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A note on distributivity of the lattice of L-ideals of a ring

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Abstract

Many studies have investigated the lattice of fuzzy substructures of algebraic structures such as groups and rings. In this study, we prove that the lattice of L-ideals of a ring is distributive if and only if the lattice of its ideals is distributive, for an infinitely $\vee$-distributive lattice L.

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1. Introduction

The lattice-theoretic aspects of algebraic substructures and L-algebraic substructures have been a topic of discussion in the literature for quite some time. It follows as a consequence of the subdirect product theorem formulated by Professor Tom Head in [9] that the properties of the lattice of algebraic substructures and that of corresponding fuzzy algebraic substructures are almost identical. However before the emergence of the subdirect product theorem, the modularity of the lattice of fuzzy normal subgroups of a group and the modularity of the lattice of fuzzy ideals of a ring have been established in [1–5, 10, 17].

The distributivity constitutes a very powerful property of a lattice. On the other hand, Tarnauceanu [15] worked on finite groups and proved that a group is cyclic iff its lattice of fuzzy subgroups is distributive. Majumdar and Sultana [13] proved that the lattice of fuzzy ideals of a ring is distributive. However, Kumar [12] has obtained just the opposite of this result. Also Zhang and Meng [18] gave a counter example for the result of Majumdar and Sultana. Recently in [11] the modularity of L-ideals of a ring is established, where the subdirect product theorem of Tom Head does not apply. Finally, the lattice of L-fuzzy extended ideals is studied in [7]. We ask: is the lattice of all L-ideals of a ring distributive whose lattice of all ideals is distributive? This paper will answer the question for an infinitely $\vee$-distributive lattice. In this paper, we propose an analogous connection between the lattice of L-ideals and the lattice of ideals of a ring. We first describe some properties of the lattice of L-ideals that are tools to obtain some results. Using these results, we prove that the lattice of L-ideals is distributive when the lattice of ideals is distributive for an infinitely $\vee$-distributive lattice L.

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2. Preliminaries

In this section, we briefly recall some basic concepts of lattices, L-subsets and rings. Throughout this paper, L is a completely lattice with the least element 0 and the greatest element 1. For every family $\{b_i \mid i \in \Delta\}$, we can popularize some operations such as

$$\bigvee_{i\in\Delta} b_i = \sup\{b_i \mid i \in \Delta\}, \ \bigwedge_{i\in\Delta} b_i = \inf\{b_i \mid i \in \Delta\}.$$

A complete lattice L is called infinitely $\vee$-distributive lattice if for all $\alpha \in L$ and $\Delta \subseteq L$,

$$\alpha \wedge (\bigvee_{\beta\in\Delta} \beta) = \bigvee_{\beta\in\Delta} (\alpha \wedge \beta).$$

For a nonempty set X, an L-subset is any function from X into L, which is introduced by Goguen [8] as a generalization of the notion of Zadeh's fuzzy subset [16]. The class of L-subsets of X will be denoted by $F(X, L)$. In particular, if $L = [0, 1]$, it is appropriate to replace fuzzy subset with L-subset. In this case the set of all fuzzy subsets of X is denoted by $F(X)$. Let μ and ν be L-subsets of X. We say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$, denoted $\mu \leq \nu$. Then $\leq$ is a partial ordering on $F(X, L)$.

For each $\alpha \in L$, we define the level subset

$$\mu_\alpha \ = \{x \in X \mid \alpha \leq \mu(x)\}$$

Let μ_i $(i \in \Delta)$ be an L-subset of X. Define the intersection as follows:

$$(\textstyle\bigcap_{i\in\Delta} \mu_i)(x) \ = \textstyle\bigwedge_{i\in\Delta} \mu_i(x)$$

for all $x \in X$. The characteristic function of a set $A \subseteq X$ is denoted by 1_A.

Throughout this paper, R stands for a commutative ring with identity. $I(R)$ stands for all ideals of R, is a complete lattice with respect to set inclusion, called the ideals lattice of R. Note that $I(R)$ has initial element $\{0\}$ and final element R, and its binary operations $\wedge, \vee$ are defined by $I \wedge J = I \cap J$ and $I \vee J = I + J$, for all $I, J \in I(R)$. $I(R)$ may not be a distributive lattice. For example, let $R = \mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}$ is the ring of integers, we define the operations as follows:

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b) \cdot (c, d) = (0, 0)$$

for any $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. Then $(R, +, \cdot)$ form a ring with zero $(0, 0)$.

$$\{(x, x) \mid x \in \mathbb{Z}\} \cap ((\mathbb{Z} \times \{0\}) + (\{0\} \times \mathbb{Z})) = \{(x, x) \mid x \in \mathbb{Z}\},$$

whereas

$$(\{(x, x) \mid x \in \mathbb{Z}\} \cap (\mathbb{Z} \times \{0\})) + (\{(x, x) \mid x \in \mathbb{Z}\} \cap (\{0\} \times \mathbb{Z})) = \{(0, 0)\}.$$

The further knowledge about lattices and rings required in this paper can be found in [6, 14].

3. L-ideals

In this section, we investigate the lattice structure of L-ideals of a ring R.

Definition 3.1. [14] Let μ be an L-subset in a ring R. Then μ is called an L-ideal of R if

$$\mu(x - y) \geq \mu(x) \wedge \mu(y) \text{ and } \mu(xy) \geq \mu(x) \vee \mu(y)$$

for all $x, y \in R$. The family of all L-ideals is denoted by $FI(R, L)$. In particular, when $L = [0, 1]$, an L-ideal of R is referred to as a fuzzy ideal of R. The family of all fuzzy ideals is denoted by $FI(R)$.

The following lemma easily obtained from Proposition 2.2.[17].

Lemma 3.2. *Let $\mu \in F(R, L)$. Then μ is an L-ideal of R iff $\mu_\alpha = \emptyset$ or μ_α is a classical ideal of R, for any $\alpha \in L$.*

Theorem 3.3. [11] *Let μ_i $(i \in \Delta)$ be an L-ideal of a ring R. Then $\bigcap_{i\in\Delta} \mu_i$ is an L-ideal of R.*

By the Theorem 3.3., we immediately get the next corollary.

Corollary 3.4. *$FI(R, L)$ is a complete lattice under the ordering of L-set inclusion such that $\bigwedge_{i\in\Delta} \mu_i = \bigcap_{i\in\Delta} \mu_i$ for all $\mu_i \in FI(R, L)$ $(i \in \Delta)$.*

Lemma 3.5. [2] *Let A, B be subsets of R. Then*

(1) *A is an ideal of R if and only if 1_A is an L-ideal of R,*
(2) *If A, B are ideals of R, then $1_A \vee 1_B = 1_{A+B}$ and $1_A \wedge 1_B = 1_{A\cap B}$.*
(3) *$\{1_A \mid A$ is an ideal of $R\}$ is a sublattice of $FI(R, L)$*

4. The distributivity of $FI(R, L)$

In this section we will investigate some conditions related to distributivity of the lattice of L-ideals of a ring R.

Definition 4.1. Let μ and ν be L-subsets of a ring R. Define $\mu \oplus \nu$ as follows:

$$(\mu \oplus \nu)(x) = \mu(x) \vee \nu(x) \vee \bigvee_{x=y+z} \mu(y) \wedge \nu(z)$$

for all $x \in R$.

Lemma 4.2. *Let L be an infinitely $\vee$-distributive lattice and $\mu, \nu \in FI(L, R)$. Then $\mu \vee \nu = \mu \oplus \nu$.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}
&\mu \oplus \nu(x) \wedge \mu \oplus \nu(y)\\
&= [\mu(x) \vee \nu(x) \vee (\bigvee_{x=a+b} \mu(a) \wedge \nu(b))] \wedge [\mu(y) \vee \nu(y) \vee (\bigvee_{y=c+d} \mu(c) \wedge \nu(d))]\\
&= [(\mu(x) \vee \nu(x)) \wedge (\mu(y) \vee \nu(y))] \vee [(\mu(x) \vee \nu(x)) \wedge (\bigvee_{y=c+d} \mu(c) \wedge \nu(d))]\\
&\vee[(\mu(y) \vee \nu(y)) \wedge (\bigvee_{x=a+b} \mu(a) \wedge \nu(b))] \vee [(\bigvee_{y=c+d} \mu(c) \wedge \nu(d)) \wedge (\bigvee_{x=a+b} \mu(a) \wedge \nu(b))]\\
&= (\mu(x) \wedge \mu(y)) \vee (\mu(x) \wedge \nu(y)) \vee (\nu(x) \wedge \mu(y)) \vee (\nu(x) \wedge \nu(y))\\
&\vee(\bigvee_{y=c+d} \mu(x) \wedge \mu(c) \wedge \nu(d)) \vee (\bigvee_{y=c+d} \nu(x) \wedge \mu(c) \wedge \nu(d)) \vee (\bigvee_{x=a+b} \mu(a) \wedge \nu(b) \wedge \mu(y))\\
&\vee(\bigvee_{x=a+b} \mu(a) \wedge \nu(b) \wedge \nu(y)) \vee (\bigvee_{\substack{x=a+b\\ y=c+d}} \mu(a) \wedge \nu(b) \wedge \mu(c) \wedge \nu(d))\\
&\le \mu(x+y) \vee \nu(x+y) \vee (\mu(x) \wedge \nu(y)) \vee (\mu(y) \wedge \nu(x)) \vee (\bigvee_{y=c+d} \mu(x+c) \wedge \nu(d))\\
&\vee(\bigvee_{y=c+d} \mu(c) \wedge \nu(d+x)) \vee (\bigvee_{x=a+b} \mu(a+y) \wedge \nu(b))\\
&\vee(\bigvee_{x=a+b} \mu(a) \wedge \nu(b+y)) \vee (\bigvee_{\substack{x=a+b\\ y=c+d}} \mu(a+c) \wedge \nu(b+d))\\
&\le \mu(x+y) \vee \nu(x+y) \vee (\bigvee_{x+y=u+v} \mu(u) \wedge \nu(v))\\
&= \mu \oplus \nu(x+y)
\end{aligned}$$

Hence $\mu \oplus \nu(x) \wedge \mu \oplus \nu(y) \leq \mu \oplus \nu(x+y)$.

$$\begin{aligned} \mu \oplus \nu(-x) &= \mu(-x) \vee \nu(-x) \vee (\bigvee_{-x=a+b} \mu(a) \wedge \nu(b)) \\ &= \mu(-x) \vee \nu(-x) \vee (\bigvee_{x=(-a)+(-b)} \mu(-(-a)) \wedge \nu(-(-b)) \\ &\leq \mu(x) \vee \nu(x) \vee (\bigvee_{x=u+v} \mu(u) \wedge \nu(v))] \\ &= \mu \oplus \nu(x) \end{aligned}$$

Hence $\mu \oplus \nu(-x) \leq \mu \oplus \nu(x)$.

$$\begin{aligned} \mu \oplus \nu(x) &= \mu(x) \vee \nu(x) \vee (\bigvee_{x=a+b} \mu(a) \wedge \nu(b)) \\ &\leq \mu(xy) \vee \nu(xy) \vee (\bigvee_{x=a+b} \mu(ay) \wedge \nu(by)) \\ &\leq \mu(xy) \vee \nu(xy) \vee (\bigvee_{xy=u+v} \mu(u) \wedge \nu(v))] \\ &= \mu \oplus \nu(xy) \end{aligned}$$

Similarly, we have $\mu \oplus \nu(y) \leq \mu \oplus \nu(xy)$. Thus $\mu \oplus \nu \in FI(R, L)$. It is clear that $\mu \leq \mu \oplus \nu$ and $\nu \leq \mu \oplus \nu$.
Let $\theta \in FI(R, L)$ such that $\mu \leq \theta$ and $\nu \leq \theta$. Then

$$\mu(a) \wedge \nu(b) \leq \theta(a) \wedge \theta(b) \leq \theta(a+b) = \theta(x)$$

for all $x = a+b$. By the definition of $\mu \oplus \nu$, it folows that $\mu \oplus \nu \leq \theta$. Hence $\mu \vee \nu = \mu \oplus \nu$ □

The following theorem gives the main results of this section.

Theorem 4.3. *If L is an infinitely $\vee$-distributive lattice, then the following conditions are equivalent:*

(1) *$I(R)$ is a distributive lattice,*
(2) *$FI(R, L)$ is a distributive lattice.*

Proof. (2) ⇒ (1) By Lemma 3.5, it is clear.
(1) ⇒ (2) Let $\mu, \nu, \theta \in FI(R)$. Since the distributive inequality is valid for every lattice, we have

$$(\mu \wedge \nu) \vee (\mu \wedge \theta) \leq \mu \wedge (\nu \vee \theta).$$

And by Lemma 4.2 and Corollary 3.4,

$$\begin{aligned} (\mu \wedge (\nu \vee \theta))(x) &= (\mu \wedge (\nu \oplus \theta))(x) \\ &= \mu(x) \wedge [\nu(x) \vee \theta(x) \vee (\bigvee_{x=a+b} \nu(a) \wedge \theta(b))] \\ &= (\mu(x) \wedge \nu(x)) \vee (\mu(x) \wedge \theta(x)) \vee (\bigvee_{x=a+b} \nu(a) \wedge \theta(b) \wedge \mu(x)) \end{aligned}$$

Let $\lambda = \nu(a) \wedge \theta(b) \wedge \mu(x)$ for some $a, b \in R$ such that $x = a+b$
Thus we have $x \in \mu_\lambda,\ a \in \nu_\lambda,\ b \in \theta_\lambda$. Then $x \in \mu_\lambda \cap (\nu_\lambda + \theta_\lambda)$. Due to distributivity of $I(R)$,

$$x \in (\mu_\lambda \cap \nu_\lambda) + (\mu_\lambda \cap \theta_\lambda).$$

It follows that there exist $u, v \in R$ such that $x = u + v$,

$$u \in \mu_\lambda \cap \nu_\lambda \ \text{ and } \ v \in \mu_\lambda \cap \theta_\lambda.$$

Thus we have $\lambda \leq \mu(u),\ \lambda \leq \nu(u),\ \lambda \leq \mu(v),\ \lambda \leq \theta(v)$. Hence,

$$\lambda \leq (\mu \wedge \nu)(u) \wedge (\mu \wedge \theta)(v).$$

Now it follows that

$$\lambda \leq \bigvee_{x=u+v} (\mu \wedge \nu)(u) \wedge (\mu \wedge \theta)(v).$$

Hence we obtain

$$\bigvee_{x=a+b} \nu(a) \wedge \theta(b) \wedge \mu(x) \leq \bigvee_{x=u+v} (\mu \wedge \nu)(u) \wedge (\mu \wedge \theta)(v).$$

Therefore,

$$\begin{aligned}(\mu \wedge (\nu \vee \theta))(x) &= (\mu(x) \wedge \nu(x)) \vee (\mu(x) \wedge \theta(x)) \vee (\bigvee_{x=a+b} \nu(a) \wedge \theta(b) \wedge \mu(x)) \\ &\leq (\mu \wedge \nu)(x) \vee (\mu \wedge \theta)(x) \vee (\bigvee_{x=u+v} (\mu \wedge \nu)(u) \wedge (\mu \wedge \theta)(v)) \\ &= ((\mu \wedge \nu) \oplus (\mu \wedge \theta))(x) \\ &= (\mu \wedge \nu) \vee (\mu \wedge \theta)(x).\end{aligned}$$

and the proof is completed. □

By the Theorem 4.3, we immediately get the next corollary.

Corollary 4.4. *$I(R)$ is a distributive lattice if and only if $FI(R)$ is a distributive lattice.*

5. Conclusion

Many researches studied the lattice structure (distributive or modular) of fuzzy algebraic substructures. In future work, the same results could also be studied under a t-norm operation on L. Also, we will try to expose some classes of algebra whose lattices of L-subalgebras constitute distributive lattice.

References

[1] N. Ajmal and K.V. Thomas, *The lattices of fuzzy subgroups and fuzzy normal subgroups*, Inform. Sci. **76**, 1–11, 1994.

[2] N. Ajmal and K.V. Thomas, *The lattice of fuzzy ideals of a ring*, Fuzzy Sets and Systems **74**, 371–379, 1995.

[3] N. Ajmal and K.V. Thomas, *The lattices of fuzzy normal subgroups is modular*, Inform. Sci. **83**, 199–218, 1995.

[4] D. Bayrak and S. Yamak, *The lattice of generalized normal L-subgroups*, J. Intell. Fuzzy Syst. **27**, 1143–1152, 2014.

[5] D. Bayrak and S. Yamak *Distributivity and pseudocomplementation of lattices of generalized L-subgroups*, Int. J. Algebra Stat. **5**, 107–114, 2016.

[6] G. Birhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., Rhode Island, 1967.

[7] N. Gao, Q. Li and Z. Li, *When do L-fuzzy ideals of a ring generate a distributive lattice?*, Open Math. **4** (1), 531–542, 2016.

[8] J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18**, 145–174, 1967.

[9] T. Head, *A meta theorem for deriving fuzzy theorems from crisp versions*, Fuzzy Sets and Systems **73**, 349–358, 1995.

[10] I. Jahan, *Modularity of Ajmal for the lattices of fuzzy ideals of a ring*, Iran. J. Fuzzy Syst. **5**, 71–78, 2008.

[11] I. Jahan, *The lattice of L-ideals of a ring is modular*, Fuzzy Sets and Systems **199**, 121–129, 2012.

[12] R. Kumar, *Non-distributivity of the lattice of fuzzy ideals of a ring*, Fuzzy Sets and Systems **97**, 393–394, 1998.

[13] S. Majumdar and Q.S. Sultana, *The lattice of fuzzy ideals of a ring*, Fuzzy Sets and Systems **81**, 271–273, 1996.

[14] J.N. Mordeson and D.S. Malik, *Fuzzy Commutative Algebra*, World Scientific, 1998.

[15] M. Tarnauceanu, *Distributivity in lattices of fuzzy subgroups*, Inform. Sci. **179**, 1163–1168, 2009.
[16] L.A. Zadeh, *Fuzzy sets*, Inform. Control **8**, 338–353, 1965.
[17] Q. Zhang, *The lattice of fuzzy (left, right) ideals of a ring is modular*, Fuzzy Sets and Systems **125**, 209–214, 2002.
[18] Q. Zhang and G. Meng, *On the lattice of fuzzy ideals of a ring*, Fuzzy Sets and Systems **112**, 349–353, 2000.