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Comparison of loss functions for estimating the scale parameter of log-normal distribution using non-informative priors

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Abstract

The estimation of parameters of distributions is a core topic in the literature on Statistical methodology. Many Bayesian and classical approaches have been derived for estimating parameters. In this study, Bayesian estimation technique is adopted for the comparison of two non-informative priors and six loss functions to estimate the scale parameter of Log-Normal distribution assuming fixed values of location parameter. The main purpose of this study is to search for a suitable prior when no prior information is available and to look for an appropriate loss function for estimation of the scale parameter of Log-Normal distribution. Through simulation study, comparisons are made on the basis of the posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. The simulation results are verified through a real data set of lung cancer patients.

Keywords: Prior distribution, Posterior distribution, Log-Normal distribution, Inverted Gamma distribution.

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1. Introduction

Suppose a random variable Y is normally distributed with mean θ and variance ϕ then $X = exp(Y)$ is distributed Log-Normally with location and scale parameters θ and ϕ, respectively. The probability density function (pdf) of Log-Normal random variable X is:

$$f(x;\theta,\phi) = \frac{1}{x\sqrt{2\pi\phi}}\, e^{-\frac{1}{2\phi}(\ln x-\theta)^2},\ x > 0,\ -\infty < \theta < \infty,\ \phi > 0 \tag{1.1}$$

where, θ is location and ϕ is scale parameter.

The cumulative distribution function of this distribution is given by

$$F(x) = \frac{1}{2} + \frac{1}{2}Erf\left(\frac{\ln x - \theta}{\sqrt{2\phi}}\right) \tag{1.2}$$

The Log-normal distribution, defined in equation (1.1), has become a convenient model for different biological, social and life testing phenomena. This distribution has wide applications in business and economics such as modeling of firm sizes, incomes, stock prices, lengths of service in labor turnover contexts and many other fields. Finney [2] obtained formulae for efficient estimation of the mean and variance of a population using sample information from the Log-Normal distribution. Tiku [13] found the estimators of parameters of Log-Normal distribution using type-II censored sample data. He obtained the asymptotic variances and covariances of the estimators. Zellner [14] used Bayesian and non-Bayesian methods for estimating parameters of the log-normal distribution and of log-normal regression processes. He derived posterior distributions for parameters of interest and described their statistical properties. Stedinger [12] evaluated efficiency of different methods for fitting two-parameter and three-parameter log-normal distributions. He made the comparison using mean square error of estimators. Alternatively, Shen [10] combined the orthogonal transformation and the Rao-Blackwell Theorem for deriving uniform minimum variance unbiased estimators (UMVUEs) for the parameters of the Log-Normal distribution. Limpert et al. [5] discussed the use of Log-Normal distribution in different fields of science, specially in biological sciences. For the two-parameter log-normal distribution, Khan et al. [4] derived the prediction of future responses assuming a non-informative prior and an informative prior for the parameters under type-II censored sampling and type-II median censored sampling. Mehta et al. [7] proposed a simple and novel method to approximate the sum of several log-normal random variables with a single log-normal random variable. Martín and Pérez [6] presented generalized form of the log-normal distribution and analyzed it through Bayesian tools. Saleem and Aslam [9] used Bayesian tools of inference to estimate the parameters of two-component mixture of Rayleigh distributions assuming the uniform and the Jeffreys priors. Rupasov et al. [8] showed that trial to trial neuronal variability of electromyographic (EMG) signals can be well described by the Log-Normal distribution. They also found that the variability of temporal parameters of handwriting duration and response time can also be well described by the Log-Normal distribution. Sindhu and Aslam [11] estimated the parameters of Inverse Weibul distribution under different loss functions and different priors.

Loss function plays a vital rule in Bayesian estimation problems. Loss function is the penalty for not getting the actual value. Zellner [15] discussed estimation of parameters of different models including Log-Normal under Varian's asymmetric LINEX loss function. Fabrizi and Trivisano [1] proposed a generalized inverse Guassian prior for the variance parameter of Log-Normal distribution and discussed the estimation of its mean under Quadratic Loss Function (QLF).

Estimation of scale parameter of Log-Normal distribution is not yet been considered under different loss functions using non-informative priors. Keeping in mind the above discussion and motivated by importance of the Log-Normal distribution in different fields, this study is made to look for best loss function and appropriate non-informative prior. In this paper, the posterior distributions for the scale parameter ϕ are derived under Uniform and Jeffreys priors. Also, Bayes estimators (BEs) and Bayes risks (BRs) are obtained using Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Weighted Loss Function (WLF), Precautionary Loss Function (PLF), Simple Asymmetric Precautionary Loss Function (SAPLF) and DeGroot Loss Function (DLF).

Comparisons of the priors and the loss functions are made on the basis of posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. For these comparisons, different sample sizes, different loss functions and different choices of location parameter θ have been considered. The rest of the paper is designed as follows.

In section 2, the posterior distributions of the scale parameter have been derived under Uniform and Jeffreys priors. A simulation study is presented in section 3 to compare the performance of the two priors on the basis of posterior variance, skewness and ex-kurtosis. Bayes estimators (BEs) and Bayes risks (BRs) under the considered loss functions are given in section 4. To look for best non-informative prior and loss function for the estimation of the scale parameter, a simulation study is carried out in section 5. A real data set of lung cancer patients is used in section 6 to draw graphs of the posterior distributions for different values of the location parameter and to verify the simulation results discussed in section 5.

2. The Posterior Distributions of the Scale Parameter under Non-Informative Priors

The Likelihood function of the Log-normal distribution can be written as under.

$$L(\phi) = \frac{(2\pi\phi)^{-\frac{n}{2}}}{\prod_{i=1}^{n} x_i}\, e^{-\frac{1}{2\phi}\sum_{i=1}^{n}(\ln x_i-\theta)^2} \tag{2.1}$$

We assume the improper Uniform prior ($U(0,\infty)$) for ϕ which can be written as

$$\Phi_U(\phi) \propto 1,\ \phi > 0. \tag{2.2}$$

The Posterior Distribution of ϕ given the data, under the above prior is given by

$$p(\phi|\mathbf{x}) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\phi^{-(\alpha_1+1)}e^{-\frac{\beta_1}{\phi}},\ \ \phi > 0 \tag{2.3}$$

where $\alpha_1 = \frac{n}{2} - 1$ and $\beta_1 = \frac{1}{2}\sum_{i=1}^{n}(\ln x_i - \theta)^2$.

The expression in 2.3 can be identified as Inverted Gamma distribution.

The Jeffreys prior for ϕ is

$$\Phi_J(\phi) \propto \frac{1}{\phi},\ \phi > 0. \tag{2.4}$$

The posterior distribution of ϕ given the data, using Jeffreys prior is given by

$$p(\phi|\mathbf{x}) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\phi^{-(\alpha_2+1)}e^{-\frac{\beta_2}{\phi}},\ \phi > 0 \tag{2.5}$$

where $\alpha_2 = \frac{n}{2}$ and $\beta_2 = \frac{1}{2}\sum_{i=1}^{n}(\ln x_i - \theta)^2$.

The expression in 2.5 can be identified as Inverted Gamma distribution.

3. Simulation Study for Comparison of Priors on the basis of Posterior Variances, Co-Efficient of Skewness and Ex-Kurtosis

Consider the generation of random samples of sizes $n = 30, 50, 100, 200$ and 500 from the Log-Normal distribution assuming the location parameter $\theta = 1, 2, 3$ and the scale parameter $\phi = 1, 4, 7$. The simulation process is repeated $10,000$ times and the results have then been averaged.

The results of posterior variance, skewmess and ex-kurtosis are showcased in the following tables.

Table 1. Posterior Variance for Different Values of θ and ϕ

n	Prior	$\theta = 1$			$\theta = 2$			$\theta = 3$		
		$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$
30	Uniform	0.11885	30.4636	283.5813	0.11764	30.2204	281.9675	0.11712	30.2193	284.025
50		0.05619	14.2796	133.6523	0.05604	14.3247	134.2029	0.05569	14.2144	134.045
100		0.02360	6.01455	56.68350	0.02352	6.03511	56.61045	0.02347	5.99707	56.4623
200		0.01085	2.77936	26.00710	0.01087	2.77255	26.01307	0.01083	2.76955	25.9318
500		0.00414	1.05660	9.922510	0.00413	1.05823	9.913580	0.00413	1.05553	9.91950
30	Jeffreys	0.09363	24.2955	226.2839	0.09394	24.1826	225.9903	0.09475	24.0629	225.914
50		0.04923	12.5365	118.0433	0.04864	12.5922	117.5760	0.04932	12.5574	118.546
100		0.02210	5.66527	53.18104	0.02198	5.67742	52.97626	0.02220	5.68842	53.2441
200		0.01054	2.69131	25.18842	0.01052	2.69231	25.19030	0.01051	2.69701	25.3189
500		0.00409	1.04692	9.805566	0.00408	1.04212	9.804080	0.00408	1.04248	9.79537

Table 2. Posterior Skewness for Different Values of θ and ϕ

n	Prior	$\theta = 1$			$\theta = 2$			$\theta = 3$		
		$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$
30	Uniform	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492
50		0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335
100		0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904
200		0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259
500		0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389
30	Jeffreys	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046
50		0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799
100		0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741
200		0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206
500		0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376

Table 3. Posterior Ex-Kurtosis for Different Values of θ and ϕ

n	Prior	$\theta=1$			$\theta=2$			$\theta=3$		
		$\phi=1$	$\phi=4$	$\phi=7$	$\phi=1$	$\phi=4$	$\phi=7$	$\phi=1$	$\phi=4$	$\phi=7$
30	Uniform	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818
50		1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714
100		0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826
200		0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842
500		0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285
30	Jeffreys	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909
50		1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052
100		0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327
200		0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508
500		0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235

To search for a suitable prior for the scale parameter ϕ of the Log-Normal distribution, different properties of the posterior distributions have been checked under the two assumed priors and for different values of the location parameter θ.

It is clear from Tables 1, 2 and 3 that as the sample size increases, posterior variances decrease. From Table 1, it can be seen that the posterior variances for Jeffreys prior are smaller for all the values of θ, considered in the simulation study. Specifically, for $\theta = 3$, the posterior variances are minimum. Tables 2 and 3 show that both the skewness and ex-kurtosis are positive. Therefore, both the posteriors are positively skewed and are lepto-kurtic. Skewness and ex-kurtosis decrease with the increase in sample size. The choices of location parameter θ and the scale parameter ϕ put no effect on these two quantities. Both co-efficients of skewness and ex-kurtosis for posterior distribution obtained under Jeffreys prior are minimum.

It can be concluded, when no prior information is in hand, that Jeffreys prior performs better than the Uniform prior for estimating the scale parameter ϕ of the Log-Normal model.

4. Bayes Estimators and Bayes Risks under Different Loss Functions

In this section, Bayes estimators (BE) and Bayes risks (BR) are derived for different loss functions under the considered priors.

4.1. BE and BR under Squared Error Loss Function (SELF). For an estimator ϕ^* of ϕ, the SELF is defined as follows.

$$L(\phi, \phi^*) = (\phi - \phi^*)^2 \tag{4.1}$$

The BE under this loss function is:

$$\phi^* = E_{\phi|\mathbf{x}}(\phi) \tag{4.2}$$

where $E_{\phi|\mathbf{x}}$ is the Expectation over the posterior distribution.

The BR under SELF is given by:

$$\rho(\phi^*) = E_{\phi|\mathbf{x}}(\phi^2) - (E_{\phi|\mathbf{x}}(\phi))^2 \tag{4.3}$$

The BEs and BRs under SELF are given in the following table.

Table 4. BEs and BRs under SELF

Prior	$BE = \frac{\beta}{\alpha-1}$	$BR = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-4}$	$\frac{2\left\{\sum_{i=1}^{n}(\ln x_i-\theta)^2\right\}^2}{(n-4)^2(n-6)}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-2}$	$\frac{2\left\{\sum_{i=1}^{n}(\ln x_i-\theta)^2\right\}^2}{(n-2)^2(n-4)}$

4.2. BE and BR under Quadratic Loss Function (QLF). The QLF, for an estimator ϕ^* of the parameter ϕ, is defined as:

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi^2} \tag{4.4}$$

The BE under the above loss function is given below.

$$\phi^* = \frac{E_{\phi|\mathbf{x}}\left(\phi^{-1}\right)}{E_{\phi|\mathbf{x}}\left(\phi^{-2}\right)} \tag{4.5}$$

The BR under QLF is of the following form.

$$\rho(\phi^*) = 1 - \frac{\left(E_{\phi|\mathbf{x}}(\phi)^{-1}\right)^2}{E_{\phi|\mathbf{x}}(\phi)^{-2}} \tag{4.6}$$

The BEs and BRs, using the two priors, under QLF are given in the following table.

Table 5. BEs and BRs under QLF

Prior	$BE = \frac{\beta}{\alpha+1}$	$BR = \frac{1}{\alpha+1}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n}$	$\frac{2}{n}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n+2}$	$\frac{2}{n+2}$

4.3. BE and BR under Weighted Loss Function (WLF). The mathematical form of this loss function, for an estimator ϕ^* of the parameter ϕ, is as under.

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi} \tag{4.7}$$

The BE under WLF is of the following form.

$$\phi^* = \left\{E_{\phi|x}\left(\frac{1}{\phi}\right)\right\}^{-1} \tag{4.8}$$

The BR under WLF is written as under.

$$\rho(\phi^*) = E_{\phi|\mathbf{x}}(\phi) - \left\{E_{\phi|\mathbf{x}}\left(\frac{1}{\phi}\right)\right\}^{-1} \tag{4.9}$$

The following table contains BEs and BRs, using the two priors, under WLF.

Table 6. BEs and BRs under WLF

Prior	$BE = \frac{\beta}{\alpha}$	$BR = \frac{\beta}{\alpha(\alpha-1)}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-2}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{(n-2)(n-4)}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n(n-2)}$

4.4. DeGroot Loss Function (DLF). For an estimator ϕ^* of the parameter ϕ, the DLF is written mathematically as follows.

$$L(\phi, \phi^*) = \frac{(\phi - \phi^*)^2}{\phi^{*2}} \tag{4.10}$$

The BE under this loss function is as under.

$$\phi^* = \frac{E_{\phi|\mathbf{x}}(\phi^2)}{E_{\phi|\mathbf{x}}(\phi)} \tag{4.11}$$

Under DLF, the BR is of the form.

$$\rho(\phi^*) = 1 - \frac{\left\{E_{\phi|\mathbf{x}}(\phi)\right\}^2}{E_{\phi|\mathbf{x}}(\phi)^2} \tag{4.12}$$

The BEs and BRs, using the two priors, under DLF are contained in the following table.

Table 7. BEs and BRs under DLF

Prior	$BE = \frac{\beta}{\alpha-2}$	$BR = \frac{1}{\alpha-1}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-6}$	$\frac{2}{n-4}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-4}$	$\frac{2}{n-2}$

4.5. BE and BR under Precautionary Loss Function (PLF). Let ϕ^* be an estimator of a parameter ϕ, then PLF can be defined through the following equation.

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi^*} \tag{4.13}$$

The BE under PLF is given below.

$$\phi^* = \sqrt{E_{\phi|\mathbf{x}}(\phi^2)} \tag{4.14}$$

The BR under PLF is as under.

$$\rho(\phi^*) = 2\left\{\sqrt{E_{\phi|\mathbf{x}}(\phi^2)} - E_{\phi|\mathbf{x}}(\phi)\right\} \tag{4.15}$$

The BEs and BRs, using the two priors, under PLF are presented in the following table.

Table 8. BEs and BRs under PLF

Prior	$BE = \frac{\beta}{\sqrt{(\alpha-1)(\alpha-2)}}$	$BR = 2\left\{\frac{\beta}{\sqrt{(\alpha-1)(\alpha-2)}} - \frac{\beta}{\alpha-1}\right\}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-4)(n-6)}}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-4)(n-6)}} - \frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-4}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}} - \frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-2}$

4.6. BE and BR under Simple Asymmetric Precautionary Loss Function (SAPLF). The SAPLF, for an estimator ϕ^* of a parameter ϕ^*, is defined as follows

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi\phi^*} \tag{4.16}$$

The BE under SAPLF is as under.

$$\phi^* = \sqrt{\frac{E_{\phi|\mathbf{x}}(\phi)}{E_{\phi|\mathbf{x}}(\phi^{-1})}} \tag{4.17}$$

The BR under SAPLF is as follows.

$$\rho(\phi^*) = 2\left\{\sqrt{E_{\phi|\mathbf{x}}(\phi)\,E_{\phi|\mathbf{x}}(\phi^{-1})} - 1\right\} \tag{4.18}$$

The BEs and BRs, using the two priors, under SAPLF are shown in the following table.

Table 9. BEs and BRs under SAPLF

Prior	$BE = \frac{\beta}{\sqrt{\alpha(\alpha-1)}}$	$BR = 2\left\{\sqrt{\frac{\alpha}{\alpha-1}} - 1\right\}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}}$	$2\left\{\sqrt{\frac{n-2}{n-4}} - 1\right\}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{n(n-2)}}$	$2\left\{\sqrt{\frac{n}{n-2}} - 1\right\}$

It can easily be depicted from the expressions of BRs in Tables 4−9 that Jeffreys prior requires less number of observations than the Uniform prior.

5. Simulation Study for Bayes Estimators and Bayes Risks under Different Loss Functions

A simulation study is carried out to obtain the BEs and BRs under different loss functions using different priors. The simulation process is repeated 10,000 times considering generation of random samples of sizes 30, 50, 100, 200 and 500 from Log-Normal distribution assuming $\phi = 1, 4, 7$ and $\theta = 1, 2, 3$, and the results have then been averaged. These results are presented in the following tables.

Table 10. BE and BR under SELF for different values of θ and ϕ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.15610	0.11885	18.52246	30.5066	56.6850	285.3746	1.15067	0.11764	18.4261	30.18859	56.4148	283.1549	1.14740	0.11712	18.3917	30.07997	56.6257	284.8302
50		1.08966	0.05619	17.37730	14.2768	53.2779	134.0584	1.08900	0.05604	17.3905	14.30565	53.2108	133.8097	1.08531	0.05569	17.4230	14.3490	53.3653	134.5817
100		1.04295	0.02360	16.70100	6.05753	51.0567	56.58021	1.04104	0.02352	16.7038	6.053948	50.9718	56.40364	1.03978	0.02347	16.7008	6.05122	51.0835	56.62360
200		1.02092	0.01085	16.2807	2.75962	49.9615	25.98787	1.02177	0.01087	16.3434	2.781177	49.9828	26.01366	1.02010	0.01080	16.3055	2.76811	49.9114	25.94280
500		1.00884	0.00414	16.1477	1.05986	49.3953	9.918169	1.00857	0.00413	16.1233	1.056755	49.3473	9.898489	1.00763	0.00413	16.1270	1.05722	49.3409	9.89536
30	Jeffreys	1.06848	0.09363	17.0768	23.9193	52.3748	225.4040	1.06970	0.09394	17.1377	24.0990	52.4906	225.9511	1.07458	0.09475	17.1855	24.2322	52.5921	226.905
50		1.04362	0.04923	16.6159	12.4797	50.9967	117.5658	1.03705	0.04864	16.6749	12.5786	51.0691	117.9625	1.04420	0.04932	16.6611	12.5524	50.9845	117.615
100		1.01979	0.02210	16.3062	5.65119	49.9702	53.07748	1.01701	0.02198	16.3365	5.67291	50.0336	53.21180	1.02176	0.02220	16.2968	5.64316	50.0850	53.2689
200		1.01145	0.01054	16.1601	2.69165	49.5063	25.2586	1.01029	0.01052	16.1555	2.68966	49.4593	25.21420	1.00971	0.01051	16.1558	2.6904	49.5561	25.3112
500		1.00457	0.00409	16.0601	1.04411	49.1995	9.80001	1.00423	0.00408	16.0619	1.04442	49.1835	9.792590	1.00390	0.00408	16.0679	1.04520	49.2420	9.81679

Table 11. BE and BR under QLF for different values of θ and ϕ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	0.99683	0.06667	15.9892	0.06667	49.1442	0.06667	1.00001	0.06667	16.0197	0.06667	49.0209	0.06667	0.99872	0.06667	15.9924	0.06667	48.9732	0.06667
50		1.00010	0.04000	15.9874	0.04000	49.0865	0.04000	0.99895	0.04000	15.9871	0.04000	49.0378	0.04000	0.99732	0.04000	16.0079	0.04000	49.0887	0.04000
100		0.99876	0.02000	15.9954	0.02000	49.0019	0.02000	1.00020	0.02000	15.9781	0.02000	49.1506	0.02000	1.00026	0.02000	16.0037	0.02000	48.9685	0.02000
200		1.00048	0.01000	15.9864	0.01000	49.0171	0.01000	0.99951	0.01000	16.0114	0.01000	48.9663	0.01000	1.00032	0.01000	16.0211	0.01000	49.0125	0.01000
500		0.99990	0.00400	16.0153	0.00400	49.0476	0.00400	0.99996	0.00400	15.9822	0.00400	49.0226	0.00400	1.00059	0.00400	16.0141	0.00400	49.0054	0.00400
30	Jeffreys	0.93610	0.06250	14.9223	0.06250	45.8692	0.06250	0.93545	0.06250	15.0120	0.06250	45.8977	0.06250	0.93735	0.06250	14.9988	0.06250	45.9757	0.06250
50		0.96133	0.03846	15.3730	0.03846	47.0391	0.03846	0.96016	0.03846	15.3760	0.03846	47.3039	0.03846	0.96199	0.03846	15.3717	0.03846	46.9715	0.03846
100		0.98191	0.01961	15.6737	0.01961	47.9365	0.01961	0.97958	0.01961	15.6617	0.01961	48.0052	0.01961	0.98258	0.01961	15.6731	0.01961	48.0314	0.01961
200		0.99101	0.00990	15.8256	0.00990	48.5146	0.00990	0.99037	0.00990	15.8430	0.00990	48.5811	0.00990	0.98921	0.00990	15.8360	0.00990	48.5113	0.00990
500		0.99697	0.00398	15.9414	0.00398	48.7507	0.00398	0.99711	0.00398	15.9281	0.00398	48.7708	0.00398	0.99693	0.00398	15.9442	0.00398	48.7972	0.00398

Table 12. BE and BR under DLF for different values of θ and ϕ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.24656	0.07692	19.9602	0.076923	61.1146	0.076923	1.24301	0.07692	19.9337	0.076923	61.3169	0.076923	1.24796	0.07692	19.9892	0.076923	61.1388	0.076923
50		1.13850	0.04348	18.1615	0.043478	55.5427	0.043478	1.13462	0.04348	18.1893	0.043478	55.6496	0.043478	1.13137	0.04348	18.1539	0.043478	55.5593	0.043478
100		1.06318	0.02083	17.0610	0.020833	52.2324	0.020833	1.06191	0.02083	17.0426	0.020833	52.1436	0.020833	1.06028	0.02083	17.0164	0.020833	52.1689	0.020833
200		1.03232	0.01020	16.4914	0.010204	50.5055	0.010204	1.03061	0.01020	16.4988	0.010204	50.5139	0.010204	1.03112	0.01020	16.4932	0.010204	50.5463	0.010204
500		1.01192	0.00403	16.1892	0.004032	49.5339	0.004032	1.01224	0.00403	16.2136	0.004032	49.6621	0.004032	1.01274	0.00403	16.1880	0.004032	49.6351	0.004032
30	Jeffreys	1.15722	0.07143	18.4422	0.071429	56.6918	0.071429	1.15580	0.07143	18.4283	0.071429	56.3957	0.071429	1.14590	0.07143	18.4731	0.071429	56.6648	0.071429
50		1.08962	0.04167	17.3342	0.041667	53.2016	0.041667	1.08643	0.04167	17.3816	0.041667	53.4161	0.041667	1.09065	0.04167	17.3178	0.041667	53.2236	0.041667
100		1.04305	0.02041	16.6493	0.020408	50.8967	0.020408	1.04361	0.02041	16.6594	0.020408	51.0393	0.020408	1.04216	0.02041	16.6128	0.020408	51.0043	0.020408
200		1.02001	0.01010	16.3188	0.010101	49.9748	0.010101	1.02134	0.01010	16.3336	0.010101	49.9855	0.010101	1.01959	0.01010	16.3251	0.010101	50.0521	0.010101
500		1.00817	0.00402	16.1329	0.004016	49.4114	0.004016	1.00777	0.00402	16.1279	0.004016	49.4766	0.004016	1.00817	0.00402	16.1195	0.004016	49.4526	0.004016

Table 13. BE and BR under PLF for different values of θ and ϕ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30		1.19900	0.09412	19.2689	1.50411	58.9176	4.62280	1.20447	0.09406	19.1396	1.50174	58.8260	4.61562	1.20300	0.09400	19.1579	1.50317	58.7515	4.60977
50		1.10650	0.04890	17.8331	0.78000	54.1936	2.38243	1.11411	0.04889	17.8105	0.78298	54.4042	2.39169	1.11085	0.04892	17.7623	0.78086	54.2870	2.38653
100	Uniform	1.04918	0.02208	16.8483	0.35191	51.4916	1.07839	1.05408	0.02203	16.8532	0.35296	51.7212	1.08320	1.05465	0.02206	16.8587	0.35307	51.5888	1.08042
200		1.02585	0.01051	16.4034	0.16813	50.3456	0.51505	1.02526	0.01049	16.4234	0.16802	50.2900	0.51448	1.02659	0.01050	16.4062	0.16784	50.2758	0.51433
500		1.01034	0.00408	16.1673	0.06524	49.4900	0.19976	1.01074	0.00408	16.1734	0.06528	49.5453	0.19998	1.01002	0.00408	16.1553	0.06521	49.4755	0.19970
30		1.10422	0.08083	17.7628	1.29227	54.4858	3.96394	1.11366	0.08065	17.8432	1.29813	54.7219	3.98112	1.10893	0.08077	17.7381	1.29048	54.7425	3.98262
50		1.06769	0.04483	17.0319	0.71721	52.1369	2.19548	1.06427	0.04496	17.0239	0.71688	52.1163	2.19462	1.06592	0.04502	17.0836	0.71939	52.0445	2.19159
100	Jeffreys	1.03147	0.02115	16.5329	0.33915	50.4415	1.03473	1.03123	0.02111	16.4857	0.33818	50.5669	1.03730	1.03206	0.02114	16.5087	0.33865	50.5803	1.03757
200		1.01443	0.01029	16.2330	0.16439	49.6779	0.50307	1.01520	0.01028	16.2528	0.16459	49.7859	0.50417	1.01576	0.01029	16.2391	0.16445	49.8353	0.50466
500		1.00522	0.00404	16.0991	0.06472	49.3332	0.19832	1.00423	0.00405	16.0972	0.06471	49.3168	0.19826	1.00528	0.00405	16.0805	0.06465	49.2540	0.19801

Table 14. BE and BR under WLF for different values of θ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30		1.07093	0.08238	17.0964	1.31511	52.3845	4.02958	1.07238	0.08249	17.1583	1.31987	52.2952	4.02271	1.06978	0.08229	17.1576	1.31982	52.7499	4.05769
50		1.04137	0.04528	16.7085	0.72646	51.1505	2.22394	1.04207	0.04531	16.6322	0.72314	51.1362	2.22332	1.04296	0.04535	16.6528	0.72404	51.1519	2.22400
100	Uniform	1.02233	0.02130	16.3509	0.34065	49.9914	1.04149	1.02088	0.02127	16.3070	0.33973	49.8856	1.03928	1.01842	0.02122	16.3514	0.34066	49.9854	1.04136
200		1.01161	0.01032	16.1406	0.16470	49.4889	0.50499	1.01038	0.01031	16.1580	0.16488	49.4440	0.50453	1.01029	0.01031	16.1529	0.16483	49.5732	0.50585
500		1.01161	0.01032	16.0685	0.06479	49.1605	0.19823	1.01038	0.01031	16.0550	0.06474	49.1999	0.19839	1.01029	0.01031	16.0480	0.06471	49.2116	0.19843
30		0.99817	0.07130	16.0086	1.14347	48.8909	3.49221	1.00151	0.07154	15.9995	1.14283	48.9981	3.49986	0.99775	0.07127	15.9993	1.14281	49.0448	3.50320
50		1.00190	0.04175	15.9760	0.66567	49.1387	2.04745	1.00016	0.04167	15.9847	0.66603	49.0197	2.04249	0.99927	0.04164	16.0619	0.66925	48.8018	2.03341
100	Jeffreys	0.99962	0.02040	16.0068	0.32667	48.9493	0.99897	1.00114	0.02043	16.0117	0.32677	49.1360	1.00278	1.00192	0.02045	15.9928	0.32638	49.0456	1.00093
200		1.00433	0.00405	16.0147	0.16177	48.9206	0.49415	1.00342	0.00405	16.0418	0.16204	49.0315	0.49527	1.00444	0.00405	16.0020	0.16164	48.9709	0.49466
500		1.00061	0.00402	15.9865	0.06420	48.9984	0.19678	1.00041	0.00402	15.9710	0.06414	48.9624	0.19664	0.99911	0.00401	16.0096	0.06430	49.0144	0.19685

Table 15. BE and BR under SAPLF for different values of θ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.10882	0.07550	17.8537	0.07550	54.2717	0.07550	1.10565	0.07550	17.7617	0.07550	54.1956	0.07550	1.11006	0.07550	17.8116	0.07550	54.4152	0.07550
50		1.06607	0.04302	17.0353	0.04302	52.1882	0.04302	1.06244	0.04302	17.0284	0.04302	52.1682	0.04302	1.05939	0.04302	17.0585	0.04302	52.2884	0.04302
100		1.03035	0.02073	16.5174	0.02073	50.6166	0.02073	1.02912	0.02073	16.4974	0.02073	50.4981	0.02073	1.02754	0.02073	16.5164	0.02073	50.5237	0.02073
200		1.01661	0.01018	16.2418	0.01018	49.7278	0.01018	1.01493	0.01018	16.2476	0.01018	49.7359	0.01018	1.01543	0.01018	16.2452	0.01018	49.8430	0.01018
500		1.00585	0.00403	16.1019	0.00403	49.2450	0.00403	1.00647	0.00403	16.1005	0.00403	49.2936	0.00403	1.00464	0.00403	16.0890	0.00403	49.3088	0.00403
30	Jeffreys	1.03686	0.07020	16.5563	0.07020	50.5246	0.07020	1.02791	0.07020	16.5673	0.07020	50.6038	0.07020	1.03678	0.07020	16.6341	0.07020	50.6590	0.07020
50		1.02010	0.04124	16.2593	0.04124	50.1087	0.04124	1.02411	0.04124	16.3179	0.04124	49.8674	0.04124	1.02081	0.04124	16.3662	0.04124	50.0359	0.04124
100		1.01204	0.02031	16.1777	0.02031	49.4667	0.02031	1.01064	0.02031	16.1720	0.02031	49.4841	0.02031	1.01039	0.02031	16.1187	0.02031	49.4309	0.02031
200		1.00596	0.01008	16.0850	0.01008	49.1865	0.01008	1.00423	0.01008	16.0760	0.01008	49.2391	0.01008	1.00501	0.01008	16.0741	0.01008	49.3851	0.01008
500		1.00243	0.00401	16.0383	0.00401	49.0779	0.00401	1.00275	0.00401	16.0324	0.00401	49.1231	0.00401	1.00125	0.00401	16.0218	0.00401	49.0598	0.00401

From Tables 10− 15, it is clear that Bayes estimates approach to the true value and Bayes risks approach to zero with increase in sample size. Jeffreys prior requires less number of observations than the Uniform prior for efficient estimation. Also, Bayes risks under SELF, QLF, DLF, PLF, WLF and SAPLF using Jeffreys prior are minimum. The results under QLF are much better than the results of the remaining assumed loss functions. Also, the results show that the parameter is over and under estimated and the degree of over and under estimation is minimum under Jeffreys and QLF. For large scale, the degree of over estimation is significant and SELF gives much poor estimates under both the priors.

6. Application

To illustrate the proposed methodology, we used a published data on 137 inoperable lung cancer patients. The data were published in a book titled "The Statistical Analysis of Failure Time Data", by Kalbfleisch and Prentice [3]. For our purpose, obviously these are anonymous data since we don't know the patients' identifications. Also, no identification of the patients are given by the authors in their book. In our study, these data are analyzed anonymously. There is no ethics committee/institutional review board (or data production agency/commissioner) that approved this retrospective study.

The BEs and BRs under the assumed loss functions, for the real data set, are showcased in the tables given below.

Table 16. BEs and BRs under SELF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.65964	2.075529	6.316937	0.6092167	3.034387	0.1405726
Jeffreys	11.48690	1.984194	6.223353	0.582408	2.989433	0.1343866

Table 17. BEs and BRs under QLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.31921	0.014598	6.132501	0.0145985	2.945792	0.014598
Jeffreys	11.15634	0.014389	6.0442630	0.014389	2.903407	0.014388

Table 18. BEs and BRs under DLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.83765	0.015037	6.413379	0.015037	3.080714	0.015037
Jeffreys	11.65964	0.014815	6.316937	0.014815	3.034387	0.014815

Table 19. BEs and BRs under PLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.748300	25.77324	6.364975	13.36979	3.057463	6.258806
Jeffreys	11.572950	25.31613	6.269970	13.15072	3.011826	6.16111

Table 20. BEs and BRs under WLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.48690	0.1727354	6.223353	0.0935842	2.989433	0.0449539
Jeffreys	11.31921	0.167692	6.1325010	0.090852	2.94579	0.043641

Table 21. BEs and BRs under SAPLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.572950	0.0149815	6.269970	0.0149815	3.011826	0.0149815
Jeffreys	11.402750	0.0147603	6.1777600	0.0147603	2.967532	0.0147603

After examining the results presented in tables 16−21, it can easily be concluded that the performance of Jeffreys prior is better. Also, QLF performs much better than the rest of the assumed loss functions for estimating the scale parameter of Log-Normal distribution.

6.1. Graphical Presentation. The graphs of the posterior distributions, under the two priors, are sketched for different values of the location parameter using the real data set.

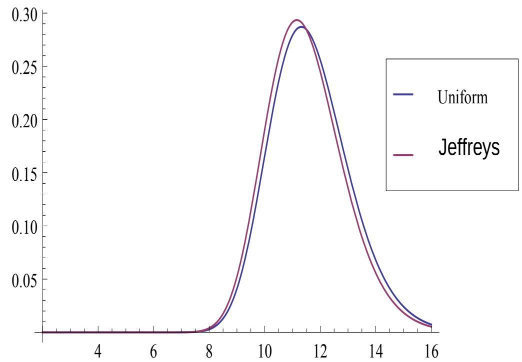

Figure 1. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 1$

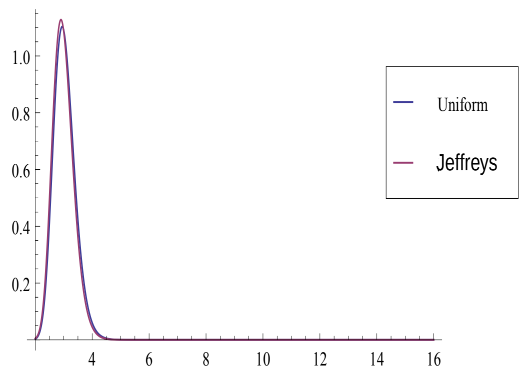

Figure 2. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 2$

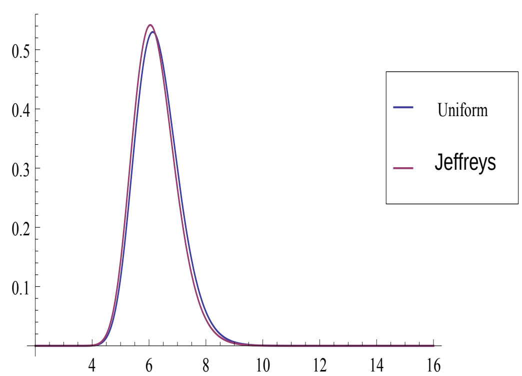

Figure 3. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 3$

The figures 1, 2 and 3 clearly favor Jeffreys as the best non-informative prior for the scale parameter, as the curves of the posterior distribution under this prior are less skewed. For $\theta = 3$, the graph has least posterior skewness.

Conclusions

In this paper, comparisons of non-informative priors for estimating the scale parameter of log-normal distribution have been presented. The two non-informative priors, Uniform and Jeffreys, have been considered and then their posterior distributions were derived. Comparisons have been made on the basis of posterior variance, coefficients of skewness, ex-kurtosis and Bayes risks. In these comparisons, we have observed that Jeffreys prior gives the less posterior variances, less posterior skewness, less ex-kurtosis and less Bayes risks. The simulation results for large value of ϕ are not much convincing. The degree of over estimation significantly increases for large values of scale. SELF performs poorly for large values of ϕ. Also, the results under Quadratic Loss Function (QLF) are efficient with minimum risks. A real data set of lung cancer patients had also been analyzed which verified the simulation results.

Therefor, it is concluded that Jeffreys prior is the appropriate prior when no prior information is available. Also, the Quadratic Loss Function (QLF) is recommended to be used for the estimation of the scale parameter of Log-Normal distribution.

This work can further be extended by considering different (weakly) informative priors and different loss functions. The location parameter can also be estimated for known scale parameter in future research work. Both the parameters can be assumed unknown and can be estimated simultaneously.

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