frac{(C_{r1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10}) + C_{10}}{2\lambda_1} + t$$

$$\left[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1}\right]$$

$$= K_1 + K_2 t$$

$$\text{where } K_1 = a_1(\mu + b_{10}) - b_1 - b_{11} + \frac{(C_{r1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10})}{2\lambda_1}$$

$$K_2 = \left[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1}\right]$$

The complementary function of the Eq. (4.4) is $C_1' e^{(b_{10} + \mu)t} + C_2' e^{-b_{10}t}$, where C_1' and C_2' are arbitrary constants and the particular integral is given by the $\frac{1}{D^2 - \mu D - (\mu + b_{10})b_{10}} H_1(t)$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(4.4) can be represented as

$$q_1(t) = C_1' e^{(b_{10} + \mu)t} + C_2' e^{-b_{10}t} - \frac{1}{K_3^2}\big[K_1 K_3 + K_2(K_3 t - \mu)\big]$$

and the corresponding rate is

(4.5)
$$P_1(t) = K_4 e^{(b_{10} + \mu)t} + K_5 t + K_7,$$

$$\text{where, } K_3 = b_{10}(b_{10} + \mu), K_4 = C_1'(2b_{10} + \mu), K_5 = (b_1 + b_{11} - \frac{b_{10}K_2}{K_3}),$$

$$\text{and } K_7 = \frac{1}{K_3^2}(a_1 K_3^2 + b_{10}K_2\mu - b_{10}K_1 K_3 - K_2 K_3)$$

Using the boundary conditions given with $q_1(0) = -S_1$ and $q_1(T) = 0$ in the expression of $q_1(t)$, we can get the value of C_1' and C_2'. Substituting the value of $q_1(t)$ and $P_1(t)$ in the expression of (4.2), the corresponding profit function can be expressed as

$$Z_p = \int_0^T e^{-\mu t}\big[S_{p_1}D_1 - (C_{r_1} + C_{d_1}r_1)(\dot{q}_1 + D_1 + B_1) - \chi(r_1) - \lambda_1(\dot{q}_1 + D_1 + B_1)^2$$

$$- (C_{10} + C_{11}t)q_1 - \{C_{30} + C_{31}(\dot{q}_1 + D_1 + B_1)\}/T - S_{h_1}S_1\big]dt$$

$$= S_{p_1}\Big[-a_1\frac{(e^{-\mu T}-1)}{\mu}-b_1\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\Big]-(C_{r_1}+r_1C_{d_1})\Big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}}$$

$$+ K_5\Big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\Big)+\frac{K_7}{\mu}(1-e^{-\mu T})\Big]+\Big[N_1+N_2$$

$$e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\Big]\Big(\frac{e^{-\mu T}-1}{\mu}\Big)+\lambda_1\Big[(K_4e^{(b_{10}+\mu)T}+K_5T+K_7)^2\frac{e^{-\mu T}}{\mu}-$$

$$\frac{(K_4+K_7)^2}{\mu}+\frac{2}{\mu^2}\{\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4K_5}{b_{10}}(e^{b_{10}T}-1)+\frac{K_4K_5}{b_{10}}$$

$$(b_{10}+\mu)\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\}+\frac{K_4K_7(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1)-\frac{K_5K_7}{\mu}(e^{-\mu T}-1)\}\Big]$$

$$-C_{10}\Big[\frac{C_1'}{b_{10}}(e^{b_{10}T}-1)-\frac{C_2'}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1K_3$$

$$+ K_2(K_3T-\mu))\frac{e^{-\mu T}}{\mu}+\frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{\mu K_3^2}(K_1K_3-K_2\mu)\Big]$$

$$- C_{11}\Big[C_1'(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2})+C_2'(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)}-\frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2}+\frac{1}{(b_{10}+\mu)^2})$$

$$- \frac{K_1}{K_3}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})-\frac{K_2\mu}{K_3^2}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{K_2}{K_3}(\frac{-T^2e^{-\mu T}}{\mu}$$

$$- \frac{2Te^{-\mu T}}{\mu^2}-\frac{2}{\mu^3}e^{-\mu T}+\frac{2}{\mu^3})\Big]-\frac{1}{T}\Big[\frac{C_{30}}{\mu}(1-e^{-\mu T})+C_{31}(\frac{K_4}{b_{10}}(e^{b_{10}T}-1)$$

$$(4.6) \quad + K_5(\frac{-Te^{\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{K_7}{\mu}(1-e^{-\mu T}))\Big]-S_{h_1}S_1(\frac{1-e^{-\mu T}}{\mu})$$

4.2. Model-2a: Model with single stock-dependent breakable item. In the above Model-2, if we take the the parametric values of breakability/deterioration which are directly related to the time equal to zero i.e. $b_{11}=0$, then we get another Model-2a. Therefore, the Model-2 reduces to a production-inventory model for deteriorating items with stock dependent breakability/deterioration. So, the total profit can be obtained by optimizing the Eq. (4.6) with $b_{11}=0$

4.3. Model-2b: Model with single non-breakable item. In the above Model-2, if we take the the parametric value of deterioration which is directly related to stock and time is equal to zero i.e. $b_{10}=0$, $b_{11}=0$, then we get a another model-2b. Therefore, the Model-2 reduces to a production-inventory model with out deteriorating item. As b_{10}, b_{11} appears in the denominator of the expression of (4.6), So the total profit can not obtain by optimizing the Eq. (4.6) by directly putting with $b_{10}=0$, $b_{11}=0$. Thus, for the total profit of Model-2b can be obtain by omitting the reliability term from the expression of (4.6) and processing the same way as before in Model-2.

4.4. Model-2c: Model with single breakable item with constant demand. In the above Model-2, if we take the the parametric value of demand which is directly related to the time is equal to zero i.e. $b_1=0$, then we get a another Model-2c. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant demand . So, the total profit can be obtain by optimizing the Eq. (4.6) with $b_1=0$.

4.5. Model-2d: Model with single breakable item with constant holding cost.
In the above Model-2, if we take the the parametric value of holding cost which is directly related to the time is equal to zero i.e. $C_{11} = 0$, then we get a another Model-2d. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant holding cost. So, the total profit can be obtain by optimizing the Eq. (4.6) with $C_{11} = 0$.

4.6. Model-2e: Model with single breakable item with constant set-up cost.
In the above Model-2, if we take the the parametric value of setup cost which is directly related to the production rate is equal to zero i.e. $C_{31} = 0$, then we get a another Model-2e. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant set up cost. So, the total profit can be obtain by optimizing the Eq. (4.6) with $C_{31} = 0$.

5. Solution procedure:

In section 3.2, we already prove that there exists a path $q = q_1(t)$ and $q = q_2(t)$ lying between the interval $[0, T]$ for which Z_p is maximum. In this problem, only the reliability indicator is the decision variable and others parameters are known, so the profit function Z_p given by (3.20) and (4.6) are the function of a two variable r_1 and r_2 for Model-1 and single variable r_1 for Model-2 respectively. So, there are two method for finding the optimal value of r_1 and r_2. First we discussed the analytical method for finding the optimal value of r_1 and r_2. To find the optimal value of r_1 and r_2, the first order partial derivative of the profit function with respect to r_1 and r_2 are made equal to zero. Thus for the Model-1, we get two different transcendental equation on r_1 and r_2 and for Model-2, we get one transcendental equation on r_1 and solve using Newton-Raphson method. Now to find the second order derivative of Z_p with respect to r_1 and r_2 are calculate separately for both the models. Both the value of second order derivative with to the calculated r_1 and r_2 value are less than zero i.e. $\frac{\partial^2 Z_T}{\partial r_1^2} < 0, \frac{\partial^2 Z_T}{\partial r_2^2} < 0, \frac{\partial^2 Z_T}{\partial r_1^2}\frac{\partial^2 Z_T}{\partial r_2^2} - \left(\frac{\partial^2 Z_T}{\partial r_1 \partial r_2}\right)^2 > 0$ for model-1 and $\frac{d^2 Z_T}{d r_1^2} < 0$ for Model-2. So for both the model, we conclude that both the profit function are maximized and the corresponding profit can be calculated by putting the value of r_1 and r_2 respectively for both the models. Also the profit functions are optimized using LINGO-13 software and the result obtained are same as those obtained by analytical method. Therefore, we conclude that the result obtained by the above mentioned procedure is a global optimal solution for different models.

6. Numerical Experiment:

Model-1: The following parametric value have been used to validate the model:
$a_1 = 60$; $b_1 = 50$; $\lambda_1 = 0.05$; $C_{r_1} = 4$; $C_{d_1} = 4$; $C_{10} = 1$, $C_{11} = 0.02$; $C_{30} = 10$; $C_{31} = 0.02$; $C_A = 0.002$; $S_{p_1} = 75$; $b_{10} = 0.05$; $b_{11} = 1.5$; $r_{1max} = 0.9$; $r_{1min} = 0.1$ $N_1 = 200$; $N_2 = 30$; $T = 12$; $S_{h1} = 1.02$; $S_1 = 10$; $S_2 = 20$; $\mu = 0.03$; $M = 2000$; $a_2 = 65$; $b_2 = 55$; $\lambda_2 = 0.06$; $C_{r_2} = 5$; $C_{d_2} = 5$; $C_{20} = 2$, $C_{21} = 0.03$; $C_{40} = 11$; $C_{41} = 0.03$; $S_{p_2} = 76$; $b_{20} = 0.06$; $b_{21} = 1.6$; $r_{2max} = 0.9$; $r_{2min} = 0.1$, $S_{h2} = 1.03$; $N_3 = 205$; $N_4 = 32$;
Model-2: In this model i.e.,only one item is considered. In this case, we consider the inputs of 1st item and all the parameters are same as Model 1.
With the above input data, the optimum values of r_1 and r_2 and the corresponding value of profit function for both the models are obtained and presented in Tables-1 and 2.

Table-1: Optimum results of Models-1

	Model-1	Model-1a	Model-1b	Model-1c	Model-1d	Model-1e
r_1	0.1065489	0.1066123	0.107523	0.107528	0.101572368	0.124578
r_2	0.1064327	0.1066529	0.107439	0.157423	0.1792436	0.14132563
Z_p	609575.45	611359.89	702204.56	120451.97	617561.24	624578.57

Table-2: Optimum results of Models -2

	Model-2	Model-2a	Model-2b	Model-2c	Model-2d	Model-2e
r_1	0.1065357	0.1066117	0.1081474	0.0059423	0.1065271	0.1065355
r_2	$--$	$--$	$--$	$--$	$--$	$--$
Z_p	307580.8	307731.6	398616.8	50141.1	315486	307629

With the optimal values of r_1 and r_2, different pictorial representations of inventory, production and demand against time, profit and development cost against reliability indicator , unit production cost and set-up cost against time for Model-1 are depicted in Figs.1-6, respectively. Similar graphical representation for Model-2 are given in Figs.7-9.
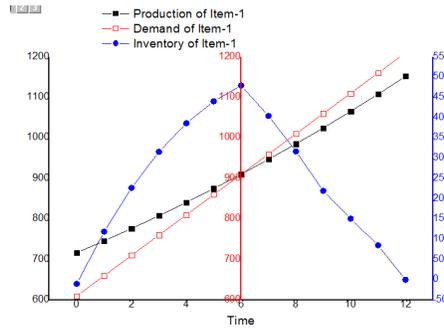
Figure 1. Time vs. production, demand and inventory of item-1 in Model-1
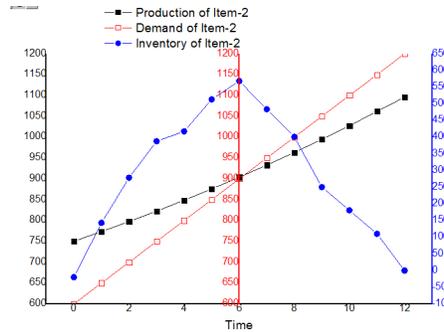
Figure 2. Time vs. production, demand and inventory of item-2 in Model-1
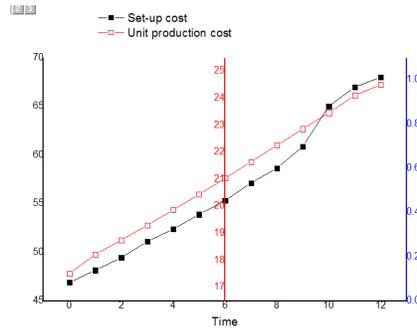
Figure 3. Time vs. unit production cost and set-up cost of item-1 in Model-1
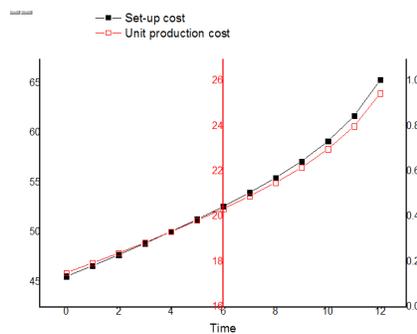
Figure 4. Time vs. unit production cost and set-up cost of item-2 in Model-1
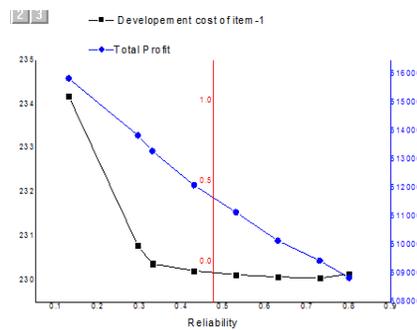
Figure 5. Reliability vs.developement cost and Profit of item-1 (Model-1)

7. Discussion:

For the presumed parametric values, it is very clear that profits for the models without damageability i.e. Model-1b and Model-2b gives the more profits than the corresponding models with damageability models such as Model-1, Model-1a, Model-2 and Model-2a. It is as per expectation of the real life phenomena. It occurs because profits decreases
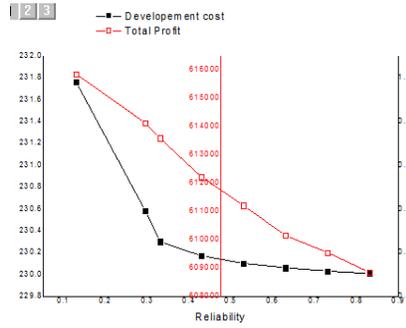
Figure 6. Reliability vs.developement cost and Profit of item-2 (Model-1)
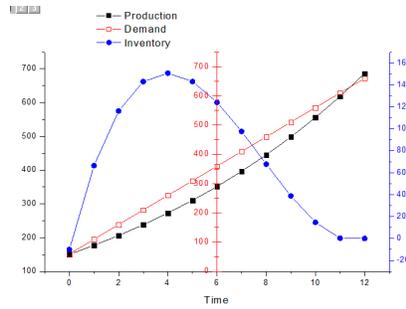
Figure 7. Time vs. production, demand and inventory for Model-2
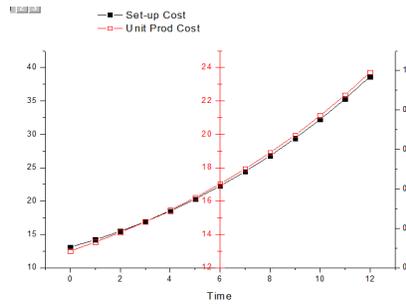
Figure 8. Time vs. set-up cost and unit production cost for Model-2

due to damageability of the units. Also from the Tables-1 and -2, it is observed that the models with stock dependent breakable items i.e. Model-1a and Model-2a give more profits than the corresponding models of breakable items i.e. Model-1 and Model-2. It is because the damageability rates for breakable items with both stock and time dependent breakability are higher than that of stock dependent breakability.

From Tables -1 and -2 it can be observe that the profit for the time dependent demand i.e Model -1 and -2 is greater than the constant demand i.e Model -1c and -2c, it can be explained from the real life situation that if demand increases with time then the profit will be more.It is also observed from Tables -1 and -2 that the profits for the constant
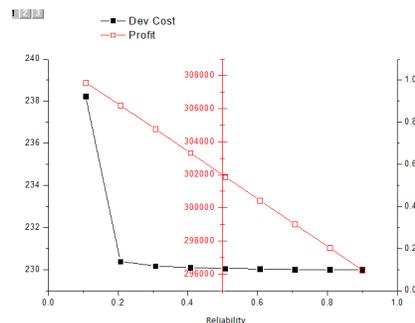
Figure 9. Reliability vs. profit and development cost for Model-2

holding cost and constant set up cost i.e profits for Model -1d,Model -2d,Model -1e and Model -2e are more than the profits for Model -1 and -2.It can be justified from the real fact that if unit holding cost and set up cost are constant than the retailer has to pay less amount for holding cost and set up cost and as a result the retailer gets more profit. Again process reliability indicators plays an important role for the profit making imperfect production system. Reliability indicator of a imperfect production process can be controlled using high quality machineries and skilled and efficient manpowers workers.

In our present investigation, demand of both models are time dependent. For both the models, production rate increases with time as demand increases with time. This phenomenon is justified by our pictorial representation i.e. Figs.-1,-2 and -7. It is observed from the Figs.1,2,7 that as the terminal conditions for stock are $q_1(0) = -10$, $q_1(T) = 0$, $q_2(0) = -20$, $q_2(T) = 0$ for Model-1 and $q_1(0) = -10$ and $q_1(T) = 0$ for Model-2, initially when time $t = 0$ shortages occurs at maximum level and as the demand and production dependent on time t and due to their combine effect, the shortages reach to zero after certain time. Due to this effect, the inventory is built-up as production is greater than the combine effect of demand and damageability. But after some time when considerable stock is built-up i.e., when the stock level becomes highest,production is discontinued. After this, to meet the demand, after allowing breakability, stock gradually reduces and ultimately becomes zero at $t = T$. Again from the Figs.-5,-6 and-9, it is observed that optimum profit Z_T is attained for a particular value of the process reliability r. Also it is noticed that the profit deceases with increasing process reliability,since reliability is defined as the ratio of number of damageable item with total items. Since the breakability increases with time i.e. damageability increases with time, so the profit is decreases with increasing reliability. This Phenomenon is also agree with the real life situation. Again from this figure, we observed that for some initial increasing value of r, the development cost sharply decreases and then become almost constant for higher values of r.Initially the profit become maximum and then decreases with increasing reliability. As set-up cost and production cost are partially production dependent, and production is time dependent, set-up cost and production cost increases with increasing time.These observation are found from the Figs.-3,-4 and-8.

8. Conclusions and Future Research work:

In this paper, for the first time, a multi-item production-inventory model with imperfect production process is considered for a breakable or deteriorating item over finite time horizon, where the process reliability indicator of the production process together with

the production rate is controllable. For the present models, we observed that an optimum reliability indicator lures the maximum profit for an item having time dependent demand. Also it is found from our findings that minimum unit production cost for an item does not guaranty for giving maximum profit always. From the present models, it can be concluded that optimal control of production rate reduces holding cost as well as damageability which in turn increases profit separately for breakable/deteriorating items. The present investigation reveals that process reliability indicator is an important factor which determines the production rate and thus determining the optimal production path, unit production cost and optimal profit for the production-inventory managers. Here we formulate two types of models with shortages. First model is for two items with shortages and second model is for single item with shortages. The unit production cost is a function of production rate, raw material cost, labour charge, wear and tear cost and product reliability indicator. The first model is formulated as optimal control problems for the maximization of total profits over the planning horizon with budget constraint and optimum profit with profits along with optimum reliability indicator(r) are obtained using Euler-Lagrange equations based on variational principle.The second model is also solved under the same assumptions and using the same technique. Both the problems have been solved using a non-linear optimization technique -GRG (LINGO-13.0) and illustrated with some numerical data. Several particular cases are derived and the results are presented in both tabular and graphical forms. Finally, some sensitivity analyses can be made with respect to different parameters.The present models can be extended to fuzzy environment taking constant part of holding cost, set-up cost, etc as fuzzy in nature.Now a days due to inherent various and highly uncertinity of real life informations/data , impreciseness of fuzzy set i.e type-2 fuzzy sets in quite popular. Hence the present problem can be solved with type-2 fuzzy inventory cost,etc.This is a new area of research in which integrand of a finite integral is fuzzy or type-2 fuzzy and variational principle is applied.

More-over with the deterministic integrand and the limits of a finite integral as fuzzy, models can be formulated and solved using Fuzzy Riemann integral, not using variational principle.

The present model can also be extended to multi-period models where period starts with inventory and end with shortages or starts with inventory and end with inventory or different variations can be done with respect to shortages.

9. Acknowledgement:

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